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A novel method for selecting a single efficient unit in data envelopment analysis without explicit inputs/outputs

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Abstract Data Envelopment Analysis (DEA) is a non-parametric technique for evaluating a set of homogeneous decision-making units (DMUs) with multiple inputs and multiple outputs. Various DEA methods have been proposed to rank all the DMUs or to select a single efficient DMU with a single constant input and multiple outputs [i.e., *without explicit inputs* (WEI)] as well as multiple inputs and a single constant output [i.e., *without explicit outputs* (WEO)]. However, the majority of these methods are computationally complex and difficult to use. This study proposes an efficient method for finding a single efficient DMU, known as the most efficient DMU, under WEI and WEO conditions. Two compact forms are introduced to determine the most efficient DMU without solving an optimization model under the DEA-WEI and DEA-WEO conditions. A comparative analysis shows a significant reduction in the computational complexity of the proposed method over previous studies. Four numerical examples from different contexts are presented to demonstrate the applicability and exhibit the effectiveness of the proposed compact forms.

Keywords Data envelopment analysis (DEA) · Most efficient unit · Explicit inputs · Explicit outputs · Computational complexity

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1 Introduction

During the last two decades, data envelopment analysis (DEA) has been widely utilized in many operations research applications such as banking, education, healthcare, and manufacturing among others. DEA is a mathematical method for measuring the relative efficiency of decision-making units (DMUs) with multiple inputs and multiple outputs. The relative efficiency score of a DMU is a positive number less than or equal to one. A DMU is called efficient if its efficiency score is equal to one and otherwise it is inefficient. This definition is often used in DEA to group all the DMUs into two mutually exclusive sets: efficient and inefficient groups. Obviously, the performance of efficient DMUs is better than the inefficient ones. Subsequently, the inefficient DMUs are easily discriminated by their efficiency scores while the efficient DMUs are not.

A large number of methods have been proposed in order to more clearly discriminate between efficient DMUs. Some ranking methods have been proposed to fully rank efficient and inefficient DMUs, with aim of improving the differential capabilities of DEA. The cross-efficiency (Sexton et al. 1986) and super-efficiency (Andersen and Petersen 1993) approaches are two main ranking methods. In some problems, such as the supplier selection problem and the technology selection problem, the decision maker has to practically select a single efficient DMU. One way to address this issue is to rank all of the efficient DMUs and then select the one with the highest ranking score. Nevertheless, there is no need to rank all the efficient DMUs in order to identify only a single efficient DMU. On the other hand, in order to establish an identical situation for all the DMUs, the common weight approaches are widely utilized [(for more details see Toloo (2012)].

Farzipoor Saen (2007) and Toloo (2014) proposed a method for finding the best suppliers in the presence of both cardinal and ordinal data. Baker and Talluri (1997) utilized the concept of cross-efficiency in DEA and suggested a methodology for dealing with the advanced manufacturing technology selection problem and then applied it to the robot selection process. Farzipoor Saen (2011) formulated a DEA model for handling the media selection problem considering both flexible factors and imprecise data [for a deeper discussion of imprecise data in DEA, we refer the reader to Zhu (2003, 2004) and Cook and Zhu (2005)]. Amin et al. (2011) proposed a maximum discrimination DEA approach for selecting and ranking discovered association rules from data mining without requiring the solution of any optimization problem. Wang and Jiang (2012) considered constant returns to scale and variable returns to scale assumptions with both input and output orientations. The authors suggested three alternative mixed integer linear programming models which are simpler and more succinct than previous approaches, in order to determine the most efficient unit. Toloo and Ertay (2014) and Toloo (2016) proposed a method for determining the most cost efficient DMU with price (un)certainty which was applied to 73 vendors in the Turkish automotive industry.

DEA models are generally developed for data sets with multiple inputs and multiple outputs; however, there are some applications with multiple outputs and without explicit inputs (WEI) as well as applications with multiple inputs and without explicit outputs (WEO). Several studies have been conducted in the DEA literature to solve problems under the WEI or WEO conditions. Lovell and Pastor (1999) investigated DEA models with a single constant input and multiple outputs or with multiple inputs and a single constant output and achieved some simplified versions of DEA models with fewer constraints and variables. Other performance evaluation studies have been conducted for problems with multiple outputs and WEI [for more details, we refer the reader to Fernandez-Castro and Smith (1994) and Despotis (2005)].

Seiford and Zhu (2003) incorporated value judgements into the attractiveness and progress measures and applied their extended context-dependent DEA model to measure the attractiveness of 32 computer printers with six inputs and a single output. Chen (2004) improved the super-efficiency DEA model to fully rank the efficient DMUs and ranked the 20 largest Japanese companies with three inputs and one output. Chen and Lin (2006) proposed a mutual fund performance evaluation method using data envelopment analysis by capturing the pervasive skewness and leptokurtosis in return distributions of mutual funds. Chen and Lin (2006) considered a numerical example involving 14 Chinese mutual funds with six inputs and a single output to validate their approach. Lozano et al. (2009) made a connection between operational efficiency and environmental impacts and provided a joint application of life cycle assessment and DEA and provided a numerical sample of 62 mussel cultivation sites (rafts) with 14 inputs and a single output. Liu et al. (2011) provided an axiomatic foundation for DEA-WEI models and discussed how to incorporate value judgments of the decision-makers into these models. Lee and Zhu (2012) investigated super-efficiency infeasibility and zero data issues in DEA and considered a real data set of 15 Illinois strip mines with seven inputs and a single output.

Asmild et al. (2012) proposed DEA models for reallocations of police personnel resources and demonstrated the applicability of their model through a data set with a single input. Asmild et al. (2013) confirmed if the distribution of efficiency scores in DEA is independent of the inputs or outputs. The authors utilized 169 different demolition projects with three inputs and a single output. Moreno and Lozano (2014) proposed a network DEA approach to assess the efficiency of 30 National Basketball Association teams with two inputs and one output. Shokouhi et al. (2014) modified a prevailing robust DEA model with a unified production possibility set to overcome some issues and examined seven manufacturing industries with respect to two inputs (capital and labor) and one output (gross output value).

Other researchers have proposed approaches for ranking efficient DMUs and/or finding a single efficient DMU, known as the best/most efficient DMU, with a pure output data set. Karsak and Ahiska (2005, 2008) and Amin et al. (2006) suggested some common weight multi-criteria decision-making approaches for the technology selection problem and applied it to select the best robot. Amin (2009) obtained a compact form which determined the optimal solution of the technology selection model without the need to solve any optimization problem. Foroughi (2011) proposed a two-stage method to set up a full ranking of DMUs with a pure data set using association rules from data mining. Ramón et al. (2012) proposed a DEA methodology which minimized the differences between flexible weights in traditional DEA models and a common set of weights (CSW) in integrated DEA models. It was showed that different norms might lead to different results and illustrated their approach via ranking professional tennis players. Toloo (2013) noticed a drawback in some previous studies and proposed a simplified DEA-WEI approach for finding the most efficient DMU. Toloo and Kresta (2014) utilized the method of Toloo (2013) and developed a DEA-WEO method to select the best alternative for asset financing.

In this study, we investigate previous DEA-WEI and DEA-WEO methods which are suggested to rank the efficient DMUs and/or select a single efficient DMU, known as the best/most efficient DMU. We introduce a significant efficient approach which identifies the most efficient DMU without the need for solving any optimization problem. From a theoretical point of view, we state and prove some theorems to validate our approach; from a practical point of view, we utilize some applications from the DEA literature to illustrate the applicability of the suggested approach.

The remainder of this paper is organized as follows: Sect. 2 reviews some proposed methods to deal with DEA-WEI and DEA-WEO. In Sect. 3, we explain the role of the non-

Archimedean epsilon in finding the most efficient DMU and we propose two compact forms for finding the most efficient unit using the DEA-WEI and DEA-WEO approaches. More importantly, in this section, we prove that the most efficient unit obtained from the proposed compact forms is identical to the one achieved from previous studies. Section 4 presents four numerical examples from different contexts to demonstrate the effectiveness of the proposed compact forms. The conclusions and remarks are given in Sect. 5.

2 Literature review

Suppose that there are n DMUs ($DMU_j, j = 1, \ldots, n$) each having a single input x_j with s outputs $\boldsymbol{y}_j = \left(y_{1j}, \ldots, y_{sj}\right)$. In contrast to previous studies in DEA, Amin (2009) imposed different non-Archimedean epsilon values for evaluating each DMU and considered the following output-oriented CCR model:

$$
\begin{aligned}
& z^*(\varepsilon_o) = \min vx_o \\
& \text{s.t.} \\
& \sum_{r=1}^{s} u_r y_{ro} = 1 \\
& \sum_{r=1}^{s} u_r y_{rj} - vx_j \leq 0 \quad && j = 1, \ldots, n \\
& u_r \geq \varepsilon_o && r = 1, \ldots, s \\
& v \geq \varepsilon_o
\end{aligned}
\tag{1}
$$

where v and $\boldsymbol{u} = (u_1, \ldots, u_s)$ are the weights for the single input x_j and the output $\boldsymbol{y}_j = \left(y_{1j}, \ldots, y_{sj}\right)$, respectively, $\left(x_o, \boldsymbol{y}_o\right) = (x_o, y_{1o}, \ldots, y_{so}) \in \mathbb{R}^{1+s}$ is the input and output vector of unit under evaluation (DMU_o), and ε_o is the non-Archimedean epsilon which is added to the model for keeping the weights from being zero.

Let $X(\varepsilon)$ be the feasible region of model (1) and $z^*(\varepsilon^*)$ be the optimal objective value of model (1) for $\varepsilon_o = \varepsilon$. Moreover, suppose $\varepsilon_1 < \varepsilon_2$ and $X(\varepsilon_2)$ is a nonempty set. Obviously, the feasible region for $\varepsilon = \varepsilon_1$ is a superset of the feasible region for $\varepsilon = \varepsilon_2$, i.e., $X(\varepsilon_1) \supset X(\varepsilon_2)$, and hence $z^*(\varepsilon_1^*) \geq z^*(\varepsilon_2^*)$. As a result, increasing the value of the epsilon leads to decreasing the efficiency score, and then improving the discrimination power of the model. Cook et al. (1996) suggested utilizing the maximum value for epsilon, known as the *maximum non-Archimedean epsilon*, for improving the discrimination power of DEA models.

Amin (2009) proved that one can select $\varepsilon_o = \left(\sum_{r=1}^{s} y_{ro}\right)^{-1}$ to obtain the optimal solution of model (1) as $(v^*, \boldsymbol{u}^*) = \varepsilon_o \left(\max\left\{\sum_{r=1}^{s} u_r y_{rj}/x_j : j = 1, \ldots, n\right\}, \mathbf{1}_s\right)$ where $\mathbf{1}_s = (1, \ldots, 1) \in \mathbb{R}^s$. Meanwhile, Liu et al. (2011) ignored the role of epsilon and considered the input-oriented CCR model to deal with the DEA-WEI model.

The following common weight (CW) DEA-WEI model is suggested for determining the CW-efficient units with a pure output data set [see Toloo (2013)]:

$$
\begin{aligned}
& \min d_{\max} \\
& \text{s.t.} \\
& \sum_{r=1}^{s} u_r y_{rj} + d_j = 1 \quad && j = 1, \ldots, n \\
& d_{\max} - d_j \geq 0 && j = 1, \ldots, n \\
& u_r \geq \varepsilon && r = 1, \ldots, s \\
& d_j \geq 0 && j = 1, \ldots, n
\end{aligned}
\tag{2}
$$

where d_j is the deviation from efficiency of DMU_j and ε is the non-Archimedean infinitesimal. In this model, we have $0 \leq d_j^* \leq 1$ and $1 - d_j^*$ is the CW-efficiency score of DMU_j and hence DMU_p is CW-efficient if and only if $d_p^* = 1$.

On the other hand, the following integrated DEA-WEO model is formulated for determining the CW-efficient units with a pure input data set [see Toloo and Kresta (2014)]:

$$
\begin{aligned}
& \min d_{\max} \\
& \text{s.t.} \\
& \sum_{i=1}^{m} v_i x_{ij} - d_j = 1 \quad && j = 1, \ldots, n \\
& d_{\max} - d_j \geq 0 && j = 1, \ldots, n \\
& v_i \geq \varepsilon && i = 1, \ldots, m \\
& d_j \geq 0 && j = 1, \ldots, n
\end{aligned} \tag{3}
$$

where x_{ij} is the ith input of DMU j, v_i is the weight of the ith input, and ε is a positive Archimedean parameter. In this model, $\left(1 + d_j^*\right)^{-1}$ is the CW-efficiency score of DMU_j and consequently DMU_p is CW-efficient if and only if $d_p^* = 0$.

Some studies have been performed to find the most efficient DMU with a single input or a single output data set which are summarized in the following sections. The first three approaches find the most efficient DMU under WEI condition and the fourth approach deals with finding the most efficient DMU under WEO condition.

2.1 Karsak and Ahiska's (2005) method

Karsak and Ahiska (2005) proposed the following model for finding the most efficient DMU with a single input and s outputs:

$$
\begin{aligned}
& \min d_{\max} - k \sum_{j \in EF} d_j \\
& \text{s.t.} \\
& \frac{\sum_{r=1}^{s} u_r y_{rj}}{x_j} + d_j = 1 \quad && j = 1, \ldots, n \\
& d_{\max} - d_j \geq 0 && j = 1, \ldots, n \\
& d_j \geq 0 && j = 1, \ldots, n \\
& u_r \geq \varepsilon && r = 1, \ldots, s
\end{aligned} \tag{4}
$$

where $k \in [0, 1]$ is a discriminating parameter which must be determined via a trial-and-error approach and EF is the set of all CW-efficient DMUs, i.e., $EF = \left\{ j : d_j^* = 0 \right\}$. For a given k, DMU_p is the most efficient DMU if and only if $d_p^* = 0$ and $d_j^* > 0 \, \forall j \neq p$. A suitable value for k is a value in which EF is a singleton set. Karsak and Ahiska (2008) designed a bisection algorithm to obtain a suitable value for k in a systematic manner. Nevertheless, model (4) must be solved iteratively to obtain a single efficient DMU. The authors utilized a real data set involving 12 robots with 4 outputs and a single input to solve the robot selection problem. In order to obtain the most efficient robot, model (4) was solved for $k = 0$, $k = 0.5$, and $k = 0.25$.

2.2 Foroughi's (2011) method

Foroughi (2011) proposed the following two-stage approach to find the most efficient unit and also to fully rank all the DMUs:

Stage I First, the set of all CCR-efficient DMUs, E, must be determined via the following algorithm:

Step 0 Let $E = \emptyset$.
Step 1 Solve the following model:

$$\begin{aligned}
& d^* = \min \sum_{j=1}^{n} d_j \\
& \text{s.t.} \\
& 1 - d_j \le \sum_{r=1}^{s} u_r y_{rj} \le 2 - d_j \quad && j = 1, \ldots, n \\
& \sum_{j=1}^{n} d_j \ge n - 1 \\
& d_j = 1 && j \in E \\
& d_j \in \{0, 1\} && j = 1, \ldots, n \\
& u_r \ge \varepsilon && r = 1, \ldots, s
\end{aligned}$$

If $d^* = n$ then stop: E is the set of all efficient DMUs.
Else, assuming that $d^* = p$ in the optimal solution, set $E = E \cup \{p\}$ and go to Step 1.

Stage II The following model is suggested to obtain the reversed-normalized score $(1/z_o'^*)$:

$$\begin{aligned}
& 1/z_o'^* = \min \sum_{r=1}^{s} u_r \\
& \text{s.t.} \\
& \sum_{r=1}^{s} u_r y_{ro} = z_o^* \\
& \sum_{r=1}^{s} u_r y_{rj} \le 1 \quad && j \in E, j \ne o \\
& u_r \ge \varepsilon && r = 1, \ldots, s
\end{aligned}$$

where z_o^* is the CCR-efficiency score of DMU$_o$. In this approach, a DMU with a bigger efficiency score and a smaller reversed-normalized score is preferred. Foroughi (2011) then applied this approach to find the best association rule in data mining and fully rank all of the given 46 association rules with 4 outputs and WEI.

2.3 Toloo's (2013) method

In contrast with Karsak and Ahiska (2005), who considered a penalty function to the objective function of model (2), Toloo (2013) restricted the feasible region of the model and proposed the following integrated DEA-WEI model:

$$
\begin{aligned}
& \min d_{\max} \\
& \text{s.t.} \\
& \sum_{r=1}^{s} u_r y_{rj} + d_j = 1 && j = 1, \ldots, n \\
& d_{\max} - d_j \geq 0 && j = 1, \ldots, n \\
& \sum_{j=1}^{n} \theta_j = n - 1 \\
& d_j \leq \theta_j \leq M d_j && j = 1, \ldots, n \\
& \theta_j \in \{0, 1\} && j = 1, \ldots, n \\
& d_j \geq 0 && j = 1, \ldots, n \\
& u_r \geq \varepsilon && r = 1, \ldots, s
\end{aligned}
\tag{5}
$$

where M is a large enough positive number. In this model, a zero auxiliary binary variable, i.e., $\theta_j = 0$, leads to a zero deviation from the efficiency variable, i.e., $d_j = 0$, and similarly a positive deviation from the efficiency variable will be obtained for each auxiliary binary variable with a value of one. DMU$_p$ is the most efficient unit if and only if $d_p^* = 0$ (which leads to $d_j^* > 0, \forall j \neq p$). As a result, model (5) finds a single efficient DMU which is clearly the most efficient unit. Toloo (2013) applied his approach in a real data set containing 40 professional tennis players with 9 outputs and WEI to illustrate the applicability of model (5).

2.4 Toloo and Kresta's (2014) method

Toloo and Kresta (2014) extended model (3) and suggested the following DEA-WEO model to find the most CW-efficient DMU when there are no explicit outputs:

$$
\begin{aligned}
& \min d_{\max} \\
& \text{s.t.} \\
& \sum_{i=1}^{m} v_i x_{ij} - d_j = 1 && j = 1, \ldots, n \\
& d_{\max} - d_j \geq 0 && j = 1, \ldots, n \\
& \sum_{j=1}^{n} \theta_j = n - 1 \\
& \theta_j \leq M d_j && j = 1, \ldots, n \\
& d_j \leq N \theta_j && j = 1, \ldots, n \\
& \theta_j \in \{0, 1\} && j = 1, \ldots, n \\
& v_i \geq \varepsilon && i = 1, \ldots, m
\end{aligned}
\tag{6}
$$

where M and N are large enough positive numbers. The deviation from the efficiency variable, d_j, in model (3) might be larger than 1 and hence, in comparison to model (5), there is one more parameter (N) in model (6). In this model, DMU$_p$ is the most efficient unit if and only if $d_p^* = 0$. Toloo and Kresta (2014) applied model (6) on a real data set of 139 different alternatives for long-term asset financing provided by Czech banks and leasing companies with 4 inputs and WEO.

In the next section, we provide a compact mathematical form for the optimal solution of both DEA-WEI and DEA-WEO models.

3 Proposed approach

We first illustrate the role of the maximum non-Archimedean epsilon in finding the most efficient DMU. Toward this end, we adopt a real data set from Ramón et al. (2012) involving 53 professional tennis players with 9 outputs; Percentages of: 1*stserve* (y_1), 1*stserve points won* (y_2), 2*ndserve points won* (y_3)*, service games won* (y_4)*, break points saved* (y_5)*, points won returning 1stserve* (y_6)*, points won returning 2ndserve* (y_7)*, break points converted* (y_8), and *return games won* (y_9).

In order to evaluate the performance of the professional tennis player, we utilize the integrated DEA-WEI model (2). We first obtain the maximum non-Archimedean epsilon for model (2) which improves the discriminating power of the model. Mathematically, we are looking for a unique ε^* such that model (2) is feasible for all $\varepsilon \in (0, \varepsilon^*]$ and is infeasible for all $\varepsilon > \varepsilon^*$. To do this, we formulae the following model which obtains the maximum non-Archimedean epsilon ε^* for Model (2):

$$\begin{aligned} &\varepsilon^* = \max \varepsilon \\ &\text{s.t.} \\ &\sum_{r=1}^{s} u_r y_{rj} \le 1 \quad && j = 1, \ldots, n \\ &\varepsilon - u_r \le 0 && r = 1, \ldots, s \\ &u_r \ge 0 && r = 1, \ldots, s \end{aligned} \tag{7}$$

Here are some interesting properties of the above model (7):

Corollary 1 *Model (7) is always feasible.*

Proof It is sufficient to provide a feasible solution for model (7). Let:

$$\forall r, u_r^0 = \frac{1}{s}\min\left\{\left(y_{rj}\right)^{-1} : j = 1, \ldots, n\right\}$$
$$\varepsilon^0 = \min\left\{u_r^0 : r = 1, \ldots, s\right\}$$

Hence, $\forall j, \sum_{r=1}^{s} u_r^0 y_{rj} \le \frac{1}{\text{s}} \sum_{r=1}^{s} \frac{1}{y_{rj}}\left(y_{rj}\right) = 1, \forall r, \varepsilon^0 - u_r^0 \le 0$, and $\forall r, u_r^0 \ge 0$ which proves that $(\varepsilon^0, u_1^0, \ldots, u_\text{s}^0)$ is a feasible solution for model (7). □

Corollary 2 *In model (7),* $\varepsilon^* = \left(\max\left\{\sum_{r=1}^{s} y_{rj} : j = 1, ..., n\right\}\right)^{-1}$.

Proof From the constraints of model (7) we have:

$$\forall r, \varepsilon \le u_r \Rightarrow \forall r, \forall j, \varepsilon y_{rj} \le u_r y_{rj} \Rightarrow \forall j, \varepsilon \sum_{r=1}^{s} y_{rj} \le \sum_{\text{r}=1}^{\text{s}} u_r y_{rj} \le 1 \Rightarrow \forall j, \varepsilon \le \frac{1}{\sum_{r=1}^{s} y_{rj}}$$

Let $\forall r, u_r^* = \left(\sum_{r=1}^{s} y_{rj}\right)^{-1}, \varepsilon^* = \left(\min\left\{u_r^* : r = 1, ..., s\right\}\right) = \frac{1}{\max\left\{\sum_{r=1}^{s} y_{rj} : j=1,\ldots,n\right\}}$. An easy computation clarifies that the vector $\left(\varepsilon^*, u_1^*, \ldots, u_s^*\right)$ is an optimal solution for model (7) which completes the proof. □

Referencing to Corollary 2, the maximum non-Archimedean epsilon for the model can be determined as bellow:

$$\varepsilon = \left(\max \left\{ \sum_{r=1}^{s} y_{rj} : j = 1, \ldots, n \right\} \right)^{-1} = \left(\sum_{r=1}^{9} y_{r2} \right)^{-1} = \frac{1}{516} = 0.001938$$

Table 1 summarizes the optimal deviation from the efficiency variable, $d^*_{j,}$ obtained by solving model (2) with various values for the non-Archimedean epsilons in the interval [0, 0.001938].

As can be seen, there are three efficient players for $\varepsilon = 0$ and $= \varepsilon^*/10$, i.e., Nadal, Djokovic and Roddick, meanwhile for $\varepsilon = \varepsilon^*/2$ the number of efficient players decreases to two players, Nadal and Djokovic. Finally, if we select the maximum non-Archimedean epsilon ($\varepsilon = \varepsilon^*$) the most efficient player is determined, Nadal. Indeed, the results obtained illustrate that the maximum non-Archimedean epsilon increases the discrimination power of integrated DEA-WEI models. It is worth noticing that Table 1 indicates that Nadal is the single efficient player as the other players are inefficient and subsequently he must be selected as the best player. In other words, Nadal is selected as the single efficient player and subsequently making it unnecessary to impose a parametric penalty function for the objective function (Karsak and Ahiska 2008) or restricting the feasible region (Toloo 2013).

Now, we find a compact mathematical form for the optimal solution of the integrated DEA-WEI model (2) and then extend it in order to solve the integrated DEA-WEO model (4).

3.1 The DEA-WEI approach

In this section, we obtain a compact mathematical form for the optimal solution of the DEA-WEI model (2) and accordingly determine the most efficient DMU when there are no explicit inputs without requiring any optimization problem.

The following theorem provides a compact mathematical form for the optimal solution of model (2) using the maximum non-Archimedean epsilon ε^*.

Theorem 1 $\left(\boldsymbol{u}^*, \boldsymbol{d}^*, d^*_{\max}\right)$ *is an optimal solution of integrated DEA model (2) where* $u^*_r = \varepsilon^* \forall r$, $d^*_j = 1 - \varepsilon^* \sum_{r=1}^{s} y_{rj}$, $d_{\max} = 1 - \varepsilon^* \left(\min \left\{ \sum_{r=1}^{s} y_{rj} : j = 1, \ldots, n \right\}\right)$.

Proof A simple computation shows that $\left(\boldsymbol{u}^*, \boldsymbol{d}^*, d^*_{\max}\right)$is a feasible solution for model (2). The main idea of the proof is to utilize the Karush–Kuhn–Tucker (KKT) optimality conditions for LPs [see Bazaraa et al. (2010)]. To do this, we provide a feasible solution for the dual of model (2) and show that its objective value is $d^*_{\max}$.

Consider the following dual of model (2):

$$\begin{aligned} \max\ & \sum_{j=1}^{n} \lambda_j + \varepsilon^* \sum_{r=1}^{s} t_r \\ \text{s.t.}\ & \\ & \sum_{j=1}^{n} \eta_j = 1 \\ & \sum_{j=1}^{n} \lambda_j y_{rj} + t_r = 0 && r = 1, \ldots, s \\ & \lambda_j - \eta_j \le 0 && j = 1, \ldots, n \\ & \eta_j \ge 0 && j = 1, \ldots, n \\ & t_r \ge 0 && r = 1, \ldots, s \end{aligned} \tag{8}$$

Table 1 Optimal deviation for efficiency (d^*) for different epsilons

No.	Players	$\varepsilon = 0$	$\varepsilon = \frac{\varepsilon^*}{10}$	$\varepsilon = \frac{\varepsilon^*}{2}$	$\varepsilon = \varepsilon^*$
1	Federer	0.01956	0.01883	0.01355	0.02326
2	Nadal	0.00000	0.00000	0.00000	0.00000
3	Djokovic	0.00000	0.00000	0.00000	0.01744
4	Murray	0.02857	0.02901	0.01194	0.01550
5	del Potro	0.03931	0.03942	0.03965	0.04845
6	Davydenko	0.01045	0.00963	0.01404	0.03101
7	Roddick	0.00000	0.00000	0.02475	0.04651
8	Soderling	0.03564	0.03604	0.03109	0.04264
9	Verdasco	0.00808	0.00715	0.01539	0.02519
10	Tsonga	0.03190	0.03105	0.03879	0.06008
11	Gonzalez	0.03768	0.03570	0.03801	0.05039
12	Stepanek	0.05105	0.05229	0.06043	0.07946
13	Monfils	0.04166	0.04147	0.04701	0.06589
14	Cillic	0.06558	0.06517	0.05208	0.06395
15	Simon	0.08435	0.08520	0.07044	0.06589
16	Robredo	0.04653	0.04750	0.05119	0.06589
17	Ferrer	0.05972	0.06102	0.06138	0.06783
18	Haas	0.06052	0.06027	0.06237	0.07558
19	Youzhny	0.05846	0.05902	0.06519	0.08333
20	Berdych	0.06812	0.06941	0.07467	0.09496
21	Wawrinka	0.07044	0.06982	0.06412	0.07946
22	Hewitt	0.08148	0.08327	0.07197	0.07752
23	Ferrero	0.04637	0.04789	0.06364	0.07364
24	Ljubicic	0.06720	0.06753	0.07694	0.09690
25	Querrey	0.05559	0.05699	0.06779	0.08915
26	Almagro	0.07326	0.07479	0.08101	0.09690
27	Kohlschreiber	0.04524	0.04453	0.05000	0.06008
28	Melzer	0.06900	0.06963	0.07597	0.10271
29	Troicki	0.08279	0.08359	0.08955	0.10853
30	Montantes	0.08434	0.08647	0.08728	0.09884
31	Chardy	0.08672	0.08745	0.09683	0.11240
32	Mathieu	0.11117	0.11302	0.11666	0.12016
33	Isner	0.06850	0.06617	0.08655	0.10078
34	Andreev	0.07387	0.07429	0.08586	0.10853
35	Karlovic	0.02686	0.02580	0.05584	0.08527
36	Tipsarevic	0.09180	0.09353	0.08950	0.09302
37	Beck	0.08673	0.08713	0.09492	0.10853
38	Garcia-López	0.07389	0.07561	0.08689	0.10271
39	Blake	0.09588	0.09678	0.09871	0.10853
40	Bennetau	0.05208	0.05308	0.07729	0.10659

Table 1 continued

No.	Players	$\varepsilon = 0$	$\varepsilon = \frac{\varepsilon^*}{10}$	$\varepsilon = \frac{\varepsilon^*}{2}$	$\varepsilon = \varepsilon^*$
41	López	0.07701	0.07624	0.08664	0.11822
42	Hanescu	0.04135	0.04176	0.06951	0.09302
43	Seppi	0.08343	0.08486	0.09900	0.12984
44	Acasuso	0.10178	0.10345	0.10495	0.11047
45	Fognini	0.11117	0.11345	0.12594	0.15698
46	Gicquel	0.10145	0.10194	0.10152	0.11434
47	Serra	0.09761	0.09848	0.11139	0.12209
48	Hernandez	0.07739	0.07863	0.10666	0.13566
49	Schuettler	0.09059	0.09368	0.11321	0.12791
50	Rochus	0.11117	0.11531	0.13838	0.16279
51	Gulbis	0.07056	0.07126	0.09371	0.12403
52	Granollers	0.09536	0.09644	0.10411	0.13760
53	Vassllo Arguello	0.11117	0.11531	0.13838	0.23643

Without loss of generality, let $\sum_{r=1}^{s} y_{rq} = \min\left\{\sum_{r}^{s} y_{rj} : j = 1, \ldots, n\right\}$ and $\sum_{r=1}^{s} y_{rp} = \max\left\{\sum_{r=1}^{s} y_{rj} : j = 1, \ldots, n\right\}$ which yields $\sum_{r=1}^{s} y_{rq} \le \sum_{r=1}^{s} y_{rj} \le \sum_{r=1}^{s} y_{rp}, \forall j$. Note that these assumptions also imply that $\varepsilon^* \sum_{r=1}^{s} y_{rp} = 1$. It is clear on inspection that $(\boldsymbol{\eta}, \boldsymbol{\lambda}, \boldsymbol{t})$ satisfies all the constraints of model (8) and hence is a feasible solution, where:

$$\boldsymbol{\eta} = (\eta_1, \ldots, \eta_n) \ \text{with}\ \eta_j = \begin{cases} 1 & j = p \\ 0 & j \neq q \end{cases},$$

$$\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \ \text{with}\ \lambda_j = \begin{cases} 1 & j = q \\ \min\left\{\frac{-y_{rq}}{y_{rp}} : r = 1, \ldots, s\right\} & j = p \\ 0 & j \notin \{p, q\} \end{cases}$$

$$\boldsymbol{t} = (t_1, \ldots, t_s) \quad \text{with}\ t_r = -y_{rq} - \lambda_p y_{rp} \ge 0.$$

The objective value of $(\boldsymbol{\eta}, \boldsymbol{\lambda}, \boldsymbol{t})$ is as follows:

$$\sum_{j=1}^{n} \lambda_j + \varepsilon^* \sum_{r=1}^{s} t_r = 1 - \lambda_p - \varepsilon^* \left(\sum_{r=1}^{s} y_{rq} + \sum_{j=1}^{n} \lambda_j y_{rp} \right) = 1 - \varepsilon^* \sum_{r=1}^{s} y_{rq}$$

which is equal to the objective value of $\left(\boldsymbol{u}^*, \boldsymbol{d}^*, d^*_{\max}\right)$ for model (2). Summarizing, $\left(\boldsymbol{u}^*, \boldsymbol{d}^*, d^*_{\max}\right)$ is a feasible solution for model (2), $(\boldsymbol{\eta}, \boldsymbol{\lambda}, \boldsymbol{t})$ is a feasible solution for model (8), these models are mutually dual, and their corresponding objective values are identical. Hence, the KKT optimality conditions imply that these vectors are the optimal solutions for the related models, which completes the proof. □

The principle significance of Theorem 1 is that it gives us the optimal solution for model (2) without requiring the solution of any optimization problem.

The following corollary can be directly derived from Theorem 1:

Corollary 3 *When there are no explicit inputs, DMU_p is the efficient DMU if and only if*

$$\sum_{\mathrm{r}=1}^{\mathrm{s}} y_{rp} = \max\left\{\sum_{\mathrm{r}=1}^{\mathrm{s}} y_{rj} : j = 1, \ldots, n\right\}$$

Proof We first assume that DMU_p is efficient. In this case, we have $d_p^* = 0$ and applying Theorem 1, leads to $d_p^* = 1 - \varepsilon^* \sum_{\mathrm{r}=1}^{\mathrm{s}} y_{rp} = 0$ and hence $\sum_{\mathrm{r}=1}^{\mathrm{s}} y_{rp} = (\varepsilon^*)^{-1} = \max\left\{\sum_{\mathrm{r}=1}^{\mathrm{s}} y_{rj} : j = 1, \ldots, n\right\}$.

Conversely, we let $\sum_{\mathrm{r}=1}^{\mathrm{s}} y_{rp} = \max\left\{\sum_{\mathrm{r}=1}^{\mathrm{s}} y_{rj} : j = 1, \ldots, n\right\}$ which implies $\varepsilon^* = \left(\sum_{\mathrm{r}=1}^{\mathrm{s}} y_{rp}\right)^{-1}$. Now, from Theorem 1, $d_p^* = 0$ and therefore DMU_p is efficient. □

As a result, the efficient DMU (when there are no explicit inputs) is a unit with the maximum summation of its outputs. Now, more importantly, we will prove that the same result will be obtained when the other approaches mentioned above are applied.

Corollary 4 *DMU_p is the most efficient DMU using the approach of Karsak and Ahiska (2005) when $\varepsilon = \varepsilon^*$.*

Proof Letting $k = 0$ in model (4), converts it to model (2) and following from Corollary 3 we obtain that DMU_p is the most efficient DMU. □

The average empirical complexity of the simplex method for solving an LP in the standard form with $\bar{n}$ variables and $\bar{m}$ constraints is $O\left(\bar{m}^2\bar{n}\right)$, [see Bazaraa et al. (2010)], and consequently the method of Karsak and Ahiska (2005) is of complexity $O\left(n^2\,(n+s)\right)$,whereas the proposed compact solution of this study only requires $n\,(s-1)$ additions. In other words, instead of solving the linear programing model (4) with $2n$ constraints and $2n+s$ variables (in the standard form), the most efficient DMU is obtained using simply $n\,(s-1)$ additions. Moreover, note that model (4) must be solved with different values of the discriminating parameter k . It is important to mention that the simplex method has been shown empirically requires $2n$ to $6n$ iterations, each iteration requiring $(2n+1)\,(s+1)$ multiplications and $2n\,(s+1)$ additions.

Corollary 5 *DMU_p is the most efficient DMU using the approach of Foroughi (2011) when $\varepsilon = \varepsilon^*$.*

Proof Let $\forall r, u_r^* = \varepsilon^*$. We have $\sum_{r=1}^{s} u_r^* y_{rj} = \varepsilon^* \sum_{r=1}^{s} y_{rj}$ or equivalently:

$$\sum_{r=1}^{s} u_r^* y_{rj} = \begin{cases} = 1 & j = p \\ > 1 & j \neq p \end{cases}$$

As a result, from the constraint $1 - d_j \leq \sum_{\mathrm{r}=1}^{\mathrm{s}} u_r y_{rj} \leq 2 - d_j$ we obtain $d_p^* \in \{0, 1\}$ and $\forall j \neq p, d_j^* = 1$. On the other hand, the objective function ensures that $d_p^* = 0$ rather than $d_p^* = 1$. Hence, DMU_p is determined as the most efficient DMU. □

It should be emphasized here that the Stage I model of Foroughi (2011) is an MILP model with $2n + |E| + 1$ constraints, s continuous variables, and n binary variables and the Stage II is an LP involving $|E| + 1$ constraints and s continues variables. In general, it is not easy to measure the complexity of an MILP problem and it depends on many factors such as the utilized algorithm. However, in order to draw a possible comparison, we can compare our approach with the relaxed MILP model of Stage I. A simple computation shows that the standard form of the RMLP Stage I model of Foroughi (2011) contains $3n+s+1$ variables and $2n+|E|+1$ constraints and hence it is of complexity $O\left((n+|E|)^2(n+s)\right)$. The standard form of the Stage II model involves $s+1$ variables and $|E|+1$ constraints, which clarifies that the model is of complexity $O\left(s\,|E|^2\right)$. Furthermore, the number of computations rapidly increases because Stage I is, in fact, an MILP and must be solved n times. In addition, the Stage II model must be solved $|E|$ times. It is very important to stress that our proposed compact solution only needs $n(s-1)$ additions.

Corollary 6 *Using the approach of Toloo (2013), DMUp is the most efficient DMU, when* $\varepsilon=\varepsilon^*$.

Proof Let $\forall r, u_r^*=\varepsilon^*$. We obtain $\sum_{r=1}^{s}u_r^*y_{rp}=1$, which implies $d_p^*=0$ and from the constraint $\theta_p\le Md_p$, we have $\theta_p^*=0$. Now, from the constraint $\sum_{j=1}^{n}\theta_j=n-1$, we obtain $\forall j\ne p, \theta_j^*=1$, which leads to $\forall j\ne p, d_j^*>0$, due to the constraint $\theta_j\le Md_j$, which identifies DMU_p as the most efficient DMU. □

Model (5) is an MILP model with $4n+1$ constaints, $n+s+1$ continues variables, and n binary variables. The standard form of the relaxed MILP model (5) has $4n+1$ constraints and $5n+s$ variables which shows that the model is of complexity $O\left(n^2(n+s)\right)$.

3.2 The DEA-WEO approach

This section presents a compact mathematical form for the optimal solution of a DEA-WEO model. To this end, we state and prove that model (3) is always feasible for all non-negative values for and conclude that the epsilon is an Archimedean parameter for this model.

Theorem 2 *Model (3) is feasible for all* $\varepsilon\ge 0$.

Proof It suffices to prove that the optimal objective value of the following model is unbounded:

$$\begin{aligned}&\max \varepsilon\\&\text{s.t.}\\&\sum_{i=1}^{m}v_ix_{ij}\ge 1 \qquad j=1,\ldots,n\\&d_{\max}-d_j\ge 0 \qquad j=1,\ldots,n\\&\varepsilon-v_i\le 0 \qquad i=1,\ldots,m\end{aligned}\tag{9}$$

It easy to verify that $\left(v^0,\varepsilon^0\right)$ is a feasible solution for the model where $v_i^0=\left(\min\left\{x_{ij}:j=1,\ldots,n\right\}\right)^{-1}\forall i$ and $\varepsilon^0=\min\left\{v_i^0:i=1,\ldots,m\right\}$ and hence the feasible region of model (9) is not an empty set. On the other hand, the vector $\left(\frac{1}{m+1},\frac{1}{m+1},\ldots,\frac{1}{m+1}\right)\in\mathbb{R}^{m+1}$ is a solution for the following system:

$$\begin{aligned}&d_1x_{1j}+d_2x_{2j}+\cdots+d_mx_{mj}\ge 0 \qquad j=1,\ldots,n\\&d_1+d_2+\cdots+d_m+d_\varepsilon=1\\&d_i-d_\varepsilon\ge 0 \qquad i=1,\ldots,m\end{aligned}$$

In conclusion, $\left(\frac{1}{m+1},\frac{1}{m+1},\ldots,\frac{1}{m+1}\right)$ is a recession direction of the feasible region of model (9). Note that the inner product of the cost coefficient vector of model (9), i.e., $(c_1,c_2,\ldots,c_m,c_\varepsilon)=(0,0,\ldots,0,1)$, and the recession direction is positive,

$(0, 0, \ldots, 0, 1)\left(\frac{1}{m+1}, \frac{1}{m+1}, \ldots, \frac{1}{m+1}\right) = \frac{1}{m+1} > 0$, which indicates that the optimal objective value of model (9) is unbounded. □

As a result, any positive value can be considered for ε and hence, unlike other DEA models, ε is an Archimedean parameter in model (3). The following theorem can be proved in much the same way as Theorem 1.

Theorem 3 $\left(\boldsymbol{v}^*, \boldsymbol{d}^*, d^*_{\max}\right)$ *is an optimal solution of model (3) where* $\boldsymbol{v}^* = \left(v^*_1, \ldots, v^*_m\right)$ *and* $v^*_i = \varepsilon^* \forall i$, $d^*_j = \varepsilon^* \sum_{i=1}^m x_{ij} - 1 \forall j$, $d^*_{\max} = \varepsilon^*\left(\max\left\{\sum_{i=1}^m x_{ij} : j = 1, \ldots, n\right\}\right) - 1$.

Corollary 7 *When there are no explicit outputs, DMU$_q$ is the efficient DMU if and only if* $\sum_{i=1}^m x_{iq} = \min\left\{\sum_{i=1}^m x_{ij} : j = 1, \ldots, n\right\}$.

Proof Let DMU$_q$ be an efficient unit, which implies that $d^*_q = 0$, and let ε^* be a small positive number. Referencing to Theorem 3, $d^*_q = \left(\varepsilon^* \sum_{i=1}^m x_{iq}\right) - 1 = 0$ and hence, $\sum_{i=1}^m x_{iq} = (\varepsilon^*)^{-1}$. On the other hand, $\forall j \neq q, d^*_j = \left(\varepsilon^* \sum_{i=1}^m x_{ij}\right) - 1 \geq 0$, which leads to $\sum_{i=1}^m x_{iq} = \left(\frac{1}{\varepsilon^*}\right) \leq \left(\frac{1+d^*_j}{\varepsilon^*}\right) = \sum_{i=1}^m x_{ij}$.

Conversely, let $\sum_{i=1}^m x_{iq} = \min\left\{\sum_{i=1}^m x_{ij} : j = 1, \ldots, n\right\}$, which gives us $\forall j, \sum_{i=1}^m x_{iq} \leq \sum_{i=1}^m x_{ij}$,or equivalently $\forall j, d^*_q = \varepsilon^* \sum_{i=1}^m x_{iq} - 1 \leq \varepsilon^* \sum_{i=1}^m x_{ij} - 1 = d^*_j$. Hence, we have $d^*_q = \min\{d^*_j : j = 1, .., n\}$, and the fact that there is always at least one efficient DMU completes the proof. □

Corollary 8 *Using the approach of Toloo and Kresta (2014), DMUq is the most efficient DMU, when $\varepsilon = \varepsilon^*$.*

Proof Letting $\forall i, v^*_i = \varepsilon^*$ leads to $\sum_{i=1}^m v^*_i x_{ip} = 1$, or equivalently $d^*_p = 0$, and hence $\theta^*_p = 0$, which implies $\forall j \neq p, \theta^*_j = 1$ and then $\forall j \neq p, d^*_j > 0$. Hence, model (6) selects DMU$_p$. □

Note that the standard form of the relaxed MILP model (6) involve $4n + 1$ constraints and $5n + m$ variables and hence the model is of complexity $O\left(n^2 (n + m)\right)$. However, our proposed compact solution only needs $n\,(m - 1)$ additions.

4 Numerical examples

In this section, we consider four illustrative examples which are already utilized in various studies for finding the most efficient DMU with a single constant input and multiple outputs or multiple inputs and a single constant output.

Example 1 Association rules from data mining.

Foroughi (2011) applied his approach to a real data set presented in Table 2. The data set involves 46 association rules (AR) with respect to four outputs: *support* (y_1), *confidence* (y_2), *Itemset value* (y_3) *and cross-selling profit* (y_4).

The column labeled $\sum_{r=1}^4 y_{rj}$ shows the sum of the four output values for each AR. Hence, we have $\varepsilon^* = \left(\max\left\{\sum_{r=1}^4 y_{rj} : j = 1, \ldots, 46\right\}\right)^{-1} = 0.001282$. Moreover, d^*_j equals to $1 - 0.001282\left(\sum_{r=1}^4 y_{rj}\right)$ using Theorem 1, which implies that AR26 is the most efficient

Table 2 Data and results for the association rules

DMUs	y_1	y_2	y_3	y_4	$\sum_{j=1}^{4} y_j$	d_j^*	Rank	Foroughi (2011)
AR01	3.87	40.09	337	25.66	406.62	0.478752	40	11
AR02	1.42	18.17	501	11.63	532.22	0.317745	24	32
AR03	2.83	17.64	345	11.29	376.76	0.51703	43	26
AR04	2.34	30.83	163	19.73	215.9	0.723237	45	40
AR05	2.63	23.9	325	15.3	366.83	0.529759	44	30
AR06	1.19	55.65	436	35.61	528.45	0.322578	25	7
AR07	1.19	47.42	598	30.35	676.96	0.132203	10	5
AR08	1.19	15.7	436	52.91	505.8	0.351613	26	42
AR09	1.19	10.82	598	36.45	646.46	0.171301	15	23
AR10	1.19	12.32	436	20.08	469.59	0.398031	34	45
AR11	1.19	12.32	598	40.04	651.55	0.164776	14	24
AR12	3.87	38.08	337	103.97	482.92	0.380943	29	10
AR13	1.18	15.09	710	41.19	767.46	0.01619	6	16
AR14	2.44	15.22	554	41.56	613.22	0.213911	20	13
AR15	2.14	28.21	372	77.02	479.37	0.385494	31	31
AR16	2.51	22.81	534	62.26	621.58	0.203195	19	14
AR17	1.19	50.92	436	139.02	627.13	0.19608	17	4
AR18	1.19	45.25	598	123.52	767.96	0.015549	5	3
AR19	1.19	11.7	436	43.54	492.43	0.368752	28	44
AR20	1.19	11.7	598	62.5	673.39	0.136779	11	22
AR21	1.42	13.99	501	61.16	577.57	0.259611	22	28
AR22	1.18	12.23	710	53.45	776.86	0.004141	2	2
AR23	1.5	13.64	698	59.59	772.73	0.009435	3	6
AR24	2.83	27.82	345	78.17	453.82	0.418247	36	25
AR25	2.44	25.27	554	71	652.71	0.163289	13	12
AR26	1.25	15.97	718	44.87	780.09	0	1	1
AR27	1.22	34.89	339	98.04	473.15	0.393467	33	34
AR28	1.3	35.12	435	98.68	570.1	0.269187	23	27
AR29	1.42	33.81	534	95.01	664.24	0.148509	12	21
AR30	1.91	25.26	380	70.97	478.14	0.387071	32	35
AR31	1.43	37.14	618	104.35	760.92	0.024574	7	9
AR32	2.38	21.63	542	60.78	626.79	0.196516	18	18
AR33	1.18	30.24	366	84.98	482.4	0.38161	30	41
AR34	1.23	29.36	626	82.51	739.1	0.052545	8	19
AR35	1.58	22.65	354	63.64	441.87	0.433565	37	43
AR36	2.34	22.99	163	22.76	211.09	0.729403	46	46
AR37	2.14	22.14	372	21.92	418.2	0.463908	39	33
AR38	1.91	11.94	380	11.82	405.67	0.47997	41	38

Table 2 continued

DMUs	y_1	y_2	y_3	y_4	$\sum_{j=1}^{4} y_j$	d_j^*	Rank	Foroughi (2011)
AR39	2.03	18.42	360	18.23	398.68	0.488931	42	39
AR40	1.19	30.73	436	30.43	498.35	0.361163	27	37
AR41	2.63	25.87	325	67.52	421.02	0.460293	38	29
AR42	2.51	25.98	534	67.81	630.3	0.192016	16	15
AR43	1.5	19.16	698	50.02	768.68	0.014627	4	8
AR44	2.38	14.85	542	38.75	597.98	0.233447	21	17
AR45	2.03	26.73	360	69.78	458.54	0.412196	35	36
AR46	1.19	30.73	598	80.22	710.14	0.089669	9	20

DMU. Finally, the last column indicates the results obtained by the two-stage approach of Foroughi (2011).

As can be seen, both approaches determine AR26 as the most efficient AR, while the proposed approach in this study provides a more computational efficient solution than the method of Foroughi (2011). Furthermore, the suggested ranking approach in this study utilizes the CW-efficiency in a simplified manner, instead of developing a more complex model or models. The method of Foroughi (2011) ranks the ARs via solving 67 optimization problems [see Table 1 in Foroughi (2011)] while the proposed method obtains the ranking by solving a single optimization problem with a CSW in an identical situation. Although an identical best efficient DMU is obtained by both methods, the ranking scores are different. It is understandable that ranking approaches with different assumptions might have different results [for more details about ranking approaches, see Adler et al. (2002)].

Example 2 Research institutes at Chinese Academy of Sciences.

Liu et al. (2013) evaluated fifteen research institutions at the Chinese Academy of Sciences (CAS) using six outputs: SCI Pub.[1] per Staff[2] (y_1), SCI pub. per Res.Expen.[3] (y_2), High Pub.[4] per Staff (y_3), High Pub. per Res.Expen. (y_4), Exter. Fund.[5] per Staff (y_5), and Grad. Enroll.[6] per Staff (y_6). Table 3 represents the data set and the results obtained by the proposed method in this study.

The column labeled CW-efficiency displays the CW-efficiency score which is equal to $\left(1 + d_j^*\right)^{-1}$ by definition. Hence, the suggested simple computation gives CAS03 as the best institute, the same result obtained in the comprehensive evaluation system at CAS.

Example 3 Robot selection.

Karsak and Ahiska (2005) evaluated twelve robots using four engineering attributes as outputs: handling coefficient (y_1), load capacity (y_2), repeatability (y_3) and velocity (y_4), and a single input cost (x_1) in US$.

1 The number of international papers indexed by the Science Citation Index.

2 The number of full-time research staff.

3 Total research expenditures.

4 The number of high quality papers published in top research journals.

5 The external research funding obtained.

6 The number of graduate students enrolled.

Table 3 Data and results for the research institutes at the Chinese Academy of Sciences

DMU	y_1	y_2	y_3	y_4	y_5	y_6	$\sum_{j=1}^{6} y_j$	d_j^*	CW-efficiency	Rank
CAS 01	46.06	55.88	6.96	8.54	40.41	85.45	243.3	0.519027	0.658316	4
CAS 02	100	100	44.28	44.8	72.46	71.24	432.78	0.14445	0.873782	2
CAS 03	99.89	98.75	100	100	36.7	70.51	505.85	0	1	1
CAS 04	12.06	27.47	3.85	8.86	19.28	48.87	120.39	0.762005	0.567535	11
CAS 05	23.88	59.83	6.85	17.38	24.16	55.97	188.07	0.62821	0.614171	7
CAS 06	17.56	20.71	7.3	8.71	38.67	23.27	116.22	0.770248	0.564893	12
CAS 07	6.88	12.06	2.55	4.53	29.61	23.78	79.41	0.843017	0.542589	15
CAS 08	16.6	38.32	1.15	2.68	11.22	19.62	89.59	0.822892	0.548579	14
CAS 09	18.91	30.98	7.63	12.64	35.01	33.56	138.73	0.725749	0.579459	9
CAS 10	15.05	21.46	4.28	6.17	37.04	16.62	100.62	0.801087	0.55522	13
CAS 11	24.57	41.82	5.81	10	53.33	15.97	151.5	0.700504	0.588061	8
CAS 12	16.06	21.52	2.18	2.95	39.56	46.06	128.33	0.746308	0.572637	10
CAS 13	86.1	93.57	34.38	37.8	60.93	100	412.78	0.183987	0.844604	3
CAS 14	37.85	68.33	8.35	15.26	56.7	44.33	230.82	0.543699	0.647795	6
CAS 15	26.93	26.32	5.9	5.84	100	72.78	237.77	0.529959	0.653612	5

Table 4 Robot selection data and results (Karsak and Ahiska 2005)

Robots	x_1	y_1	y_2	y_3	y_4	d_j^*			Rank
						$k=0$	$k=0.1$	$k=0.2$	
R1	100,000	0.995	85	1.7	3	0.347	0.347	0.366	8
R2	75,000	0.933	45	2.5	3.6	0.247	0.247	0.306	6
R3	56,250	0.875	18	5	2.2	0.117	0.117	0.223	5
R4	28,125	0.409	16	1.7	1.5	0.138	0.138	0.216	4
R5	46,875	0.818	20	5	1.1	0	0	0.114	3
R6	78,125	0.664	60	2.5	1.35	0.437	0.437	0.45	11
R7	875,00	0.88	90	2	1.4	0.317	0.317	0.323	7
R8	56,250	0.633	10	8	2.5	0.369	0.369	0.45	12
R9	56,250	0.653	25	4	2.5	0.313	0.313	0.377	10
R10	87,500	0.747	100	2	2.5	0.383	0.383	0.368	9
R11	68,750	0.88	100	4	1.5	0.11	0.11	0.108	2
R12	43,750	0.633	70	5	3	0	0	0	1

As can be inferred from Table 4, there are two efficient robots, i.e., R5 and R12, with $k=0$. Karsak and Ahiska (2005) increased the value of the discriminating parameter and considered $k=0.1$. Since the results remain the same, they then selected a larger value for the discriminating parameter, $k=0.2$. In this case, R12 is the single efficient robot which clearly must be considered as the most efficient robot.

There is an important issue which must be mentioned here. If Karsak and Ahiska (2005) had considered $\varepsilon=\varepsilon^*$, then the most efficient robot would have been selected for $k=0$. Nevertheless, they ignored the role of the maximum non-Archimedean epsilon and let $\varepsilon=0.00001$.

Note, that in order to apply our approach using this data set and compare the result obtained, we first must unitize ‘cost’ to 1 under the constant returns-to-scale assumption. Hence, the outputs are normalized to values per 1 unit of cost (1 US$) which are shown in Table 5.

Table 5 also shows that R5 is not a CW-efficient robot and based on its CW-efficiency score is ranked 9th, which is acceptable. Note, that solving the model of Karsak and Ahiska by the simplex method needs 50 iterations with 65 multiplications and 60 additions, or equivalently 6100 elementary arithmetic operations, while our approach requires just 36 additions.

Example 4 Asset financing selection.

Toloo and Kresta (2014) evaluated 139 different alternatives for long-term asset financing at Czech banks and leasing companies. As shown in Table 6, they considered 4 inputs: *Down payment* (x_1), *Annuities* (x_2), *other fees* (x_3) and *Bank loan coefficient* (x_4) with no explicit outputs.

A simple computation shows that the min $\left\{\sum_{i=1}^{4} x_{ij} : j=1,\ldots,139\right\} = \sum_{i=1}^{4} x_{i56} = 11499.33$ and hence, $\varepsilon^* = (11499.33)^{-1} = 0.00087$. Reference to Table 3 shows that the 59$^{\text{th}}$ financing alternative (the bank loan from ‘Komerční banka’) is the best asset financing alternative, which is the same result obtained by Toloo and Kresta (2014).

Table 5 Normalized outputs and results

Robots	$\underline{y}_1$	$\underline{y}_2$	$\underline{y}_3$	$\underline{y}_4$	$\sum_{j=1}^{4} y_j$	d_j^*	CW-efficiency	Rank
R1	9.950E−06	8.500E−04	1.700E−05	3.000E−05	9.070E−04	0.495389	0.668722	5
R2	1.244E−05	6.000E−04	3.333E−05	4.800E−05	6.938E−04	0.613997	0.61958	8
R3	1.556E−05	3.200E−04	8.889E−05	3.911E−05	4.636E−04	0.742086	0.574024	11
R4	1.454E−05	5.689E−04	6.044E−05	5.333E−05	6.972E−04	0.612085	0.620315	7
R5	1.745E−05	4.267E−04	1.067E−04	2.347E−05	5.743E−04	0.680497	0.595062	9
R6	8.499E−06	7.680E−04	3.200E−05	1.728E−05	8.258E−04	0.540551	0.649118	6
R7	1.006E−05	1.029E−03	2.286E−05	1.600E−05	1.077E−03	0.400506	0.714028	4
R8	1.125E−05	1.778E−04	1.422E−04	4.444E−05	3.757E−04	0.790968	0.558357	12
R9	1.161E−05	4.444E−04	7.111E−05	4.444E−05	5.716E−04	0.681967	0.594542	10
R10	8.537E−06	1.143E−03	2.286E−05	2.857E−05	1.203E−03	0.330771	0.751444	3
R11	1.280E−05	1.455E−03	5.818E−05	2.182E−05	1.547E−03	0.139085	0.877898	2
R12	1.447E−05	1.600E−03	1.143E−04	6.857E−05	1.797E−03	0	1	1

Table 6 Data and results for the Czech banks and the leasing companies' asset financing alternatives

DMU	(x_1)	(x_2)	(x_3)	(x_4)	$d_j^* = \varepsilon^* \sum_{i=1}^{4} x_{ij}$	DMU	(x_1)	(x_2)	(x_3)	(x_4)	$d_j^* = \varepsilon^* \sum_{i=1}^{4} x_{ij}$
1	0	17,696	1210	1.318	0.644	71	181,500	10, 911	0	1.202	15.732
2	36,300	16,917	1210	1.326	3.733	72	181, 500	9415	0	1.245	15.602
3	72,600	16,137	1210	1.335	6.822	73	181,500	8352	0	1.288	15.510
4	108,900	15, 344	1210	1.345	9.910	74	217,800	44,091	0	1.041	21.775
5	145,200	14,565	1210	1.356	12.999	75	217,800	22,869	0	1.080	19.929
6	181,500	13,774	1210	1.368	16.087	76	217,800	15,809	0	1.120	19.315
7	217,800	13,067	1210	1.391	19.182	77	217,800	12,288	0	1.161	19.009
8	254,100	12,284	1210	1.408	22.271	78	217,800	10,184	0	1.202	18.826
9	290,400	11,502	1210	1.429	25.359	79	217,800	8787	0	1.245	18.704
10	326,700	10,719	1210	1.453	28.448	80	217,800	7795	0	1.288	18.618
11	363,000	9954	1210	1.484	31.538	81	254,100	40,941	0	1.041	24.657
12	399,300	9185	1210	1.522	34.628	82	254,100	21, 236	0	1.080	22.944
13	435,600	8526	1210	1.590	37.727	83	254,100	14,680	0	1.120	22.374
14	471,900	7763	1210	1.655	40.818	84	254,100	11,411	0	1.161	22.089
15	508,200	7012	1210	1.744	43.909	85	254,100	9456	0	1.202	21.919
16	0	16,327	1210	1.351	0.525	86	254,100	8160	0	1.245	21.807
17	36,300	15,615	1210	1.360	3.620	87	254,100	7239	0	1.289	21.727
18	72,600	14,902	1210	1.370	6.715	88	290,400	37,792	0	1.041	27.540
19	108,900	14,160	1210	1.379	9.807	89	290,400	19,602	0	1.080	25.958
20	145,200	13,448	1210	1.391	12.902	90	290,400	13,550	0	1.120	25.432
21	181,500	12,737	1210	1.406	15.997	91	290,400	10,533	0	1.161	25.170
22	217,800	12,026	1210	1.422	19.091	92	290,400	8729	0	1.202	25.013
23	254,100	11,337	1210	1.444	22.188	93	290,400	7532	0	1.245	24.909
24	290,400	10,644	1210	1.469	25.285	94	290,400	6682	0	1.289	24.835

Table 6 continued

DMU	(x_1)	(x_2)	(x_3)	(x_4)	$d_j^* = \varepsilon^* \sum_{i=1}^{4} x_{ij}$	DMU	(x_1)	(x_2)	(x_3)	(x_4)	$d_j^* = \varepsilon^* \sum_{i=1}^{4} x_{ij}$
25	326,700	9930	1210	1.495	28.379	95	0	65,268	0	1.079	4.676
26	363,000	9215	1210	1.526	31.474	96	0	34,947	0	1.155	2.039
27	399,300	8516	1210	1.568	34.570	97	0	24,865	0	1.233	1.162
28	435,600	7900	1210	1.636	37.673	98	0	19,841	0	1.312	0.726
29	471,900	7197	1210	1.704	40.768	99	0	16,842	0	1.392	0.465
30	508,200	6521	1210	1.802	43.866	100	0	14,854	0	1.473	0.292
31	0	36,060	0	1.192	2.136	101	0	13,444	0	1.556	0.169
32	36,300	34,381	0	1.196	5.147	102	145,200	52,335	0	1.081	16.178
33	72,600	32,502	0	1.194	8.140	103	145,200	28,082	0	1.160	14.069
34	108,900	30,791	0	1.198	11.148	104	145,200	20,013	0	1.240	13.367
35	145,200	29,125	0	1.204	14.160	105	145,200	15,990	0	1.321	13.017
36	181,500	27,510	0	1.213	17.176	106	145,200	13,585	0	1.403	12.808
37	217,800	25,888	0	1.223	20.192	107	145,200	11,989	0	1.486	12.670
38	254,100	24,270	0	1.234	23.208	108	145,200	10,855	0	1.570	12.571
39	290,400	22,655	0	1.248	26.224	109	181,500	49,186	0	1.084	19.061
40	326,700	21,039	0	1.265	29.240	110	181,500	26,449	0	1.166	17.084
41	363,000	19,420	0	1.284	32.256	111	181,500	18,884	0	1.249	16.426
42	399,300	17,802	0	1.308	35.272	112	181,500	15,112	0	1.332	16.098
43	435,600	16,189	0	1.338	38.288	113	181,500	12,857	0	1.417	15.902
44	471,900	14,572	0	1.376	41.304	114	181,500	11,361	0	1.502	15.772
45	508,200	12,960	0	1.428	44.321	115	181,500	10,298	0	1.589	15.679
46	145,200	29,125	0	1.204	14.160	116	217,800	46,037	0	1.087	21.944
47	145,200	20,070	0	1.244	13.372	117	217,800	24,815	0	1.172	20.098

Table 6 continued

DMU	(x_1)	(x_2)	(x_3)	(x_4)	$d_j^* = \varepsilon^* \sum_{i=1}^{4} x_{ij}$	DMU	(x_1)	(x_2)	(x_3)	(x_4)	$d_j^* = \varepsilon^* \sum_{i=1}^{4} x_{ij}$
48	145,200	15,816	0	1.307	13.002	118	217,800	17,755	0	1.258	19.484
49	145,200	14,437	0	1.342	12.882	119	217,800	14,234	0	1.344	19.178
50	145,200	13,336	0	1.378	12.787	120	217,800	12,130	0	1.432	18.995
51	145,200	11,688	0	1.449	12.643	121	217,800	10,733	0	1.521	18.874
52	145,200	105,16	0	1.521	12.541	122	217,800	9741	0	1.610	18.787
53	0	63,322	0	1.047	4.507	123	254,100	42,887	0	1.091	24.827
54	0	33,001	0	1.091	1.870	124	254,100	23,182	0	1.179	23.113
55	0	22,919	0	1.136	0.993	125	254100	16,626	0	1.268	22.543
56	0	17,895	0	1.183	0.556	126	254,100	13,357	0	1.359	22.259
57	0	14,896	0	1.231	0.295	127	254,100	11,402	0	1.450	22.089
58	0	12,908	0	1.280	0.123	128	254,100	10,106	0	1.542	21.976
59	0	11,498	0	1.330	0	129	254,100	9185	0	1.635	21.896
60	145,200	50,389	0	1.041	16.009	130	290,400	39,738	0	1.095	27.709
61	145,200	26,136	0	1.080	13.900	131	290,400	21,548	0	1.187	26.128
62	145,200	18,067	0	1.120	13.198	132	290,400	15,496	0	1.281	25.601
63	145,200	14,044	0	1.161	12.848	133	290,400	12,479	0	1.375	25.339
64	145,200	11,639	0	1.202	12.639	134	290,400	10,675	0	1.470	25.182
65	145,200	10,043	0	1.245	12.500	135	290,400	9478	0	1.567	25.078
66	145,200	8909	0	1.288	12.402	136	290,400	8628	0	1.664	25.004
67	181,500	47,240	0	1.041	18.892	137	0	16,080	5000	1.203	0.833
68	181,500	24,503	0	1.080	16.914	138	0	14,749	5000	1.226	0.718
69	181,500	16,938	0	1.120	16.257	139	0	12,760	5000	1.272	0.545
70	181,500	13,166	0	1.161	15.929						

5 Conclusions

In traditional DEA models, it is assumed that each DMU involves multiple inputs and multiple outputs. However, the case of a single constant input and multiple outputs or multiple inputs and a single constant output has interested some DEA researchers. Several DEA methods (WEI/WEO) have been proposed to fully rank the DMUs, which can be utilized for selecting a single unit. However, most of these methods are computationally complex and difficult to use. In this study, we proposed an efficient method for finding a single efficient DMU without the need for solving an optimization model. Four numerical examples from different contexts were used to illustrate the validity of the proposed approach. A comparative analysis showed a significant reduction in the computational complexity of the proposed method over the previous methods proposed in the literature.

In this study, we theoretically proved and practically illustrated that the most efficient DMU in DEA-WEI (-WEO) is a unit with the maximum (minimum) summation of outputs (inputs). Although this can determine the most efficient DMU without solving any model, it disregards the importance of the weights. Future research can tackle this issue by imposing suitable weight restrictions on the models, which in turn needs additional information concerning the importance of the factors. For instance, applying the assurance region (Thompson et al. 1986) or the most favorable weights (Green et al. 1996) approaches may emphasie the importance of the weights. Another interesting research topic is extending a method to deal with ratio data in DEA-WEI/O (Emrouznejad and Amin 2009).

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