A Two-Fold Linear Programming Model with Fuzzy Data

Saber Saati, Islamic Azad University, North Tehran Branch, Iran

Adel Hatami-Marbini, Université Catholique de Louvain, Belgium

Madjid Tavana, La Salle University, USA

Elham Hajiahkondi, Payame Noor University, Iran

ABSTRACT

Linear programming (LP) is the most widely used optimization technique for solving real-life problems because of its simplicity and efficiency. Although LP models require well-suited information and precise data, managers and decision makers dealing with optimization problems often have a lack of information on the exact values of some parameters used in their models. Fuzzy sets provide a powerful tool for dealing with this kind of imprecise, vague, uncertain or incomplete data. In this paper, the authors propose a two-fold model which consists of two new methods for solving fuzzy LP (FLP) problems in which the variables and the coefficients of the constraints are characterized by fuzzy numbers. In the first method, the authors transform their FLP model into a conventional LP model by using a new fuzzy ranking method and introducing a new supplementary variable to obtain the fuzzy and crisp optimal solutions simultaneously with a single LP model. In the second method, the authors propose a LP model with crisp variables for identifying the crisp optimal solutions. The authors demonstrate the details of the proposed method with two numerical examples.

Keywords: *Fuzzy Decision Variables, Fuzzy Linear Programming, Fuzzy Sets, Membership Function, Mathematical Programming, Optimization*

INTRODUCTION

Linear programming (LP) is the most widely used and understood mathematical optimization technique employed by the business and industrial community (Aguilar, 1973; Shamir, 1987; Sweeney et al., 2009). The conventional LP deals with crisp parameters. However, managerial decision making is subject to professional judgments usually based on imprecise, vague, uncertain or incomplete information (Leung, 1988). Fuzzy set theory has been proposed to handle such imprecision by generalizing the notion of membership in a set. Essentially, in a fuzzy set each element is associated with a point value selected from the unit interval [0,1], which is an arbitrary grade of truth referred to as the grade of membership in the set.

The main objective in fuzzy LP (FLP) is to find the best solution possible with imprecise, vague, uncertain or incomplete information. There are many sources of imprecision in FLP.

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For example, sometimes coefficient variables are not known precisely, other times constraints satisfaction limits may be vague. The challenge in FLP is to construct an optimization model that can produce the optimal solution with subjective professional judgments. In this study, we propose a two-fold model for solving FLP problems in which the variables and the coefficients of the constraints are characterized by fuzzy numbers. We transform a FLP model into a conventional LP model by applying a new fuzzy ranking method and obtaining the fuzzy and crisp optimal solutions.

This paper is organized as follows: The next section presents a brief review of the existing literature followed by some primary definitions of fuzzy sets. We then introduce a mathematical model for FLP. Following this introduction, we illustrate the details of the proposed framework followed by a numerical example to demonstrate the applicability of the proposed method. Finally, we finish the paper with our conclusions and future research directions.

LITERATURE REVIEW

The theory of fuzzy mathematical programming was first proposed by Tanaka et al. (1974) based on the fuzzy decision framework of Bellman and Zadeh (1970) to address the impreciseness and vagueness of the parameters in problems with fuzzy constraints and objective functions. Zimmermann (1978) introduced the first formulation of FLP. He constructed a crisp model of the problem and obtained its crisp results using an existing algorithm. He then used the crisp results and fuzzified the problem by considering subjective constants of admissible deviations for the goal and the constraints. Finally, he defined an equivalent crisp problem using an auxiliary variable that represented the maximization of the minimization of the deviations on the constraints. Zimmermann (1978, 1987) used Bellman and Zadeh's (1970) interpretation that a fuzzy decision is a union of goals and constraints.

In the past decade, researchers have discussed various properties of FLP problems and proposed an assortment of models (Luhandjula, 1989). Zhang et al. (2003) proposed a FLP with fuzzy numbers for the coefficients of objective functions. They introduced a number of optimal solutions to the FLP problems and developed a number of theorems for converting the FLP problems to multi-objective optimization problems with four-objective functions. Stanciulescu (2003) proposed a FLP model with fuzzy coefficients for the objectives and the constraints. He used fuzzy decision variables with a joint membership function instead of crisp decision variables and linked the decision variables together to sum them up to a constant. He considered lower-bounded fuzzy decision variables that set up the lower bounds of the decision variables. He then generalized the method to lower–upper-bounded fuzzy decision variables that set up also the upper bounds of the decision variables. Ganesan and Veeramani (2006) proposed a FLP model with symmetric trapezoidal fuzzy numbers. They proved fuzzy analogues of some important LP theorems and obtained some interesting results which in turn led to the solution for FLP problems without converting them into crisp LP problems. Ebrahimnejad (2011) showed that the method proposed by Ganesan and Veermani (2006) stops in a finite number of iterations and proposed a revised version of their method that was more efficient and robust in practice. He also proved the absence of degeneracy and showed that if an FLP problem has a fuzzy feasible solution, it also has a fuzzy basic feasible solution and if an FLP problem has an optimal fuzzy solution, it also has an optimal fuzzy basic solution.

Hosseinzadeh Lotfi et al. (2009) considered full FLP problems where all parameters and variables were triangular fuzzy numbers. They pointed out that there is no method in the literature for finding the fuzzy optimal solution of full FLP problems and proposed a new method to find the fuzzy optimal solution of full FLP problems with equality constraints. They used the concept of the symmetric triangular fuzzy numbers and introduced an approach to defuzzify a general fuzzy quantity. They first approximated the fuzzy triangular numbers to its nearest symmetric triangular numbers, with the assumption that all decision variables were symmetric triangular. They then converted every FLP model into two crisp complex LP models and used a special ranking for fuzzy numbers to transform their full FLP model into a multiobjective linear programming where all variables and parameters were crisp. Kumar et al. (2011) further studied the full FLP problems with equality introduced by Hosseinzadeh Lotfi et al. (2009) and proposed a new method for finding the fuzzy optimal solution in these problems.

Mahdavi-Amiri and Nasseri (2006) proposed a FLP model where a linear ranking function was used to order trapezoidal fuzzy numbers. They established the dual problem of the LP problem with trapezoidal fuzzy variables and deduced some duality results to solve the FLP problem directly with the primal simplex tableau. Ebrahimnejad (2010) introduced a new primal-dual algorithm for solving FLP problems by using the duality results proposed by Mahdavi-Amiri and Nasseri (2007). Ebrahimnejad (2011) has also generalized the concept of sensitivity analysis in FLP problems by applying fuzzy simplex algorithms and using the general linear ranking functions on fuzzy numbers.

AN OVERVIEW OF FUZZY SETS

Let $\mathbb{R}$ be the set of all real numbers. The fuzzy subset $\tilde{A}$ is defined by $\mu_{\tilde{A}}(x) \rightarrow [0,1]$, which is called a membership function.

Definition 1 (Fuzzy number). The fuzzy number $\tilde{A}$ is a normal and convex fuzzy subset of X and is defined as

Normality: $\exists x \in \mathbb{R},\ \vee_x\ \mu_{\tilde{A}}(x) = 1$

Convexity: $\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min(\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)), \forall x,.\ y \in \mathbb{R},\ \forall \lambda \in [0,1]$

Definition 2 (Generalized trapezoidal fuzzy number). The fuzzy number $\tilde{A} = (a^l, a^m, a^n, a^u)$ is called a generalized trapezoidal fuzzy number with membership function $\mu_{\tilde{A}}$ and has the following properties:

a) $\mu_{\tilde{A}}$ is a continuous mapping from $\mathbb{R}$ to the closed interval [0, 1],
b) $\mu_{\tilde{A}}(x) = 0$ for all $x \in (-\infty, a^l]$,
c) μ_A is strictly increasing on [a^l , a^m],
d) $\mu_{\tilde{A}}(x) = 1$ for all $x \in [a^m, a^n]$,
e) $\mu_{\tilde{A}}$ is strictly decreasing on [a^n , a^u], and
f) $\mu_{\tilde{A}}(x) = 0$ for all $x \in [a^u, +\infty)$.

The membership function $\mu_{\tilde{A}}$ of $\tilde{A}$ can be defined as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} f_a(x), & a^l \leq x \leq a^m, \\ 1, & a^m \leq x \leq a^n, \\ g_a(x), & a^n \leq x \leq a^u, \\ 0, & Otherwise. \end{cases} \tag{1}$$

where

$f_a : [a^l, a^m] \rightarrow [0,1]$

and

$g_a : [a^n, a^u] \rightarrow [0,1]$.

Particularly, a special type of trapezoidal fuzzy number, with a membership function $\mu_{\tilde{A}}$, can be expressed as:

$$\mu_{\tilde{A}}(x)=\begin{cases}\dfrac{x-a^l}{a^m-a^l}, & a^l\le x\le a^m,\\ 1, & a^m\le x\le a^n,\\ \dfrac{a^u-x}{a^u-a^n}, & a^n\le x\le a^u,\\ 0, & Otherwise.\end{cases} \tag{2}$$

If $a^l=a^m$ and $a^n=a^u$, then $\tilde{A}$ is called a crisp or simple interval. The trapezoidal fuzzy number $\tilde{A}=(a^l,a^m,a^n,a^u)$ is reduced to a real number A if $a^l=a^m=a^n=a^u$. In contrast, a real number A can be written as a trapezoidal fuzzy number $\tilde{A}=(a, a, a, a)$. If $a^{Mi}=a^m=a^n$, then, $\tilde{A}=(a^l,a^{Mi},a^u)$ is called a triangular fuzzy number.

A triangular fuzzy number has the following membership function:

$$\mu_{\tilde{A}}(x)=\begin{cases}\dfrac{x-a^l}{a^{Mi}-a^l}, & a^l\le x\le a^{Mi},\\ \dfrac{a^u-x}{a^u-a^{Mi}}, & a^{Mi}\le x\le a^u,\\ 0, & Otherwise.\end{cases} \tag{3}$$

For the sake of simplicity and without loss of generality, we assume that all fuzzy numbers used throughout the paper are trapezoidal fuzzy numbers.

Definition 3 (Fuzzy arithmetic operation). Suppose that we have two positive trapezoidal fuzzy numbers $\tilde{A}=(a^l,a^m,a^n,a^u)$ and $\tilde{B}=(b^l,b^m,b^n,b^u)$, then the arithmetic operations of these two trapezoidal fuzzy numbers are defined as follows:

$$\begin{aligned}&\tilde{A}(+)\tilde{B}=(a^l+b^l,a^m+b^m,a^n+b^n,a^u+b^u),\\ &\tilde{A}(-)\tilde{B}=(a^l-b^u,a^m-b^n,a^n-b^m,a^u-b^l),\\ &(\tilde{A})^{-1}=(\frac{1}{a^u},\frac{1}{a^n},\frac{1}{a^m},\frac{1}{a^l}),\\ &-(\tilde{A})=(a^u,a^n,a^m,a^l),\\ &k\tilde{A}=(ka^l,ka^m,ka^n,ka^u),\forall k\in R^+,\\ &k\tilde{A}=(ka^u,ka^n,ka^m,ka^l),\forall k\in R^-.\end{aligned} \tag{4}$$

THE FUZZY LINEAR PROGRAMMING MODEL

The fuzzy sets theory proposed by Zadeh (1965) and further developed by Dubois and Prade (1988) is a popular method for dealing with decision problems that are formulated as LP models with imprecise, vague or uncertain variables and coefficients of the constraints. In this section we introduce a fuzzy LP (FLP) problem where the decision variables, the coefficients of the constraints and resources (right-hand-side values) are fuzzy quantities. We then define the feasible and the optimal solution based on some fuzzy relations.

The LP problems are formulated as follows:

$$\begin{aligned}&\text{Minimize}\quad Z=cx\\ &\text{Subject to:}\\ &A.x\ge b,\quad x\ge 0.\end{aligned} \tag{5}$$

where $x\in\mathbb{R}^n, c\in\mathbb{R}^n, b\in\mathbb{R}^m$ and A is an $(m\times n)$ real matrix. Contrary to the classical LP problems, here, x, A and b are the fuzzy numbers denoted by symbols with the tilde. Let $\mu_{\tilde{b}}:R\to[0,1], \mu_{\tilde{A}}:R\to[0,1], \mu_{\tilde{x}}:R\to[0,1]$ be membership functions of the fuzzy numbers, $\tilde{b},\tilde{A}$ and $\tilde{x}$, respectively. To define a FLP problem, we will use the following proposition:

Proposition 1. Let $\tilde{x}\in F(R)$ where $F(R)$ presents the set of all fuzzy subsets. Then, the fuzzy set $c\tilde{x}$ is a fuzzy number based on the extension principle.

The FLP problem associated with the standard LP problem (5) can be expressed as follows:

$$\begin{aligned}&\text{Minimize} \quad Z = c\tilde{x}\\&\text{Subject to:}\\&\tilde{A}.\tilde{x} \geq \tilde{b}, \quad \tilde{x} \geq 0.\end{aligned} \tag{6}$$

where $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n)$ are fuzzy decision variables and $\tilde{b} = (\tilde{b}_1, \tilde{b}_2, ..., \tilde{b}_m)^t$ and $\tilde{A} = [\tilde{a}_{ij}]_{m\times n}$ represent the fuzzy parameters involved in the objective function and constraints while $c = (c_1, c_2, ..., c_n)$ are the crisp parameters in the objective function.

Definition 4 (Feasible solution). The feasible solution is a set of values of the fuzzy variables $\tilde{x}$ which satisfies all of the constraints in model (6).

Definition 5 (Optimal solution). The optimal solution for model (6) is $\tilde{x}^*$ if for all feasible solutions $\tilde{x}$, we have $c\tilde{x}^* \leq c\tilde{x}$.

Definition 6. Assuming that $\tilde{A} = (a^l, a^m, a^n, a^u)$ represents a trapezoidal fuzzy number, $\tilde{A}$ can be changed into the following crisp value:

$$A = \frac{(a^n - a^m) + (a^u - a^l)}{2} \tag{7}$$

Next, we discuss the fuzzy basic feasible solution and the optimal solution.

Consider the FLP problem (6). After using (7), let $rank(A) = m$ and define partition A as $[B \quad N]$ where B, $m \times m$, is non-singular. Let y be the solution to $By = a_j$ where a_j is the j^{th} column of the coefficient matrix. Thus, $\tilde{x}_B = (\tilde{x}_{B_1}, ..., \tilde{x}_{B_m})^T = B^{-1}\tilde{b}$ and $\tilde{x}_N = 0$ is a solution of $A\tilde{x} = \tilde{b}$. $\tilde{x} = (\tilde{x}_B^T \ \ 0)$ is called a *fuzzy basic solution* (*FBS*) corresponding to the basic B when $\tilde{x}_B \geq 0$. It is vivid that the *FBS* is feasible, therefore, the fuzzy objective value is $\tilde{z} = c_B\tilde{x}_B$ where $c_B = (c_{B_1}, ..., c_{B_m})$. Then,

$$\begin{aligned}&z_j - c_j = c_B B^{-1} a_j - c_j\\&= c_B e_j - c_j = c_{B_i} - c_j\\&= c_j - c_j = 0\end{aligned}$$

Note that $B^{-1}a_j = e_j$ where $e_j = (0, ..., 1, ..., 0)^T$. If $\tilde{x}_B > 0$, then $\tilde{x}$ is called a no degenerate fuzzy basic feasible solution, and if one component of $\tilde{x}$ is zero, then is called a degenerate basic feasible solution. A fuzzy solution is optimal if and only if $z_j = c_B B^{-1} a_j \leq c_j$. In other words, the FLP problem can be rewritten as follows:

$$\begin{aligned}&\text{Minimize} \quad \tilde{Z} = c_B\tilde{x}_B + c_N\tilde{x}_N\\&\text{Subject to:}\\&B\tilde{x}_B + N\tilde{x}_N = \tilde{b}, \quad \tilde{x}_B, \tilde{x}_N \geq 0.\end{aligned} \tag{8}$$

If $\tilde{x}^* = (\tilde{x}_B^T \ \ x_N^T) = (B^{-1}\tilde{b} \ \ 0)$ is a fuzzy basic feasible solution, then, $z^* = c_B\tilde{x}_B = c_B B^{-1}\tilde{b}$. Now, we can have

$$\begin{aligned}&\tilde{z} = c\tilde{x} = c_B\tilde{x}_B + c_N\tilde{x}_N = c_B B^{-1}\tilde{b}\\&-(c_B B^{-1} N - c_N)\ \tilde{x}_N\\&= c_B B^{-1}\tilde{b} - \sum_{j=1}^{n}(c_B B^{-1}a_j - c_j)\tilde{x}_j =\\&= c_B B^{-1}\tilde{b} - \sum_{j=1}^{n}(z_j - c_j)\tilde{x}_j = \tilde{z}^*\\&-\sum_{j\neq B_i}(z_j - c_j)\tilde{x}_j\end{aligned}$$

For each feasible $\tilde{x}$, z_j is smaller than or equal to c_j; therefore, $(z_j - c_j)\tilde{x}_j \leq 0$ and $\sum_j (z_j - c_j)\tilde{x}_j \leq 0 \rightarrow \tilde{z}^* \leq \tilde{z}$. That is to say, $\tilde{x}^*$ is an optimal solution.

THE PROPOSED TWO-FOLD MODEL

In this section, we first propose a new method for dealing with the FLP model shown in (6) where the coefficients and the resources in our constraints and decision variables are assumed to be fuzzy numbers. Using this approach, we can find the fuzzy optimal solution and the crisp optimal solution simultaneously with one LP model. We then consider a special case in which the decision variables are crisp. In this case, the obtained optimal solution is identical to the crisp optimal solution obtained with the previous approach.

Definition 7. $\tilde{A} = (x^l, x^m, x^n, x^u)$ is a non-negative trapezoidal fuzzy number if $\tilde{A}$ corresponds to the following relations:

$$x^u > x^n,$$
$$x^n > x^m,$$
$$x^m > x^l.$$

Definition 8. Based on the above definitions, a trapezoidal fuzzy variable $\tilde{x} = (x^l, x^m, x^n, x^u)$ is called non-negative trapezoidal fuzzy variable if it satisfies the following conditions:

$$\begin{aligned} &x^u \geq x^n, \\ &x^n \geq x^m, \\ &x^m \geq x^l, \\ &x^l + x^u \geq 0, \\ &x^m + x^n \geq 0. \end{aligned} \tag{9}$$

We now extend definition 8 into the constraint in model (6) via definition 9. Note that $\tilde{A}$ in model (6) is characterized as a fuzzy number that can be transformed into A using (7).

Definition 9. $A\tilde{x} \geq \tilde{b}$ if and only if $Ax^m \geq b^m$, $Ax^n \geq b^n$, $Ax^l \geq b^l$ and $Ax^u \geq b^u$, where $\tilde{x}$ satisfies the conditions defined in (9). In addition, we also define a new variable x which lays within the upper and lower bounds represented by the ranges $\left[x^m, x^n\right]$.

Definition 10. The aim of the proposed method in this paper is to obtain the fuzzy and crisp optimal solution for the objective function. Hence, to get crisp optimal solution of the objective function, we need to define a new crisp variable, namely x, which lays within the upper and lower bounds represented by the ranges $\left[x^m, x^n\right]$. Based on the definitions, model (6) can be rewritten as follows:

$$\begin{aligned} &\text{Minimize} \quad Z = cx \\ &\text{Subject to:} \\ &Ax^i \geq b^i, \qquad i = l, n, m, u, \\ &x^u - x^n \geq 0, \qquad x^n - x^m \geq 0, \\ &x^m - x^l \geq 0, \qquad x^u + x^l \geq 0, \\ &x^n + x^m \geq 0, \qquad x^m \leq x \leq x^n. \end{aligned} \tag{10}$$

where x^l, x^m, x^n, x^u and x are the decision variables and A is calculated by (7) from $\tilde{A}$. We should note that model (10) is a LP problem with crisp variables and coefficients. The interesting feature of the proposed method in this study is that we can obtain the fuzzy optimal value $\tilde{x} = (x^l, x^m, x^n, x^u)$ and the crisp optimal value x simultaneously from model (10). Thereby, we can calculate the fuzzy and crisp optimal values of the objective function using one LP model (10) as follows:

$Z^* = cx^*$ (Crisp value of the objective function)
$\tilde{Z}^* = c\tilde{x}^* = (cx^l, cx^m, cx^n, cx^u)$ (Fuzzy value of the objective function)

In order to introduce an alternative corresponding LP model (11), let us assume that the decision variables in model (6) are crisp and we defuzzify $\tilde{A}$ using formula (7). Therefore,

$$\begin{aligned} &\text{Minimize} \quad Z = cx \\ &\text{Subject to:} \\ &Ax \geq b^m, x \geq 0. \end{aligned} \tag{11}$$

where the right hand side of the constraints of (6), $\tilde{b}$, is substituted with b^m .

Remark 1. The optimal solution of objective function (11) is identical to the crisp optimal solution, x, of objective function (10). It is important to note that in the maximization problems, b^n is replaced with the right hand side of all constraints in model (11).

Let us first assume that the fuzzy coefficients $\tilde{A}$ of the constraints in (6) are converted to the crisp form using formula (7). Then, by applying the following theorem, model (6) can be reduced to a LP problem (Maleki, 2002; Maleki et al., 2000).

Theorem 1. The LP problem (5) and the model in (6) are equivalent.

Proof. If we replace fuzzy numbers $\tilde{b}$ and $\tilde{x}$ in (10) with real numbers b and x the LP problem (5) and the model in (6) are equivalent (Mahdavi-Amiri & Nasseri, 2006, p. 209). That is to say, $x^l = x^m = x^n = x^u$ and $b^l = b^m = b^n = b^u$.

Remark 2. Let $b^l = b^m = b^n = b^u = b$. Then, the feasible region of LP model (5) can envelop the feasible region of (10). The following corollaries are based on this remark.

Corollary 1. Let $b^l = b^m = b^n = b^u = b$. Then the optimal solution of (5) is always smaller than or equal to the optimal solution of (10).

Corollary 2. Let $\hat{x}$ be a feasible solution of (10), then, $\hat{x}$ is a feasible solution of (5).

Corollary 3. Model (10) does not have a solution if LP (5) does not have a solution.

Consider the following general LP model

$$\begin{aligned} &\text{Minimize} \quad Z = cx \\ &\text{Subject to:} \\ &\qquad A.x \geq b, \end{aligned} \tag{12}$$

where x and b are the crisp decision vector and the crisp parameter vector, respectively. Note that x in (12) is free in sign. The proposed model (10) is equivalent to a general LP model (12) with more constraints. In other words, model (10) involves *4m+6n* constraints and *5n* decision variables while model (12) has *m* constraints and *n* decision variables. Thus, the proposed model (10) is equivalent to the general model (12) with bigger dimension (the dimension of model (10) is 4*m*+6*n* while the dimension of model (12) is *m*). We can convert (12) into the following model using an alteration variable, $x' - x''$, where $x' \geq 0$ and $x'' \geq 0$:

$$\begin{aligned} &\text{Minimize} \quad Z = c(x' - x'') \\ &\text{Subject to:} \\ &A(x' - x'') \geq b, \quad x', x'' \geq 0. \end{aligned} \tag{13}$$

Equivalently, we have

$$\begin{aligned} &\text{Minimize} \quad Z = cx \\ &\text{Subject to:} \\ &\qquad Ax \geq b, \end{aligned} \tag{14}$$

As a result, model (10) is a common LP model with some additional variables and constraints. Thereby, in the subsequent section, we review some important properties of the proposed models.

Assume that $A = [a_{ij}]_{m\times(n+1)}$ such that *rank* $(A) = m$ and partition A as $[B \quad N]$ where rank(B)= m , is non-singular. Let y_j be a solution of $By = a_j$ where a_j presents the j^{th} column of the coefficient matrix, A . Then, $w_B = (w_{B_1}, \ldots, w_{B_m})^T = B^{-1}b$ and $w_N = 0$ is a solution of $Aw = b$ where $w = (w_B^T \ w_N^T)$ displays a basic solution. The feasible basic solution can be obtained if $w_B \geq 0$. The objective function value can be computed via $z_j = c_B w_B = c_B y_j = c_B B^{-1} a_j$ for all j. In addition, for each j, we have

$$z_j - c_j = c_B B^{-1} a_j - c_j$$
$$= c_B e_j - c_j = c_{B_i} - c_j \ .$$
$$= c_j - c_j = 0$$

where $e_j = (0,...,1,...,0)^T$

A non-degenerate basic feasible solution occurs if $w_B > 0$. Equivalently, if one component of $\tilde{x}$ is zero, then, it is called a degenerate basic feasible solution. A feasible solution can be an optimal solution if and only if $z_j = c_B B^{-1} a_j \le c_j$ for all *j*.

If a basic feasible solution is $w^* = (w_B^T \ \ w_N^T)$ where $w_B = B^{-1} b$ and $w_n = 0$, then $z^* = c_B w_B = c_B B^{-1} b$. In other words, we can rewrite model (14) as follows:

$$\begin{aligned} &\text{Minimize} \quad Z = c_B w_B + c_N w_N \\ &\text{Subject to:} \\ &B w_B + N w_N = b, w_B, w_N \ge 0. \end{aligned} \tag{15}$$

Now, if we substitute $B^{-1} b - B^{-1} N w_N$ for w_B in the objective function of (15), we have

$$z = c_B B^{-1} b - (c_B B^{-1} N - c_N)\ w_N$$
$$= c_B B^{-1} b - \sum_{j=1}^{n+1} (c_B B^{-1} a_j - c_j) w_j =$$
$$c_B B^{-1} b - \sum_{j=1}^{n+1} (z_j - c_j) w_j = z^* - \sum_{j \ne B_i} (z_j - c_j) w_j$$

Obviously, $z_j \le c_j$ for each feasible solution w_j. Therefore, $z^* \le z$ owning to $(z_j - c_j) w_j$ and $\sum_{j \ne B_i} (z_j - c_j) w_j$ are non-positive i.e., w^* is an optimal solution.

Example 1. In this example we consider the following FLP to demonstrate the proposed approach:

$$\begin{aligned} &\text{Minimize} \quad Z = 6\tilde{x}_1 + 10\tilde{x}_2 \\ &\text{Subject to:} \\ &(0,1.5,2.5,3)\tilde{x}_1 + (2,4,7,9)\tilde{x}_2 \ge (3,5,8,13) \\ &(-0.5,2.5,3.5,4.5)\tilde{x}_1 + (2,3.5,5,8.5)\tilde{x}_2 \ge (4,6,10,16) \\ &\tilde{x}_1, \tilde{x}_2 \ge 0 \end{aligned}$$

We first apply formula (7) to defuzzify $\tilde{A}$. A can be obtained as follows:

$$A = \begin{bmatrix} \frac{(2.5-1.5)+(3-0)}{2}, \frac{(7-4)+(9-2)}{2} \\ \frac{(3.5-2.5)+(4.5+0.5)}{2}, \frac{(5-3.5)+(8.5-2)}{2} \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$$

By incorporating A into the FLP model, we arrive

$$\begin{aligned} &\text{Minimize} \quad Z = 6\tilde{x}_1 + 10\tilde{x}_2 \\ &\text{Subject to:} \\ &2\tilde{x}_1 + 5\tilde{x}_2 \ge (3,5,8,13), \\ &3\tilde{x}_1 + 4\tilde{x}_2 \ge (4,6,10,16), \\ &\tilde{x}_1, \tilde{x}_2 \ge 0. \end{aligned}$$

We then mimic model (10) to solve the problem:

$$\begin{aligned} &\text{Minimize} \quad Z = 6x_1 + 10x_2 \\ &\text{Subject to:} \\ &2x_1^m + 5x_2^n \ge 5, \qquad 2x_1^n + 5x_2^n \ge 8, \\ &2x_1^l + 5x_2^u \ge 3, \qquad 2x_1^u + 5x_2^u \ge 13, \\ &3x_1^m + 4x_2^m \ge 6, \qquad 3x_1^n + 4x_2^n \ge 10, \\ &3x_1^l + 4x_2^l \ge 4, \qquad 3x_1^u + 4x_2^u \ge 16, \\ &x_1^u - x_1^n \ge 0, \qquad x_1^n - x_1^m \ge 0, \\ &x_1^m - x_1^l \ge 0, \qquad x_1^u + x_1^l \ge 0, \\ &x_1^m + x_1^n \ge 0, \qquad x_2^u - x_2^n \ge 0, \\ &x_2^n - x_2^m \ge 0, \qquad x_2^m - x_2^l \ge 0, \\ &x_2^u + x_2^l \ge 0, \qquad x_2^m + x_2^n \ge 0, \\ &x_1^m \le x_1 \le x_1^n, \qquad x_2^m \le x_2 \le x_2^n. \end{aligned}$$

The fuzzy and crisp optimal solutions for this problem are

$$\tilde{x}_1^* = (1.143, 1.429, 2.929, 5.429),$$
$$\tilde{x}_2^* = (0.143, 0.429, 0.429, \ 0.429),$$

$$x_1^* = 1.429, \quad x_2^* = 0.429 .$$

and the optimal and crisp objective function values are

$$\tilde{Z}^* = 6\tilde{x}_1^* + 10\tilde{x}_2^*$$
$$= (8.288, 12.864, 21.864, \ 36.864),$$
$$\tilde{Z} = 6x_1^* + 10x_2^* = 12.8571.$$

Example 2. In this example, we use the fuzzy optimization problem in Example 1 and apply the proposed model (11) to formulate the fuzzy program as follows:

$$\begin{aligned}
&\text{Minimize} \quad Z = 6x_1 + 10x_2 \\
&\text{Subject to:} \\
&\qquad\qquad 2x_1 + 5x_2 \geq 5, \\
&\qquad\qquad 3x_1 + 4x_2 \geq 6, \\
&\qquad\qquad x_1, x_2 \geq 0.
\end{aligned}$$

where the optimal solution and the optimal value of the objective function are as follows:

$$x_1^* = 1.429, \qquad x_2^* = 0.429,$$
$$Z^* = 6x_1^* + 10x_2^* = 12.8571.$$

As shown here, the optimal solution for Example 2 is identical to the crisp optimal solution obtained in Example 1.

CONCLUSION AND FUTURE RESEARCH DIRECTIONS

Over the past few decades, researchers have proposed many FLP models with different levels of sophistication. However, many of these models have limited real-world applications because of their methodological complexities and inflexible assumptions. In contrast, the two-fold model proposed in this study is straightforward and flexible.

The managerial implication of the proposed approach is its applicability to a wide range of real-world problems such as supply chain management, performance evaluation by means of data envelopment analysis, marketing management, failure mode and effect analysis and product development (Baykasolu & Göçken, 2008; Chen & Ko, 2010; Inuiguchi & Ramík, 2000; Peidro et al., 2010).

We proposed a two-fold model with two new methods for solving FLP problems in which the variables and the coefficients of the constraints are characterized by fuzzy numbers. In the first method, we transformed our FLP model into a conventional LP model by using a new fuzzy ranking method and introducing a new supplementary variable to obtain the fuzzy and crisp optimal solutions simultaneously with a single LP model. In the second method, we proposed a LP model with crisp variables for identifying the crisp optimal solutions. We demonstrated the details of the proposed method with two numerical examples.

Future research will concentrate on the comparison of results obtained with those that might be obtained with other methods. In addition, we plan to extend the FLP approach proposed here to deal with fuzzy nonlinear optimization problems with multiple objectives where the vagueness or impreciseness appears in all the components of the optimization problem such as the objectives, constraints and coefficients. Such an extension also implies the study of new practical experiments. Finally, we plan to focus on the use of co-evolutionary algorithms to solve fuzzy optimization problems. This approach would permit the search for solutions covering optimality, diversity and interpretability.

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Saber Saati is an Assistant Professor of Mathematics and Chairman of the Department of Mathematics at the Islamic Azad University (IAU) - Tehran North Branch in Iran. He received his PhD in Operations Research at the Science and Research Branch of IAU in 2002. His research interests include Data Envelopment Analysis, Fuzzy Programming, and Fuzzy MADM. He has published in journals such as Ricerca Operativa, Fuzzy Optimization and Decision Making, Computers & Industrial Engineering, Journal of Interdisciplinary Mathematics, Far East Journal of Applied Mathematics, Applied Mathematics and Computation, Iranian Journal of Fuzzy Systems, Advance in Fuzzy Sets and Systems, International Journal of Mathematical Analysis, and Applied Soft, Computing *among others*.

Adel Hatami-Marbini is a doctoral student and research assistant at the Louvain School of Management and the Center for Operations Research and Econometrics (CORE) of the Université catholique de Louvain (UCL) in Belgium. He holds a MSc and BSc in Industrial Engineering from Islamic Azad University (IAU) in Iran. His research interests are in Operations Research, Production Frontier, Data Envelopment Analysis, Multi Criteria Decision Making and Fuzzy Sets Theory. Adel Hatami-Marbini has published in journals such as European Journal of Operational Research, Omega, Computers & Industrial Engineering, Applied Soft Computing, Expert Systems with Applications, Mathematical and Computer Modelling, International Journal of Productivity and Quality Management, and International Journal of Industrial and Systems Engineering, *among others.*

Madjid Tavana is a Professor of Business Systems and Analytics and the Lindback Distinguished Chair of Information Systems and Decision Sciences at La Salle University where he served as Chairman of the Management Department and Director of the Center for Technology and Management. He has been a Distinguished NASA Research Fellow at Kennedy Space Center, Johnson Space Center, Naval Research Laboratory - Stennis Space Center, and Air Force Research Laboratory. He was recently honored with the prestigious Space Act Award by NASA. He holds an MBA, a PMIS, and a PhD in Management Information Systems and received his post-doctoral diploma in strategic information systems from the Wharton School of the University of Pennsylvania. He is the Editor-in-Chief for Decision Analytics*, the* International Journal of Strategic Decision Sciences*, the* International Journal of Enterprise Information Systems*, and the* International Journal of Applied Decision Sciences*. He has published over one hundred research papers in academic journals such as* Decision Sciences, Information Systems, Interfaces, Annals of Operations Research, Omega, Information and Management, Expert Systems with Applications, European Journal of Operational Research, Journal of the Operational Research Society, Computers and Operations Research, Knowledge Management Research and Practice, Computers and Industrial Engineering, Applied Soft Computing, Journal of Advanced Manufacturing Technology*, and* Advances in Engineering Software*, among others.*

Elham Hajiahkondi is a graduate student in the Department of Mathematics at Payame Noor University in Tehran, Iran. His research interests are in Data Envelopment Analysis, Mathematical Optimization, and Productivity Analysis.