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A Bounded Data Envelopment Analysis Model in a Fuzzy Environment with an Application to Safety in the Semiconductor Industry

Adel Hatami-Marbini · Madjid Tavana · Kobra Gholami · Zahra Ghelej Beigi

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Abstract Data envelopment analysis (DEA) is a mathematical programming approach for evaluating the relative efficiency of decision making units (DMUs) in organizations. The conventional DEA methods require accurate measurement of both the inputs and outputs. However, the observed values of the input and output data in real-world problems are often imprecise or vague. Fuzzy set theory is widely used to quantify imprecise and vague data in DEA models. In this paper, we propose a four-step bounded fuzzy DEA model, where the inputs and outputs are assumed to be fuzzy numbers. In the first step, we create a hypothetical fuzzy anti-ideal DMU and calculate its best fuzzy relative efficiency. In the second step, we propose a pair of fuzzy DEA models to obtain the upper- and the lower-bounds of the fuzzy efficiency, where the lower-bound

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A. Hatami-Marbini
Louvain School of Management, Center of Operations Research and Econometrics (CORE), Université catholique de Louvain, L1.03.01, 1348 Louvain-la-Neuve, Belgium

M. Tavana (✉)
Business Systems and Analytics Department, Lindback Distinguished Chair of Information Systems and Decision Sciences, La Salle University, Philadelphia, PA 19141, USA
e-mail: tavana@lasalle.edu
URL: http://www.tavana.us

M. Tavana
Business Information Systems Department, Faculty of Business Administration and Economics, University of Paderborn, 33098 Paderborn, Germany

K. Gholami
Department of Science, Islamic Azad University, Bushehr Branch, Bushehr, Iran

Z. G. Beigi
Department of Mathematics, Islamic Azad University, Mabarakeh Branch, Isfahan, Iran

is at least equal to the fuzzy efficiency of the anti-ideal DMU, and the upper-bound is at most equal to one. In step three, we use multi-objective programming to solve the proposed fuzzy programs. In the fourth step, we propose a new method for ranking the bounded fuzzy efficiency scores. We also present a case study to demonstrate the applicability of the proposed model and the efficacy of the procedures and algorithms in measuring the safety performance of eight semiconductor facilities.

Keywords Data envelopment analysis · Fuzzy data · Interval efficiency · Safety · Semiconductor industry

Mathematical Subject Classification (2010) 90B50 · 68M20 · 90C70

1 Introduction

Data envelopment analysis (DEA) is a widely used mathematical programming approach for comparing the inputs and outputs of a set of homogenous decision-making units (DMUs) by evaluating their relative efficiency. DEA was first proposed by Charnes et al. in [1] and later extended by Banker et al. in [2]. The relative efficiency performance of the DMUs in DEA is defined as the ratio of multiple weighted outputs to multiple weighted inputs. The two basic DEA models are named after the respective researchers who first introduced them: the Charnes Cooper Rhodes (CCR) and the Banker Charnes Cooper (BCC) models. Charnes et al. in [1] initiated the CCR model under constant returns to scale (CRS), and Banker et al. proposed the BCC model under variable returns to scale (VRS) in [2]. A DMU is considered efficient when no other DMUs can produce more outputs using an equal or lesser amount of inputs. DEA generalizes the usual efficiency measurement from a single-input single-output ratio to a multiple-input multiple-output ratio by using a ratio of the weighted sum of outputs to the weighted sum of inputs [3]. Unlike parametric methods that require detailed knowledge of the process, DEA is non-parametric and does not require an explicit functional form relating inputs and outputs (see [3–5] for a review of the theoretical foundations and developments in DEA).

The basic DEA methods, CCR and BCC, require accurate measurement of the input and output data. However, the observed values of the input and output data in real-world problems are often imprecise or vague. Imprecise evaluations may be the result of unquantifiable, incomplete, and non-obtainable information. Fuzzy logic and fuzzy sets are used to represent ambiguous, uncertain, or imprecise data in DEA by formalizing the inaccuracies inherent in human decision-making. Fuzzy set algebra developed in [6] is the formal body of theory that allows the treatment of imprecise estimates in uncertain environments. In general, fuzzy DEA methods can be classified into four primary categories, namely, the tolerance approach [7], the alpha-level-based approach [8,9], the fuzzy ranking approach [10], and the possibility approach [11]. An exhaustive review and taxonomy of various fuzzy DEA models in the literature can be found in [12]. The stochastic approach is also used in the DEA literature as an alternative approach to deal with uncertainty. This approach requires an identification of a probability distribution function (e.g., normal) for the error process [7]. However, two shortcomings of the stochastic approach have been highlighted in [7] as: (i) small sample sizes in DEA complicate the use of stochastic models; and (ii) in stochastic approaches, the decision maker is required to assume a specific error distribution (e.g., normal or exponential) to derive given results. The assumption for the error distribution may not be realistic because, in an *a priori* basis, there is very little empirical evidence to select one type of distribution over another.

Bellman et al. in [13] used the theory of fuzzy sets and the transformation of qualitative data into quantitative data. The method in [14] proposed a pair of models to calculate the lower and upper-bounds of the efficiency, where data are characterized by interval quantities. They used two different methods for the pessimistic and optimistic viewpoints to evaluate a given DMU. Wang et al. in [15] modified [14]'s method by using a fixed method to evaluate the interval efficiency of a DMU. Wang and Luo in [16] proposed a new method based on the TOPSIS approach for a comprehensive ranking of DMUs. They determined two virtual DMUs, ideal and anti-ideal DMUs, and introduced two models to estimate the best and the worst efficiency scores of each DMU. They used the efficiency scores of the ideal and the anti-ideal DMUs along with the best and the worst efficiency scores of each DMU to define a closeness index (relative to the ideal DMU) for ranking the DMUs. Wu in [17] argued that the anti-ideal point introduced in [16] was not realistic because the best possible relative efficiency is based on an "input orientation", while the worst possible relative efficiency is based on an "output orientation," if inputs are considered as outputs and outputs are considered as inputs. The authors revised the models developed in [16] for DEA efficiency evaluation based on the ideal DMU. Chen in [18] claimed that [17]'s assertion about the orientation problem was incorrect. Accordingly, the model developed in [17] was not correctly built, and the efficiency scores of the DMUs in the best and worst viewpoints were not symmetric. Hence, the method in [18] improved [16]'s method and revised [17]'s models to evaluate and render a reliable ranking order of all DMUs using the ideal and anti-ideal DMUs. Wang and Yang in [19] evaluated the efficiencies of the DMUs within the range of an interval, so that the best and the worst relative efficiencies could be measured within a unified DEA model. They introduced a virtual DMU as an anti-ideal DMU, where the anti-ideal DMU was the worst point in the observed data. They determined an interval efficiency by setting the lower-bound efficiency equal to the efficiency score of the anti-ideal DMU and the upper-bound equal to one.

Wang et al. in [20] extended the interval efficiency method in [19]. They set the lower-bound of the interval efficiency equal to the efficiency score of the ideal DMU, where the ideal DMU has the best relative efficiency compared to its peer DMUs. After computing overall interval efficiency for each DMU, they ranked the DMUs by considering the DMs' preferences with regards to the input and output weights, as well as applying the Hurwicz criterion approach. Azizi and Wang in [21] argued that the methods proposed in [19,20] do not consider the special case where there is at least one DMU with zero output value in each output because, in such a case, the output vector of the anti-ideal DMU is the zero vector, and consequently its efficiency is equal to zero. Hence, to set the lower bound of efficiency intervals, the authors improved their models by using the optimistic and pessimistic efficiencies of the DMUs instead of the virtual anti-ideal DMU [21].

Jahanshahloo et al. in [22] estimated the interval efficiency of DMUs according to the optimistic and pessimistic viewpoints. They introduced different ideal points for different DMUs, and then ranked the DMUs by their ideal points and extended the proposed method to interval data. Sun et al. in [23] proposed two models to obtain the common set of weights according to the ideal and anti-ideal DMUs for evaluating and ranking DMUs.

The literature reports on a few applications of DEA in safety studies including manufacturing safety [24,25], construction safety [26,27], road safety [28], transportation safety [29,30], and healthcare safety [31]. In this paper, we propose a four-step bounded fuzzy DEA model, where the inputs and outputs are assumed to be fuzzy numbers. In the first step, we create a hypothetical fuzzy anti-ideal DMU and calculate its best fuzzy relative efficiency. In the second step, we propose a pair of fuzzy DEA models to obtain the upper- and the lower-bounds of the fuzzy efficiency, where the lower-bound is at least equal to the fuzzy efficiency of the anti-ideal DMU, and the upper-bound is at most equal to one. In step three, we use multi-objective programming to solve the proposed fuzzy programs. In the fourth step, we propose a new method for ranking the bounded fuzzy efficiency scores. We also present a case study to demonstrate the applicability of the proposed model and the efficacy of the procedures and algorithms in measuring the safety performance of the eight semiconductor facilities.

The remainder of this paper is organized as follows. In Sect. 2, we review the conventional DEA models and some relevant fuzzy set definitions. In Sect. 3, we present the proposed fuzzy-bounded DEA model for measuring the interval efficiencies of DMUs in the fuzzy environment. In Sect. 4, we propose our procedure for ranking the interval data. In Sect. 5, we present a case study to demonstrate the applicability of the proposed models and exhibit the efficacy of the procedures and algorithms. Section 6 presents the conclusive remarks and future research directions.

2 Performance Evaluation

The purpose of DEA as a relative "data-oriented" approach is to measure the efficiencies of a set of homogenous DMUs within a [0,1] range. Entani et al. in [32] formulated a DEA model with an interval efficiency which consisted of efficiencies obtained from the optimistic and pessimistic viewpoints. They used the two end points to construct interval efficiency. The authors in [19] showed that the method in [32] cannot identify inefficient DMUs and introduced the Anti-ideal DMU (ADMU) to reasonably provide an overall evaluation of the efficiencies for all the DMUs. The input–output quantities in the proposed models were assumed to be deterministic and precise. However, the real-world problems often exhibit some form of ambiguity and impreciseness. These imprecise data can be represented as linguistic variables characterized by fuzzy numbers. Many researchers have developed fuzzy DEA models to capture the inherent uncertainty in the real-world performance evaluation problems. In this section, we first briefly review the method in [19] and then present some basic fuzzy set definitions used throughout the paper.

2.1 Precise Formulation

Suppose there are n DMUs to be evaluated, where each DMU$_j$, j=*1,…,n*, produces an output vector $y_j = (y_{1j,\dots,} y_{sj}) \in \mathbb{R}_s^+ j = 1, \dots, n$, using a vector of inputs, x_j=$(x_{ij}, \dots, x_{mj}) \in \mathbb{R}_m^+ \ \ j = 1, \dots, n$. The efficiency of DMU$_o$ (o=1,…, n), the DMU under evaluation, is the ratio of the weighted sum of outputs to the weighted sum of inputs represented as uy_o/vx_o , where $u \in \mathbb{R}_s^+$ and $v \in \mathbb{R}_m^+$ are the vector of output and input weights, respectively.

The [input-oriented] CCR model [also known as the *constant returns to scale (CRS) model* or the *multiplier model*] for evaluating the best relative efficiency of DMU_o is formulated as follows [1]:

$$\max \theta = \frac{uy_o}{vx_o} \quad \text{s.t.} \quad \frac{uy_j}{vx_j} \leq 1, \quad \forall j; \quad u, v \geq \varepsilon, \tag{1}$$

where ε is a positive non-Archimedean infinitesimal number.

Let us consider the precise DEA model proposed in [19]. A virtual ADMU, with the worst performance amongst its peer DMUs, uses the most amount of inputs $x^{\max} = \max_j\{x_j\} \in \mathbb{R}_m^+$ to generate the least amount of output, $y^{\min} = \min_j\{y_{rj}\} \in \mathbb{R}_s^+$. The following linear programming (LP) model is used to find the best relative efficiency of the ADMU, denoted by θ_A:

$$\max \quad \theta_A = uy^{\min} \quad \text{s.t.} \quad vx^{\max} = 1; \quad uy_j - vx_j \leq 0, \quad \forall j; \quad u, v \geq \varepsilon. \tag{2}$$

It should be noted that the efficiency of none of the DMUs can be less than the optimal solution of the objective function in (2), denoted as θ_A^*, and is measured within the range of interval [θ_A^*, 1]. Accordingly, the following pair of models are used to obtain the upper-bound θ_o^{U} (maximization model) and the lower-bound θ_o^{L} (minimization model) of the efficiency with respect to $\theta_A^* \leq \theta_j \leq 1$:

$$\max / \min \frac{uy_o}{vx_o} \quad \text{s.t.} \ \theta_A^* \leq \frac{uy_j}{vx_j} \leq 1, \quad \forall j; \quad u, v \geq \varepsilon. \tag{3}$$

Model (3) can be transformed into a pair of LP models using the transformation introduced in [33] as follows:

$$\max / \min uy_o \quad \text{s.t.} \ vx_o = 1; \quad uy_j - vx_j \leq 0, \quad \forall j; \quad uy_j - \theta_A^* vx_j \geq 0, \quad \forall j; \quad u, v \geq \varepsilon. \tag{4}$$

We assume that $\theta_o^{\text{U}^*}$and $\theta_o^{\text{L}^*}$ ($\theta_o^{\text{L}^*} \leq \theta_o^{\text{U}^*}$) are the optimal values of the maximal and minimal objective functions, respectively, for estimating the best and the worst relative efficiency of DMU$_o$. Clearly, constraints (4) are redundant in the maximal problem. A DMU$_o$ is said to be "DEA efficient" and "DEA inefficient" if $\theta_o^{\text{U}^*} = 1$ and $\theta_o^{\text{L}^*} = \theta_A^*$, respectively; and a DMU$_o$, which is both DEA efficient and DEA inefficient, has the largest efficiency interval involving the highest uncertainty. Otherwise, the DMU$_o$ is said to be "DEA unspecified". As a result, the DEA efficient DMUs build the efficiency frontier, while the DEA inefficient DMUs build the inefficiency frontier.

2.2 Definitions of Fuzzy Sets

Fuzzy set theory introduced in [6] is a powerful tool to handle imprecise or vague data, where precise analysis is either difficult or impossible. In this section, we review some basic definitions of fuzzy sets [34–37].

Definition 2.1 Let U be a universe set. A fuzzy set $\tilde{A}$ of U is defined by a membership function $\mu_{\tilde{A}}(x) \Rightarrow [0, 1]$, where $\mu_{\tilde{A}}(x), \forall x \in U$, indicates the degree of membership of x in $\tilde{A}$.

Definition 2.2 A fuzzy subset $\tilde{A}$ of the real numbers $\mathbb{R}$ is convex if and only if $\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq (\mu_{\tilde{A}}(x) \wedge \mu_{\tilde{A}}(y)), \quad \forall x, y \in \mathbb{R}, \quad \forall \lambda \in [0, 1]$, where "$\wedge$" denotes the minimum operator.

Definition 2.3 The γ-level of fuzzy set $\tilde{A}$, $\tilde{A}_\gamma$, is the crisp set $\tilde{A}_\gamma = \{x \,|\mu_{\tilde{A}}(x) \geq \gamma\}$. The support of $\tilde{A}$ is the crisp set $\text{Supp}(\tilde{A}) = \{x \,|\mu_{\tilde{A}}(x) > 0\}$. $\tilde{A}$ is normal if and only if $\text{Supp}_{x \in U} \mu_{\tilde{A}}(x) = 1$, where U is the universal set.

Definition 2.4 $\tilde{A}$ is a fuzzy number if and only if $\tilde{A}$ is a normal and convex fuzzy subset of $\mathbb{R}$.

Definition 2.5 A real fuzzy number $\tilde{A}$, denoted by $\tilde{A} = (a^{\mathrm{L}}, a^{\mathrm{M1}}, a^{\mathrm{M2}}, a^{\mathrm{U}}, w)$, is defined as any fuzzy subset of the real line $\mathbb{R}$ with membership function $\mu_{\tilde{A}}$, which satisfies the following properties:

- $\mu_{\tilde{A}}$ is a semi continuous mapping from $\mathbb{R}$ to the closed interval $[0, w]$, $0 \leq w \leq 1$,
- $\mu_{\tilde{A}}(x) = 0$, for all $x \in [-\infty, a^{\mathrm{L}}]$,
- $\mu_{\tilde{A}}$ is increasing on $[a^{\mathrm{L}}, a^{\mathrm{M1}}]$,
- $\mu_{\tilde{A}}(x) = w$ for all $x \in [a^{\mathrm{M1}}, a^{\mathrm{M2}}]$, where w is a constant and $0 < w \leq 1$,
- $\mu_{\tilde{A}}$ is decreasing on $[a^{\mathrm{M2}}, a^{\mathrm{U}}]$,
- $\mu_{\tilde{A}}(x) = 0$, for all $x \in [a^{\mathrm{U}}, \infty]$,

The membership function $\mu_{\tilde{A}}$ of $\tilde{A}$ can be expressed as:

$$\mu_{\tilde{A}}(x) = \begin{cases} f^{\mathrm{L}}(x), & a^{\mathrm{L}} \leq x \leq a^{\mathrm{M1}}, \\ w, & a^{\mathrm{M1}} \leq x \leq a^{\mathrm{M2}}, \\ f^{\mathrm{R}}(x), & a^{\mathrm{M2}} \leq x \leq a^{\mathrm{U}}, \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

where $f^{\mathrm{L}} : [a^{\mathrm{L}}, a^{\mathrm{M1}}] \Rightarrow [0, w]$ and $f^{\mathrm{R}} : [a^{\mathrm{M2}}, a^{\mathrm{U}}] \Rightarrow [0, w]$. If $w = 1$, then $\tilde{A}$ is called a normal fuzzy number and, if $0 < w \leq 1$, then$\tilde{A}$ is called a non-normal fuzzy number.

A special type of trapezoidal fuzzy numbers is determined by quadruples $\tilde{u} = (u^{\mathrm{L}}, u^{\mathrm{M1}}, u^{\mathrm{v2}}, u^{\mathrm{U}})$, whose membership function can be described as follows:

$$\mu_{\tilde{u}}(x) = \begin{cases} \frac{x-u^{\mathrm{L}}}{u^{\mathrm{M1}}-u^{\mathrm{L}}}, & u^{\mathrm{L}} \le x \le u^{\mathrm{M1}}, \\ 1, & u^{\mathrm{M1}} \le x \le u^{\mathrm{M2}}, \\ \frac{u^{\mathrm{U}}-x}{u^{\mathrm{U}}-u^{\mathrm{M2}}}, & u^{\mathrm{M2}} \le x \le u^{\mathrm{U}}, \\ 0, & \text{Otherwise.} \end{cases} \tag{6}$$

The trapezoidal fuzzy number $\tilde{u} = (u^{\mathrm{L}}, u^{\mathrm{M1}}, u^{\mathrm{M2}}, u^{\mathrm{U}})$ is transformed into a real number if $u^{\mathrm{L}} = u^{\mathrm{M1}} = u^{\mathrm{M2}} = u^{\mathrm{U}}$. In contrast, a real number can be described as a trapezoidal fuzzy number $\tilde{u} = (u, u, u, u)$. If $u^{\mathrm{M}} = u^{\mathrm{M1}} = u^{\mathrm{M2}}$, then $\tilde{u} = (u^{\mathrm{L}}, u^{\mathrm{M}}, u^{\mathrm{U}})$ is said to be a triangular fuzzy number. A triangular fuzzy number has the following membership function:

$$\mu_{\tilde{u}}(x) = \begin{cases} \frac{x-u^{\mathrm{L}}}{u^{\mathrm{M}}-u^{\mathrm{L}}}, & u^{\mathrm{L}} \le x \le u^{\mathrm{M}}, \\ \frac{u^{\mathrm{U}}-x}{u^{\mathrm{U}}-u^{\mathrm{M}}}, & u^{\mathrm{M}} \le x \le u^{\mathrm{U}}, \\ 0, & \text{Otherwise.} \end{cases} \tag{7}$$

The triangular fuzzy numbers are most-widely used in practical problems in the fuzzy environment. Hence, for the sake of simplicity, it is presumed that all fuzzy numbers used throughout the paper are triangular fuzzy numbers. There is any loss of generality in assuming that the triangular fuzzy numbers are all not less than 0 here.

Definition 2.6 Consider two positive triangular fuzzy numbers $\tilde{A} = (a^{\mathrm{L}}, a^{\mathrm{M}}, a^{\mathrm{U}})$ and $\tilde{B} = (b^{\mathrm{L}}, b^{\mathrm{M}}, b^{\mathrm{U}})$; then the arithmetic operations of these two triangular fuzzy numbers are defined as follows:

Addition: $\tilde{A}(+)\tilde{B} = (a^{\mathrm{L}} + b^{\mathrm{L}}, a^{\mathrm{M}} + b^{\mathrm{M}}, a^{\mathrm{U}} + b^{\mathrm{U}})$,
Subtraction: $\tilde{A}(-)\tilde{B} = (a^{\mathrm{L}} - b^{\mathrm{U}}, a^{\mathrm{M}} - b^{\mathrm{M}}, a^{\mathrm{U}} - b^{\mathrm{L}})$,
Multiplication: $\tilde{A}(\times)\tilde{B} \approx (a^{\mathrm{L}}b^{\mathrm{L}}, a^{\mathrm{M}}b^{\mathrm{M}}, a^{\mathrm{U}}b^{\mathrm{U}})$,

$$k\tilde{A} = (ka^{\mathrm{L}}, ka^{\mathrm{M}}, ka^{\mathrm{U}}), \quad \forall k \in \mathbb{R}^{+},$$

Inverse: $(\tilde{B})^{-1} = (\frac{1}{b^{\mathrm{U}}}, \frac{1}{b^{\mathrm{M}}}, \frac{1}{b^{\mathrm{L}}})$,
Division: $\tilde{A}(\div)\tilde{B} \approx \tilde{A}(\times)\tilde{B}^{-1} = (\frac{a^{\mathrm{L}}}{b^{\mathrm{U}}}, \frac{a^{\mathrm{M}}}{b^{\mathrm{M}}}, \frac{a^{\mathrm{U}}}{b^{\mathrm{L}}})$.

Definition 2.7 Linguistic variables are those variables whose values are not real numbers, but are widely used in daily life. For example, "very cold", "good", or "hot" are linguistic variables since their values are represented by linguistic terms rather than numerical terms. The concept of linguistic variable is useful in handling situations that are complex or ill-structured to be reasonably described with quantitative values. Fuzzy numbers are often used to represent linguistic variables.

Definition 2.8 Given a set of triangular fuzzy numbers $\{\tilde{u}_j = (u_j^{\mathrm{L}}, u_j^{\mathrm{M}}, u_j^{\mathrm{U}}),\ j = 1, 2, \ldots, n\}$ and a set of positive real values $\{\lambda_j \in \mathbb{R}^{+},\ j = 1, 2, \ldots, n\}$, the combination $\lambda_1\tilde{u}_1 \oplus \lambda_2\tilde{u}_2 \oplus \ldots \oplus \lambda_n\tilde{u}_n$ will be denoted by $\sum_{j=1}^{n} \lambda_j\tilde{u}_j$, that is:

$$\sum_{j=1}^{n} \lambda_j \tilde{u}_j = \left(\sum_{j=1}^{n} \lambda_j u_j^{\mathrm{L}}, \sum_{j=1}^{n} \lambda_j u_j^{\mathrm{M}}, \sum_{j=1}^{n} \lambda_j u_j^{\mathrm{U}} \right). \tag{8}$$

Several methods have been proposed in the literature for ranking fuzzy numbers. The ranking methods are inherently context-dependent and are not applicable to all situations and all problems. We extend the ranking method proposed in [38] to compare the fuzzy efficiency of DMUs. Dubois and Prade in [39] defined the following ordering relation between two fuzzy numbers based on the "fuzzy max" operator:

Definition 2.9 [38] Suppose that $\tilde{A}$ and $\tilde{B}$ are two fuzzy numbers:

$$\tilde{A} \underset{\sim}{>} \tilde{B} \Leftrightarrow \tilde{A} \vee \tilde{B} = \tilde{A}. \tag{9}$$

Ramík and Rímánk in [38] proposed an operative characterization of (9) based on the $\alpha-$cut sets.

Lemma 2.1 [38] *Suppose that $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers. Then, $\tilde{A} \vee \tilde{B} = \tilde{A}$ if and only if, for each $\alpha \in [0, 1]$, the two following statements hold:*

$$\begin{aligned} \inf \left\{ s : \mu_{\tilde{A}}(s) \geq \alpha \right\} &\geq \inf \left\{ t : \mu_{\tilde{B}}(t) \geq \alpha \right\}, \\ \sup \left\{ s : \mu_{\tilde{A}}(s) \geq \alpha \right\} &\geq \sup \left\{ t : \mu_{\tilde{B}}(t) \geq \alpha \right\}. \end{aligned} \tag{10}$$

Particularly, for two triangular fuzzy numbers $\tilde{A} = (a^{\mathrm{L}}, a^{\mathrm{M}}, a^{\mathrm{U}})$ and $\tilde{B} = (b^{\mathrm{L}}, b^{\mathrm{M}}, b^{\mathrm{U}})$, (10) becomes:

$$\tilde{A} \geq \tilde{B}, \Leftrightarrow a^{\mathrm{L}} \geq b^{\mathrm{L}}, a^{\mathrm{M}} \geq b^{\mathrm{M}}, a^{\mathrm{U}} \geq b^{\mathrm{U}}.$$

3 Proposed Bounded Fuzzy DEA Models

The problems of uncertainty, imprecision, and vagueness in real-world problems have been studied for many years by mathematicians, statisticians, and philosophers. The pioneering fuzzy logic work introduced in [6] has been used as a powerful tool for dealing with uncertainty and ambiguity in real-world problems. In this section, we propose a fuzzy DEA framework, which uses the concept of ADMU to measure the interval relative efficiencies using the efficiency and inefficiency frontiers.

Suppose there are n DMUs and each DMU consumes m inputs to produce s outputs. Furthermore, let us assume that the input and output values cannot be measured precisely, and they are expressed with triangular fuzzy numbers. The vector of fuzzy input is denoted by $\tilde{x}_j = (\tilde{x}_{ij}, \ldots, \tilde{x}_{mj})$ $j = 1, \ldots, n$, where $\tilde{x}_{ij} = (x_{ij}^{\mathrm{L}}, x_{ij}^{\mathrm{M}}, x_{ij}^{\mathrm{U}})$ $(i = 1, \ldots, m;\ j = 1, \ldots, n)$ and the vector of fuzzy output is denoted by $\tilde{y}_j = (\tilde{y}_{1j}, \ldots, \tilde{y}_{sj})$ $j = 1, \ldots, n$, where $\tilde{y}_{rj} = (y_{rj}^{\mathrm{L}}, y_{rj}^{\mathrm{M}}, y_{rj}^{\mathrm{U}})$ $(r = 1, \ldots, s;\ j = 1, \ldots, n)$. We define the fuzzy ADMU containing the worst efficiency among the DMUs.

Definition 3.1 A fuzzy ADMU is a hypothetical DMU consuming the most fuzzy inputs $\tilde{x}^{\max}$ to produce the least fuzzy outputs $\tilde{y}^{\min}$. The inputs and outputs of the fuzzy ADMU are defined as:

$$\tilde{x}^{\max} = \max_j\{\tilde{x}_j\} = \max_j \left\{\left(x_j^{\mathrm{L}}, x_j^{\mathrm{M}}, x_j^{\mathrm{U}}\right)\right\} = \left\{\left(\max_j x_j^{\mathrm{L}}, \max_j x_j^{\mathrm{M}}, \max_j x_j^{\mathrm{U}}\right)\right\},$$

$$\tilde{y}^{\min} = \min_j\{\tilde{y}_j\} = \min_j \left\{\left(y_j^{\mathrm{L}}, y_j^{\mathrm{M}}, y_j^{\mathrm{U}}\right)\right\} = \left\{\left(\min_j y_j^{\mathrm{L}}, \min_j y_j^{\mathrm{M}}, \min_j y_j^{\mathrm{U}}\right)\right\}.$$

The best relative efficiency of the fuzzy ADMU, $\tilde{\theta}_A^*$, under the CRS technology can be formulated as:

$$\max \quad \tilde{\theta}_A = \frac{u\tilde{y}^{\min}}{v\tilde{x}^{\max}} \quad \text{s.t.} \ \frac{u\tilde{y}_j}{v\tilde{x}_j} \leq \tilde{1}, \quad \forall j; \quad u, v \geq \varepsilon. \tag{11}$$

Mathematically, solving the above fuzzy fractional programming model is a challenging problem. Fuzzy input and output data have been widely studied in the DEA literature. The recent comprehensive survey has categorized the fuzzy DEA models in the literature into four distinct classes [12]: (i) the tolerance approach; (ii) the α-level-based approach; (iii) the fuzzy-ranking approach; and (iv) the possibility approach. Most fuzzy DEA models disregard the fact that a fuzzy fractional programming model cannot be converted into a LP model using Charnes–Cooper's transformation [33] characterized with crisp fractional programming [40]. Consequently, Model (11) can be described as the following multi-objective programming model using fuzzy arithmetic (see definitions 2.6 and 2.8):

$$\max \tilde{\theta}_A = \left(\frac{uy^{\mathrm{L}\min}}{vx^{\mathrm{U}\max}}, \frac{uy^{\mathrm{M}\min}}{vx^{\mathrm{M}\max}}, \frac{uy^{\mathrm{U}\min}}{vx^{\mathrm{L}\max}}\right) \quad \text{s.t.} \quad \left(\frac{uy_j^{\mathrm{L}}}{vx_j^{\mathrm{U}}}, \frac{uy_j^{\mathrm{M}}}{vx_j^{\mathrm{M}}}, \frac{uy_j^{\mathrm{U}}}{vx_j^{\mathrm{L}}}\right) \leq (1, 1, 1),$$
$$\forall j; \quad u, v \geq \varepsilon. \tag{12}$$

Note that, since we assume that x_{ij}^{L} and y_{rj}^{L} to be positive, the values of x_j^{L} and y_j^{L} are positive in (12). In order to solve this problem, Wang et al. in [40] used fuzzy arithmetic and constructed three fractional- programming models to approximate the three components of fuzzy triangular efficiency for each DMU. In this paper, we propose a new method to optimize three objective functions in Model (12) simultaneously. Besides, this proposed method is computationally economical for obtaining the fuzzy efficiency of the DMUs. Over the past three decades, several methods have been proposed to solve multi-objective problems in the optimization literature (see e.g., [41,42]). In this study, we use the goal programming (GP) method developed in [43,44] to solve Model (12). GP is used to minimize the sum of the negative and positive deviations from the present goals for each objective function. The difference between each objective function and its goal is defined as the negative and positive deviations, denoted by $s_{\mathrm{L}}^{+}, s_{\mathrm{M}}^{+}, s_{\mathrm{U}}^{+}$ and $s_{\mathrm{L}}^{+}, s_{\mathrm{M}}^{+}, s_{\mathrm{U}}^{+}$, respectively. The GP is used here to minimize these deviations from the present goals for three objective functions. Due to the efficiency context, the goal of each DMU or objective function is to become efficient (or to reach unity). To achieve the unit value, a DMU should decrease its inputs and/or increase its outputs as much as possible. Therefore, we define the slack variables s_{L}^{+} and s_{L}^{-} for the first objective function, $uy^{\mathrm{L\,min}}/vx^{\mathrm{U\,max}}$. The corresponding constraint is formulated as $(uy^{\mathrm{L\,min}}+s_{\mathrm{L}}^{+})/(vx^{\mathrm{U\,max}}-s_{\mathrm{L}}^{-})=1$. The second and third objective functions are treated similarly in Model (13). This representation thus results in the following LP model with a single objective function:

$$\begin{aligned}
&\min \theta_A = s_{\mathrm{L}}^{-}+s_{\mathrm{L}}^{+}+s_{\mathrm{M}}^{-}+s_{\mathrm{M}}^{+}+s_{\mathrm{U}}^{-}+s_{\mathrm{U}}^{+} \quad \text{s.t.} \quad \frac{uy^{\mathrm{L\,min}}+s_{\mathrm{L}}^{+}}{vx^{\mathrm{U\,max}}-s_{\mathrm{L}}^{-}}=1;\ \frac{uy^{\mathrm{M\,min}}+s_{\mathrm{M}}^{+}}{vx^{\mathrm{M\,max}}-s_{\mathrm{M}}^{-}}=1;\\
&\frac{uy^{\mathrm{U\,min}}+s_{\mathrm{U}}^{+}}{vx^{\mathrm{L\,max}}-s_{\mathrm{U}}^{-}}=1;\quad \frac{uy_j^{\mathrm{L}}}{vx_j^{\mathrm{U}}}\le 1,\quad \forall j;\quad \frac{uy_j^{\mathrm{M}}}{vx_j^{\mathrm{M}}}\le 1,\quad \forall j;\quad \frac{uy_j^{\mathrm{U}}}{vx_j^{\mathrm{L}}}\le 1,\quad \forall j;\\
&\quad u,v\ge \varepsilon,\ s_{\mathrm{L}}^{-},s_{\mathrm{L}}^{+},s_{\mathrm{M}}^{-},s_{\mathrm{M}}^{+},s_{\mathrm{U}}^{-},s_{\mathrm{U}}^{+}\ge 0.
\end{aligned} \tag{13}$$

Model (13) can be easily transformed into the following model:

$$\begin{aligned}
&\min \theta_A = s_{\mathrm{L}}^{-}+s_{\mathrm{L}}^{+}+s_{\mathrm{M}}^{-}+s_{\mathrm{M}}^{+}+s_{\mathrm{U}}^{-}+s_{\mathrm{U}}^{+}\\
&\text{s.t.}\ uy^{\mathrm{L\,min}}-vx^{\mathrm{U\,max}}+s_{\mathrm{L}}^{+}+s_{\mathrm{L}}^{-}=0;\\
&uy^{\mathrm{M\,min}}-vx^{\mathrm{M\,max}}+s_{\mathrm{M}}^{+}+s_{\mathrm{M}}^{-}=0;\quad uy^{\mathrm{U\,min}}-vx^{\mathrm{L\,max}}+s_{\mathrm{U}}^{+}+s_{\mathrm{U}}^{-}=0;\\
&uy_j^{\mathrm{L}}-vx_j^{\mathrm{U}}\le 0,\quad \forall j;\\
&uy_j^{\mathrm{M}}-vx_j^{\mathrm{M}}\le 0,\quad \forall j;\quad uy_j^{\mathrm{U}}-vx_j^{\mathrm{L}}\le 0,\quad \forall j;\quad u,v\ge\varepsilon;\\
&s_{\mathrm{L}}^{-},s_{\mathrm{L}}^{+},s_{\mathrm{M}}^{-},s_{\mathrm{M}}^{+},s_{\mathrm{U}}^{-},s_{\mathrm{U}}^{+}\ge 0.
\end{aligned} \tag{14}$$

We further substitute s_{L} for $s_{\mathrm{L}}^{-}+s_{\mathrm{L}}^{+}$, s_{M} for $s_{\mathrm{M}}^{-}+s_{\mathrm{M}}^{+}$, and s_{U} for $s_{\mathrm{U}}^{-}+s_{\mathrm{U}}^{+}$ to simplify Model (14). The outcome of this substitution is Model (15) as follows:

$$\begin{aligned}
&\min \theta_A = s_{\mathrm{L}}+s_{\mathrm{M}}+s_{\mathrm{U}} \quad \text{s.t.}\quad uy^{\mathrm{L\,min}}-vx^{\mathrm{U\,max}}+s_{\mathrm{L}}=0;\\
&uy^{\mathrm{M\,min}}-vx^{\mathrm{M\,max}}+s_{\mathrm{M}}=0;\\
&uy^{\mathrm{U\,min}}-vx^{\mathrm{L\,max}}+s_{\mathrm{U}}=0;\quad uy_j^{\mathrm{L}}-vx_j^{\mathrm{U}}\le 0,\quad \forall j;\quad uy_j^{\mathrm{M}}-vx_j^{\mathrm{M}}\le 0,\quad \forall j;\\
&uy_j^{\mathrm{U}}-vx_j^{\mathrm{L}}\le 0,\quad \forall j;\\
&u,v\ge\varepsilon;\quad s_{\mathrm{L}},s_{\mathrm{M}},s_{\mathrm{U}}\ge 0.
\end{aligned} \tag{15}$$

Let $(u^*,v^*,s_{\mathrm{L}}^*,s_{\mathrm{M}}^*,s_{\mathrm{U}}^*)$ be the optimal solution of Model (15). Then, the fuzzy efficiency score of the fuzzy ADMU representing the worst performance can be calculated as follows:

$$\begin{aligned}\tilde{\theta}_A^* &= \left(\theta_A^{\mathrm{L}*}, \theta_A^{\mathrm{M}*}, \theta_A^{\mathrm{U}*}\right) = \left(\frac{u^* y^{\mathrm{L}\min}}{v^* x^{\mathrm{U}\max}}, \frac{u^* y^{\mathrm{M}\min}}{v^* x^{\mathrm{M}\max}}, \frac{u^* y^{\mathrm{U}\min}}{v^* x^{\mathrm{L}\max}}\right)\\ &= \left(1 - \frac{s_{\mathrm{L}}^*}{v^* x^{\mathrm{U}\max}}, 1 - \frac{s_{\mathrm{M}}^*}{v^* x^{\mathrm{M}\max}}, 1 - \frac{s_{\mathrm{U}}^*}{v^* x^{\mathrm{L}\max}}\right). \end{aligned} \tag{16}$$

Note here that the efficiencies of the DMUs $\tilde{\theta}_j = \left(\theta_j^{\mathrm{L}}, \theta_j^{\mathrm{M}}, \theta_j^{\mathrm{U}}\right)$ are always within the range of $\tilde{\theta}_A^* = \left(\theta_A^{\mathrm{L}*}, \theta_A^{\mathrm{M}*}, \theta_A^{\mathrm{U}*}\right)$ and $\tilde{1} = (1, 1, 1)$. We accordingly use a pair of the following models to obtain the upper-bound $\tilde{\theta}_o^{\mathrm{U}}$ (maximization model) and lower-bound $\tilde{\theta}_o^{\mathrm{L}}$ (minimization model) of the fuzzy-bounded efficiency with respect to $\tilde{\theta}_A^* \le \tilde{\theta}_j \le \tilde{1}$:

$$\max/\min \frac{u\tilde{y}_o}{v\tilde{x}_o} \quad \text{s.t.} \quad \in \tilde{\theta}_A^* \le \frac{u\tilde{y}_j}{v\tilde{x}_j} \le \tilde{1}, \quad \forall j; \quad u, v \ge \varepsilon. \tag{17}$$

The following multi-objective fractional program is used to measure the fuzzy-bounded efficiency of each DMU after incorporating the corresponding fuzzy numbers into Model (17):

$$\begin{aligned}&\max/\min \left(\frac{uy_o^{\mathrm{L}}}{vx_o^{\mathrm{U}}}, \frac{uy_o^{\mathrm{M}}}{vx_o^{\mathrm{M}}}, \frac{uy_o^{\mathrm{U}}}{vx_o^{\mathrm{L}}}\right) \quad \text{s.t.} \quad \left(\theta_A^{\mathrm{L}*}, \theta_A^{\mathrm{M}*}, \theta_A^{\mathrm{U}*}\right) \le \\ &\left(\frac{uy_j^{\mathrm{L}}}{vx_j^{\mathrm{U}}}, \frac{uy_j^{\mathrm{M}}}{vx_j^{\mathrm{M}}}, \frac{uy_j^{\mathrm{U}}}{vx_j^{\mathrm{L}}}\right) \le (1, 1, 1), \quad \forall j; \quad u, v \ge \varepsilon. \end{aligned} \tag{18}$$

We similarly use GP to compute the lower-bound (minimum program) and the upper-bound (maximum program) of the fuzzy efficiency. Therefore, the maximization problem can be presented as follows:

$$\begin{aligned}&\min s_{\mathrm{L}}^- + s_{\mathrm{L}}^+ + s_{\mathrm{M}}^- + s_{\mathrm{M}}^+ + s_{\mathrm{U}}^- + s_{\mathrm{U}}^+ \quad \text{s.t.} \quad \frac{uy_j^{\mathrm{L}} + s_{\mathrm{L}}^+}{vx_j^{\mathrm{U}} - s_{\mathrm{L}}^-} = 1;\\ &\frac{uy_j^{\mathrm{M}} + s_{\mathrm{M}}^+}{vx_j^{\mathrm{M}} - s_{\mathrm{M}}^-} = 1; \quad \frac{uy_j^{\mathrm{U}} + s_{\mathrm{U}}^+}{vx_j^{\mathrm{L}} - s_{\mathrm{U}}^-} = 1; \qquad \text{con. } (I)\\ &\theta_{\mathrm{ADMU}}^{\mathrm{L}*} \le \frac{uy_j^{\mathrm{L}}}{vx_j^{\mathrm{U}}} \le 1, \quad \forall j; \quad \theta_{\mathrm{ADMU}}^{\mathrm{M}*} \le \frac{uy_j^{\mathrm{M}}}{vx_j^{\mathrm{M}}} \le 1, \quad \forall j; \quad \theta_{\mathrm{ADMU}}^{\mathrm{U}*} \le \frac{uy_j^{\mathrm{U}}}{vx_j^{\mathrm{L}}} \le 1,\\ &\quad \forall j; u, v \ge \varepsilon; \quad s_{\mathrm{L}}^-, s_{\mathrm{L}}^+, s_{\mathrm{M}}^-, s_{\mathrm{M}}^+, s_{\mathrm{U}}^-, s_{\mathrm{U}}^+ \ge 0. \end{aligned} \tag{19}$$

We then substitute $s_{\mathrm{L}}^- + s_{\mathrm{L}}^+$ for s_{L}, $s_{\mathrm{M}}^- + s_{\mathrm{M}}^+$ for s_{M} and $s_{\mathrm{U}}^- + s_{\mathrm{U}}^+$ for s_{U} and transform Model (19) into the LP Model (20) as follows:

$$\begin{aligned}
&\min s_{\mathrm{L}}+s_{\mathrm{M}}+s_{\mathrm{U}} \quad \text{s.t.} \quad u y_o^{\mathrm{L}}-v x_o^{\mathrm{U}}+s_{\mathrm{L}}=0; \quad u y_o^{\mathrm{M}}-v x_o^{\mathrm{M}}+s_{\mathrm{M}}=0;\\
&u y_o^{\mathrm{U}}-v x_o^{\mathrm{L}}+s_{\mathrm{U}}=0;\\
&u y_j^{\mathrm{L}}-v x_j^{\mathrm{U}} \leq 0, \quad \forall j; \quad u y_j^{\mathrm{M}}-v x_j^{\mathrm{M}} \leq 0, \quad \forall j; \quad u y_j^{\mathrm{U}}-v x_j^{\mathrm{L}} \leq 0, \quad \forall j;\\
&u y_j^{\mathrm{L}}-\theta_{\mathrm{ADMU}}^{\mathrm{L}*} v x_j^{\mathrm{U}} \geq 0, \quad \forall j;\\
&u y_j^{\mathrm{M}}-\theta_{\mathrm{ADMU}}^{\mathrm{M}*} v x_j^{\mathrm{M}} \geq 0, \quad \forall j; \quad u y_j^{\mathrm{U}}-\theta_{\mathrm{ADMU}}^{\mathrm{U}*} v x_j^{\mathrm{L}} \geq 0, \quad \forall j; \quad u, v \geq \varepsilon;\\
&s_{\mathrm{L}}, s_{\mathrm{M}}, s_{\mathrm{U}} \geq 0.
\end{aligned} \tag{20}$$

After we calculate the optimal solutions of Model (20), $(u^*, v^*, s_{\mathrm{L}}^*, s_{\mathrm{M}}^*, s_{\mathrm{U}}^*)$, we determine the upper-bound of the fuzzy efficiency for $DMU_j,\ j=1,\ldots,n$ by using (21) as follows:

$$\begin{aligned}
\tilde{\theta}_j^{\mathrm{U}*}=\left(\theta_j^{\mathrm{LU}*}, \theta_j^{\mathrm{MU}*}, \theta_j^{\mathrm{UU}*}\right) &=\left(\frac{u^* y_j^{\mathrm{L}}}{v^* x_j^{\mathrm{U}}}, \frac{u^* y_j^{\mathrm{M}}}{v^* x_j^{\mathrm{M}}}, \frac{u^* y_j^{\mathrm{U}}}{v^* x_j^{\mathrm{L}}}\right)\\
&=\left(1-\frac{s_{\mathrm{L}}^*}{v^* x_j^{\mathrm{U}}}, 1-\frac{s_{\mathrm{M}}^*}{v^* x_j^{\mathrm{M}}}, 1-\frac{s_{\mathrm{U}}^*}{v^* x_j^{\mathrm{L}}}\right).
\end{aligned} \tag{21}$$

The minimization problem is identical to the minimization Model (19) with the exception of the constraints represented by *con. (I)* in model (19), which are changed as follows:

$$\frac{u y_j^{\mathrm{L}}-s_{\mathrm{L}}^{+}}{v x_j^{\mathrm{U}}+s_{\mathrm{L}}^{-}}=\theta_{\mathrm{ADMU}}^{\mathrm{L}*}, \frac{u y_j^{\mathrm{M}}-s_{\mathrm{M}}^{+}}{v x_j^{\mathrm{M}}+s_{\mathrm{M}}^{-}}=\theta_{\mathrm{ADMU}}^{\mathrm{M}*}, \frac{u y_j^{\mathrm{U}}-s_{\mathrm{U}}^{+}}{v x_j^{\mathrm{L}}+s_{\mathrm{U}}^{-}}=\theta_{\mathrm{ADMU}}^{\mathrm{U}*},$$

and the corresponding constraints in model (20) are formed as follows:

$$\begin{aligned}
&u y_o^{\mathrm{L}}-\theta_{\mathrm{ADMU}}^{\mathrm{L}*} v x_o^{\mathrm{U}}-s_{\mathrm{L}}=0; \quad u y_o^{\mathrm{M}}-\theta_{\mathrm{ADMU}}^{\mathrm{M}*} v x_o^{\mathrm{M}}-s_{\mathrm{M}}=0;\\
&u y_o^{\mathrm{U}}-\theta_{\mathrm{ADMU}}^{\mathrm{U}*} v x_o^{\mathrm{L}}-s_{\mathrm{U}}=0.
\end{aligned}$$

We should note that the purpose of the minimization and maximization models is to minimize the total gaps needed for reaching the goal. The goal of the maximization model is to reach the 100 % efficiency score whereas the goal of the minimization model is to reach the efficiency score of the ADMU. Constraints *con. (I)* ensure that the required goals are achieved in both models.

When computing the optimal solutions, (22) is used to determine the lower-bound of the fuzzy efficiency for DMU_j, $j=1,\ldots,n$.

$$\begin{aligned}
\tilde{\theta}_j^{\mathrm{L}*} &=\left(\theta_j^{\mathrm{LL}*}, \theta_j^{\mathrm{ML}*}, \theta_j^{\mathrm{UL}*}\right)=\left(\frac{u^* y_j^{\mathrm{L}}}{v^* x_j^{\mathrm{U}}}, \frac{u^* y_j^{\mathrm{M}}}{v^* x_j^{\mathrm{M}}}, \frac{u^* y_j^{\mathrm{U}}}{v^* x_j^{\mathrm{L}}}\right)\\
&=\left(\theta_{\mathrm{ADMU}}^{\mathrm{L}*}+\frac{s_{\mathrm{L}}^*}{v^* x_j^{\mathrm{U}}}, \theta_{\mathrm{ADMU}}^{\mathrm{M}*}+\frac{s_{\mathrm{M}}^*}{v^* x_j^{\mathrm{M}}}, \theta_{\mathrm{ADMU}}^{\mathrm{U}*}+\frac{s_{\mathrm{U}}^*}{v^* x_j^{\mathrm{L}}}\right).
\end{aligned} \tag{22}$$

Next, we present the following definition for the interval efficiency $[\tilde{\theta}_j^{\mathrm{L}*}, \tilde{\theta}_j^{\mathrm{U}*}]$, calculated by the bounded FDEA models:

Definition 3.2 A DMU_j is called *FDEA* efficient if and only if $\theta_j^{\mathrm{UU}*} = 1$ using (21), and a DMU_j is called *FDEA* inefficient if and only if $\theta_j^{\mathrm{LL}*} = \theta_{\mathrm{ADMU}}^{\mathrm{L}*}$ using (22). Note that, if the DMU_j is neither FDEA efficient nor FDEA inefficient, then it is called *FDEA unspecified*.

The following theorem presents the relation between the fuzzy efficiency score $\tilde{\theta}_j = (\theta_j^{\mathrm{L}}, \theta_j^{\mathrm{M}}, \theta_j^{\mathrm{U}})$, the lower and upper bounds of the fuzzy efficiency score, denoted by $\tilde{\theta}_j^{\mathrm{L}*} = (\theta_j^{\mathrm{LL}*}, \theta_j^{\mathrm{ML}*}, \theta_j^{\mathrm{UL}*})$ and $\tilde{\theta}_j^{\mathrm{U}*} = (\theta_j^{\mathrm{LU}*}, \theta_j^{\mathrm{MU}*}, \theta_j^{\mathrm{UU}*})$, respectively.

DEA model propose an interval efficiency for each DMU.

Theorem 3.1 *If* $\tilde{\theta}_j^{\mathrm{U}*} = (\theta_j^{\mathrm{LU}*}, \theta_j^{\mathrm{MU}*}, \theta_j^{\mathrm{UU}*})$ *and* $\tilde{\theta}_j^{\mathrm{L}*} = (\theta_j^{\mathrm{LL}*}, \theta_j^{\mathrm{ML}*}, \theta_j^{\mathrm{UL}*})$ *are the solutions for* (19) *and* (22)*, respectively, then* $\tilde{\theta}_j^* \in \left[\tilde{\theta}_j^{\mathrm{L}*}, \tilde{\theta}_j^{\mathrm{U}*}\right]$, *where* $\tilde{\theta}_j^{\mathrm{L}*} = (\theta_j^{\mathrm{LL}*}, \theta_j^{\mathrm{ML}*}, \theta_j^{\mathrm{UL}*})$, $\tilde{\theta}_j^* = (\theta_j^{\mathrm{L}*}, \theta_j^{\mathrm{M}*}, \theta_j^{\mathrm{U}*})$*, and* $\tilde{\theta}_j^{\mathrm{U}*} = (\theta_j^{\mathrm{LU}*}, \theta_j^{\mathrm{MU}*}, \theta_j^{\mathrm{UU}*})$.

Proof Let us assume for a contradiction that $\tilde{\theta}_j^* \notin \left[\tilde{\theta}_j^{\mathrm{L}*}, \tilde{\theta}_j^{\mathrm{U}*}\right]$. Thereby, either (I) $\tilde{\theta}_j^* > \tilde{\theta}_j^{\mathrm{U}*}$ or (II) $\tilde{\theta}_j^* < \tilde{\theta}_j^{\mathrm{L}*}$ occurs:

(I) $\tilde{\theta}_j^* > \tilde{\theta}_j^{\mathrm{U}*}$: Let $[u^*, v^*]$ be the optimal solution set for (18) and $[\bar{u}, \bar{v}]$ be the optimal solution set for (17). Then, we have

$$\exists[\bar{u}, \bar{v}] : \quad \frac{u^* y_j^{\mathrm{L}}}{v^* x_j^{\mathrm{U}}} < \frac{\bar{u} y_j^{\mathrm{L}}}{\bar{v} x_j^{\mathrm{U}}} \le 1; \quad \frac{u^* y_j^{\mathrm{M}}}{v^* x_j^{\mathrm{M}}} < \frac{\bar{u} y_j^{\mathrm{M}}}{\bar{v} x_j^{\mathrm{M}}} \le 1; \quad \frac{u^* y_j^{\mathrm{U}}}{v^* x_j^{\mathrm{L}}} < \frac{\bar{u} y_j^{\mathrm{U}}}{\bar{v} x_j^{\mathrm{L}}} \le 1,$$

which can be simplified as follows:

$$\begin{aligned} &u^* y_j^{\mathrm{L}} - v^* x_j^{\mathrm{U}} < \bar{u} y_j^{\mathrm{L}} - \bar{v} x_j^{\mathrm{U}} \le 0; \quad u^* y_j^{\mathrm{M}} - v^* x_j^{\mathrm{M}} < \bar{u} y_j^{\mathrm{M}} - \bar{v} x_j^{\mathrm{M}} \le 0; \\ &u^* y_j^{\mathrm{U}} - v^* x_j^{\mathrm{L}} < \bar{u} y_j^{\mathrm{U}} - \bar{v} x_j^{\mathrm{L}} \le 0, \end{aligned}$$

The inequalities above are transformed into the following equalities using the slacks

$$\begin{aligned} &u^* y_j^{\mathrm{L}} - v^* x_j^{\mathrm{U}} + s_{\mathrm{L}}^* = 0, \quad \bar{u} y_j^{\mathrm{L}} - \bar{v} x_j^{\mathrm{U}} + \bar{s}_{\mathrm{L}} = 0 \Rightarrow \quad 0 \le \bar{s}_{\mathrm{L}} < s_{\mathrm{L}}^*, \\ &u^* y_j^{\mathrm{M}} - v^* x_j^{\mathrm{M}} + s_{\mathrm{M}}^* = 0, \quad \bar{u} y_j^{\mathrm{M}} - \bar{v} x_j^{\mathrm{M}} + \bar{s}_{\mathrm{M}} = 0 \Rightarrow \quad 0 \le \bar{s}_{\mathrm{M}} < s_{\mathrm{M}}^*, \\ &u^* y_j^{\mathrm{U}} - v^* x_j^{\mathrm{L}} + s_{\mathrm{U}}^* = 0, \quad \bar{u} y_j^{\mathrm{U}} - \bar{v} x_j^{\mathrm{L}} + \bar{s}_{\mathrm{U}} = 0 \Rightarrow \quad 0 \le \bar{s}_{\mathrm{U}} < s_{\mathrm{U}}^*. \end{aligned}$$

We thus get $0 \le \bar{s}_t < s_t^*$ $t = \mathrm{L}, \mathrm{M}, \mathrm{U}$, which implies that $(u^*, v^*, s_{\mathrm{L}}^*, s_{\mathrm{M}}^*, s_{\mathrm{U}}^*)$ is not the optimal solution. This contradiction implies that $\tilde{\theta}_j^* < \tilde{\theta}_j^{\mathrm{U}*}$.

(II) $\tilde{\theta}_j^* < \tilde{\theta}_j^{L*}$: Suppose $[u^*, v^*]$ is the optimal solution set to model (18) and $[\bar{u}, \bar{v}]$ is the optimal solution set to model (17). Then, we have

$$\exists\,[\bar{u},\bar{v}] : \quad \theta_{\mathrm{ADMU}}^{L*} \le \frac{\bar{u}y_j^{L}}{\bar{v}x_j^{U}} < \frac{u^*y_j^{L}}{v^*x_j^{U}}; \quad \theta_{\mathrm{ADMU}}^{M*} \le \frac{\bar{u}y_j^{M}}{\bar{v}x_j^{M}} < \frac{u^*y_j^{M}}{v^*x_j^{M}}; \\ \theta_{\mathrm{ADMU}}^{U*} \le \frac{\bar{u}y_j^{U}}{\bar{v}x_j^{L}} < \frac{u^*y_j^{U}}{v^*x_j^{L}},$$

which can be simplified as follows::

$$\begin{aligned} u^*y_j^{L} - \theta_{\mathrm{ADMU}}^{L*}v^*x_j^{U} &> \bar{u}y_j^{L} - \theta_{\mathrm{ADMU}}^{L*}\bar{v}x_j^{U} \ge 0;\\ u^*y_j^{M} - \theta_{\mathrm{ADMU}}^{M*}v^*x_j^{M} &> \bar{u}y_j^{M} - \theta_{\mathrm{ADMU}}^{M*}\bar{v}x_j^{M} \ge 0;\\ u^*y_j^{U} - \theta_{ADMU}^{M*}v^*x_j^{L} &> \bar{u}y_j^{U} - \theta_{\mathrm{ADMU}}^{M*}\bar{v}x_j^{L} \ge 0. \end{aligned}$$

The inequalities above are converted into the following equalities using slacks:

$$\begin{aligned} &u^*y_j^{L} - \theta_{\mathrm{ADMU}}^{L*}v^*x_j^{U} - s_{L}^* = 0, \quad \bar{u}y_j^{L} - \theta_{\mathrm{ADMU}}^{L*}\bar{v}x_j^{U} - \bar{s}_{L} = 0 \Rightarrow \quad \bar{s}_{L} < s_{L}^*\\ &u^*y_j^{M} - \theta_{\mathrm{ADMU}}^{M*}v^*x_j^{M} - s_{M}^* = 0, \quad \bar{u}y_j^{M} - \theta_{\mathrm{ADMU}}^{M*}\bar{v}x_j^{M} - \bar{s}_{M} = 0 \Rightarrow \bar{s}_{M} < s_{M}^*\\ &u^*y_j^{U} - \theta_{\mathrm{ADMU}}^{M*}v^*x_j^{L} - s_{U}^* = 0, \quad \bar{u}y_j^{U} - \theta_{\mathrm{ADMU}}^{M*}\bar{v}x_j^{L} - \bar{s}_{U} = 0 \Rightarrow \quad \bar{s}_{U} < s_{U}^*. \end{aligned}$$

We thus obtain $0 \le \bar{s}_t < s_t^* \quad t = \mathrm{L, M, U}$, which implies that $(u^*, v^*, s_{L}^*, s_{M}^*, s_{U}^*)$ are not the optimal solution. This contradiction implies that $\tilde{\theta}_j^* > \tilde{\theta}_j^{L*}$. From (I) and (II), the proof is complete. □

The following corollary allows us to introduce a new ranking approach in next section and rank the interval efficiency scores with the lower and upper fuzzy limits.

Corollary 3.1 *If* $\tilde{\theta}_j^* \in \left[\tilde{\theta}_j^{L*}, \tilde{\theta}_j^{U*}\right]$, *then* $\theta_j^{L*} \in \left[\theta_j^{LL*}, \theta_j^{LU*}\right]$, $\theta_j^{M*} \in \left[\theta_j^{ML*}, \theta_j^{MU*}\right]$, *and* $\theta_j^{U*} \in \left[\theta_j^{UL*}, \theta_j^{UU*}\right]$.

Proof The proof is obvious with respect to the feasible region of Model (18). □

4 Ranking of the Interval Data

The DEA model proposed in the previous section produces an interval efficiency for each DMU. The goal here is to identify the most efficient DMUs. The ranking of a set of interval numbers is the challenging issue that has not received a great deal of attention in the literature due to the lack of a unified approach. In this section, we present a two-step ranking method that can be used to rank the interval efficiency scores.

Let $A_j = [a_j^{L}, a_j^{R}],\ j = 1, \ldots, n$ be the interval efficiencies of n DMUs. The two steps of the ranking method are succinctly described as follows:

Step 1 A virtual interval $A^* = [a^{\mathrm{L}}_{\max}, a^{\mathrm{R}}_{\max}]$ (the so-called ideal interval) is defined as $a^{\mathrm{L}}_{\max} = \max_j \{a^{\mathrm{L}}_j\}$ and $a^{\mathrm{R}}_{\max} = \max_j \{a^{\mathrm{R}}_j\}$.

Step 2 Assume that α $(0 \le \alpha \le 1)$ is the decision maker's level of optimism. We define the following indicator to measure the distance between the ideal interval and the *i*th interval efficiency:

$$d(A^*, A_j) = (\alpha a^{\mathrm{R}}_{\max} + (1-\alpha) a^{\mathrm{L}}_{\max}) - (\alpha a^{\mathrm{R}}_j + (1-\alpha) a^{\mathrm{L}}_j).$$

The DMUs are ultimately identified in descending order for a given α, according to the value of the $d(A^*, A_j)$ indicating the most preferred and the least preferred units. Consider the following example with three interval efficiency scores $A_1 = [0.2, 0.8]$, $A_2 = [0.1, 0.5]$, and $A_3 = [0.3, 0.6]$. In Step 1, the ideal interval is computed as $A^* = [a^{\mathrm{L}}_{\max}, a^{\mathrm{R}}_{\max}] = [0.3, 0.8]$, where $a^{\mathrm{L}}_{\max} = \max\{0.2, 0.1, 0.3\} = 0.3$ and $a^{\mathrm{R}}_{\max} = \max\{0.8, 0.5, 0.6\} = 0.8$. In step 2, the distance between the ideal interval and each interval efficiency is calculated as follows:

$$\begin{aligned} d(A^*, A_1) &= (\alpha(0.8) + (1-\alpha)(0.3)) - (\alpha(0.8) + (1-\alpha)(0.2)),\\ d(A^*, A_2) &= (\alpha(0.8) + (1-\alpha)(0.3)) - (\alpha(0.5) + (1-\alpha)(0.1)),\\ d(A^*, A_3) &= (\alpha(0.8) + (1-\alpha)(0.3)) - (\alpha(0.6) + (1-\alpha)(0.3)), \end{aligned}$$

where $\alpha \in [0, 1]$. If $\alpha = 0.5$, then $d(A^*, A_1) = 0.05$, $d(A^*, A_2) = 0.25$, and $d(A^*, A_3) = 0.10$. A_1 with the smallest distance from the ideal point is ranked first, and A_3 and A_2 are ranked second and third, respectively.

Theorem 4.1 *Let $A_j = [a^{\mathrm{L}}_j, a^{\mathrm{R}}_j] (j = 1, \ldots, n)$ be a set of interval efficiencies. For a given level of optimism,α , if there is a ranking order $A_{j_1} \succ A_{j_2} \succ \ldots \succ A_{j_n}$[1], then there exists a robust interval for the level of optimism as $[\alpha^{\mathrm{L}}, \alpha^{\mathrm{R}}] \cap [0, 1]$, where:*

$$\alpha^{\mathrm{L}} = \max_t \left\{ \frac{a^{\mathrm{L}}_{j_{t+1}} - a^{\mathrm{L}}_{j_t}}{\left|\left(a^{\mathrm{R}}_{j_{t+1}} - a^{\mathrm{L}}_{j_{t+1}}\right) - \left(a^{\mathrm{R}}_{j_t} - a^{\mathrm{L}}_{j_t}\right)\right|} \,|\, \left(a^{\mathrm{R}}_{j_{t+1}} - a^{\mathrm{L}}_{j_{t+1}}\right) - \left(a^{\mathrm{R}}_{j_t} - a^{\mathrm{L}}_{j_t}\right) < 0 \right\},$$

$$\alpha^{\mathrm{R}} = \min_t \left\{ \frac{a^{\mathrm{L}}_{j_t} - a^{\mathrm{L}}_{j_{t+1}}}{\left|\left(a^{\mathrm{R}}_{j_{t+1}} - a^{\mathrm{L}}_{j_{t+1}}\right) - \left(a^{\mathrm{R}}_{j_t} - a^{\mathrm{L}}_{j_t}\right)\right|} \,|\, \left(a^{\mathrm{R}}_{j_{t+1}} - a^{\mathrm{L}}_{j_{t+1}}\right) - \left(a^{\mathrm{R}}_{j_t} - a^{\mathrm{L}}_{j_t}\right) > 0 \right\}.$$

In other words, the ranking order of DMUs remains unchanged when α varies within the $[\alpha^{\mathrm{L}}, \alpha^{\mathrm{R}}]$ range.

Proof Using the proposed index value, the ranking $A_{j_1} \succ A_{j_2} \succ \ldots \succ A_{j_n}$ can be expressed as follows:

[1] The subscripts 1,2,…,n present the ranking number of DMU_i

$d(A^*, A_{j_1}) < d(A^*, A_{j_2}) < \ldots < d(A^*, A_{j_n})$ or $d(A^*, A_{j_t}) - d(A^*, A_{j_{t+1}}) < 0\, t = 1, \ldots, n-1$, that is $\left[\alpha\left(a^{\mathrm{R}}_{\max} - a^{\mathrm{R}}_{j_t}\right) + (1-\alpha)\left(a^{\mathrm{L}}_{\max} - a^{\mathrm{L}}_{j_t}\right)\right] - \left[\alpha\left(a^{\mathrm{R}}_{\max} - a^{\mathrm{R}}_{j_{t+1}}\right) + (1-\alpha)\left(a^{\mathrm{L}}_{\max} - a^{\mathrm{L}}_{j_{t+1}}\right)\right] < 0,\ t = 1, \ldots, n-1,$

$$\alpha\left[\left(a^{\mathrm{R}}_{j_{t+1}} - a^{\mathrm{R}}_{i_t}\right) - \left(a^{\mathrm{L}}_{j_{t+1}} - a^{\mathrm{L}}_{j_t}\right)\right] + \left(a^{\mathrm{L}}_{j_{t+1}} - a^{\mathrm{L}}_{j_t}\right) < 0, \quad t = 1, \ldots, n-1.$$

It is obvious that, if $\left(a^{\mathrm{R}}_{j_{t+1}} - a^{\mathrm{R}}_{i_t}\right) - \left(a^{\mathrm{L}}_{j_{t+1}} - a^{\mathrm{L}}_{j_t}\right) > 0$, then $\alpha < \frac{a^{\mathrm{L}}_{j_t} - a^{\mathrm{L}}_{j_{t+1}}}{\left|\left(a^{\mathrm{R}}_{j_{t+1}} - a^{\mathrm{L}}_{j_{t+1}}\right) - \left(a^{\mathrm{R}}_{j_t} - a^{\mathrm{L}}_{j_t}\right)\right|}$, $t = 1, \ldots, n-1$, otherwise $\alpha > \frac{a^{\mathrm{L}}_{j_{t+1}} - a^{\mathrm{L}}_{j_t}}{\left|\left(a^{\mathrm{R}}_{j_{t+1}} - a^{\mathrm{L}}_{j_{t+1}}\right) - \left(a^{\mathrm{R}}_{j_t} - a^{\mathrm{L}}_{j_t}\right)\right|}$, $t = 1, \ldots, n-1$.

These two inequalities can be further written as follows:

$$\alpha > \alpha^{\mathrm{L}} = \max_t \left\{ \frac{a^{\mathrm{L}}_{j_{t+1}} - a^{\mathrm{L}}_{j_t}}{\left|\left(a^{\mathrm{R}}_{j_{t+1}} - a^{\mathrm{L}}_{j_{t+1}}\right) - \left(a^{\mathrm{R}}_{j_t} - a^{\mathrm{L}}_{j_t}\right)\right|} \,\middle|\, \left(a^{\mathrm{R}}_{j_{t+1}} - a^{\mathrm{L}}_{j_{t+1}}\right) - \left(a^{\mathrm{R}}_{j_t} - a^{\mathrm{L}}_{j_t}\right) < 0 \right\},$$

$$\alpha < \alpha^{\mathrm{R}} = \min_t \left\{ \frac{a^{\mathrm{L}}_{j_t} - a^{\mathrm{L}}_{j_{t+1}}}{\left|\left(a^{\mathrm{R}}_{j_{t+1}} - a^{\mathrm{L}}_{j_{t+1}}\right) - \left(a^{\mathrm{R}}_{j_t} - a^{\mathrm{L}}_{j_t}\right)\right|} \,\middle|\, (a^{\mathrm{R}}_{j_{t+1}} - a^{\mathrm{L}}_{j_{t+1}}) - (a^{\mathrm{R}}_{j_t} - a^{\mathrm{L}}_{j_t}) > 0 \right\}.$$

The final interval for α can be expressed as $[\alpha^{\mathrm{L}}, \alpha^{\mathrm{R}}] \cap [0, 1]$ since α can only take values between 0 and 1. Therefore, the proof is accomplished. □

Let us apply Theorem 4.1 in the above example to find a robust interval for the α level as $\alpha^{\mathrm{L}} = \max\left\{\frac{0.3-0.2}{|0.3-0.6|}\right\} = 0.333$ and $\alpha^{\mathrm{R}} = \min\left\{\frac{0.3-0.1}{|0.4-0.3|}\right\} = 2$.

Therefore, a robust interval is $[\alpha^{\mathrm{L}}, \alpha^{\mathrm{R}}] = [0.33, 2] \cap [0, 1] = [0.33, 1]$, where the ranking in this interval is identical. Theorem 4.1 implicitly provides the decision maker with more insightful information on his/her level of the optimism.

5 Case Study

The Semicon Technologies,[2] the largest semiconductor company in Northern California, is concerned with the safety issues at their production facilities in eight cities (i.e., Dalian, Guangzhou, Ningbo, Qingdao, Shenzhen, Suzhou, Tianjin, and Yantai) throughout China. Semicon management was interested in measuring the safety performance of the eight facilities in China. They formed a safety assessment team, which included five managers and supervisors from their Northern California plant. The team

[2] The name of the company has been changed to protect their anonymity

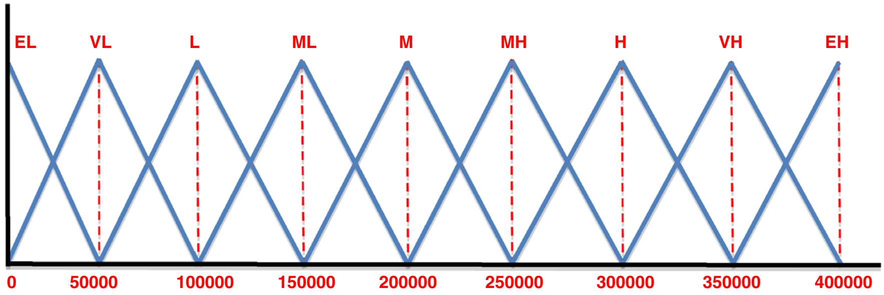

Fig. 1 Linguistic variables used for the fuzzy inputs

Table 1 Linguistic variables used to express the fuzzy inputs

Linguistic variable	Triangular fuzzy number
Extremely low (EL)	(0, 0, 50000)
Very low (VL)	(0, 50000, 100000)
Low (L)	(50000, 100000, 150000)
Medium low (ML)	(100000, 150000, 200000)
Medium (M)	(150000, 200000, 250000)
Medium high (MH)	(200000, 250000, 300000)
High (H)	(250000, 300000, 350000)
Very high (VH)	(300000, 350000, 400000)
Extremely high (EH)	(350000, 400000, 450000)

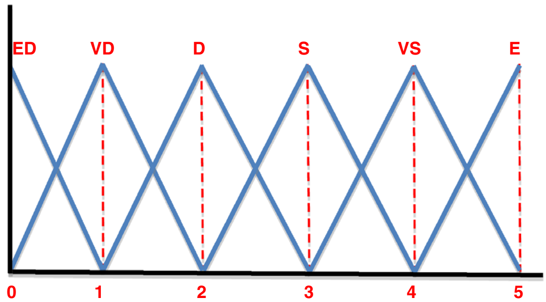

Fig. 2 Linguistic variables used for the fuzzy output

worked with two consultants and devised a performance measurement system with two outputs and two inputs. The outputs included one deterministic output of "savings from decrease in the number of non-fatal and fatal accidents" (Output 1) and one fuzzy output of "employee satisfaction score" (Output 2). The inputs included one input of "employee-related safety measurement costs" (Input 1) and one input of "equipment-related safety measurement costs" (Input 2). Fuzzy parameters were used for both inputs and output 2 due to the incompleteness of data or lack of precise information. Figure 1 and Table 1 were used to assign fuzzy triangular numbers to the fuzzy inputs, and Fig. 2 and Table 2 were used to assign fuzzy triangular numbers to the fuzzy output 2. The input and output data used in this study are presented in Tables 3 and 4.

Table 2 Linguistic variables used to express the fuzzy output

Linguistic variable	Fuzzy number
Extremely dissatisfied (ED)	(0, 0, 1)
Very dissatisfied (VD)	(0, 1, 2)
Dissatisfied (D)	(1, 2, 3)
Satisfied (S)	(2, 3, 4)
Very satisfied (VS)	(3, 4, 5)
Extremely satisfied (ES)	(4, 5, 5)

Table 3 Input and output data for the eight production facilities

DMU	Employee-related safety measurement costs (Input 1)	Equipment-related safety measurement costs (Input 2)	Savings from decrease in the number of non-fatal and fatal accidents (Output 1)	Employee satisfaction score (Output 2)
Dalian	VL	MH	\$610000	VS
Guangzhou	L	H	\$690000	D
Ningbo	MH	ML	\$720000	D
Qingdao	M	L	\$630000	VS
Shenzhen	VL	M	\$560000	ES
Suzhou	M	ML	\$640000	S
Tianjin	EH	VH	\$880000	ED
Yantai	MH	H	\$830000	VD

We implemented the following three-step procedure to calculate the fuzzy-bounded efficiencies at the eight Semicon production facilities in China:

Step 1 Obtain the fuzzy ADMU of the eight production facilities using Definition 3.1 (see the last row of Table 4).

Step 2 Identify the best relative efficiency among the fuzzy ADMUs using Model (15) and (16). The resulting objective function values ($\tilde{\theta}^*_{\text{ADMU}}$) for this example is $\tilde{\theta}^*_{\text{ADMU}} = (\theta^{\text{L}*}_{\text{ADMU}}, \theta^{\text{M}*}_{\text{ADMU}}, \theta^{\text{U}*}_{\text{ADMU}}) = (0.2328, 0.2581, 0.2914)$.

Step 3 Calculate the upper-bound of the bounded fuzzy efficiency using (21) and the lower-bound of the bounded fuzzy efficiency using (22). The results are presented in the second and third columns of Table 5.

Step 4 Rank the eight production facilities using the proposed approach.

As shown in Table 5, the facility in Tianjin was identified as an inefficient facility (based on Definition 3.2) because $\theta^{\text{LL}*}_{\text{Tianjin}}$ in the lower-bound of efficiency was equal to $\theta^{\text{L}*}_{\text{ADMU}} = 0.2328$. In addition, $\theta^{\text{UU}*}_{\text{Qingdao}}$ and $\theta^{\text{UU}*}_{\text{Shenzhen}}$ were equal to one in the upper-bound of efficiency. Therefore, the facilities in Qingdao and Shenzhen were identified as efficient facilities (based on Definition 3.2).

As indicated earlier, we have $\left(\theta^{\text{LL}*}_j, \theta^{\text{ML}*}_j, \theta^{\text{UL}*}_j\right) \leq \left(\theta^{\text{L}*}_j, \theta^{\text{M}*}_j, \theta^{\text{U}*}_j\right) \leq \left(\theta^{\text{LU}*}_j, \theta^{\text{MU}*}_j, \theta^{\text{UU}*}_j\right)$, based on the proposed bounded fuzzy DEA models. This

Table 4 Input and output data for the eight production facilities represented by fuzzy numbers

DMU	Expected annual employee-related safety measurement costs (Input 1)	Expected annual equipment-related safety measurement costs (Input 2)	Annual discount in workman compensation insurance due to decrease in the number of non-fatal and fatal accidents (Output 1)	Employee satisfaction score (Output 2)
Dalian	(0, 50000, 100000)	(200000, 250000, 300000)	610000	(3,4,5)
Guangzhou	(50000, 100000, 150000)	(250000, 300000, 350000)	690000	(1,2,3)
Ningbo	(200000, 250000, 300000)	(100000, 150000, 200000)	720000	(1,2,3)
Qingdao	(150000, 200000, 250000)	(50000, 100000, 150000)	630000	(3,4,5)
Shenzhen	(0, 50000, 100000)	(150000, 200000, 250000)	560000	(4,5,5)
Suzhou	(150000, 200000, 250000)	(100000, 150000, 200000)	640000	(2,3,4)
Tianjin	(350000, 400000, 450000)	(300000, 350000, 400000)	880000	(0,0,1)
Yantai	(200000, 250000, 300000)	(250000, 300000, 350000)	830000	(0,1,2)
Fuzzy anti-ideal	(350000, 400000, 450000)	(300000, 350000, 400000)	560000	(0,0,1)

Table 5 Lower and upper-bounds fuzzy efficiencies

DMU	Lower-bounds $(\theta_j^{LL^*}, \theta_j^{ML^*}, \theta_j^{UL^*})$	Upper-bounds $(\theta_j^{LU^*}, \theta_j^{MU^*}, \theta_j^{UU^*})$
Dalian	(0.2493, 0.3060, 0.3973)	(0.4630, 0.5903, 0.8292)
Guangzhou	(0.2377, 0.2817, 0.3460)	(0.4270, 0.5163, 0.6599)
Ningbo	(0.3160, 0.3919, 0.5164)	(0.4627, 0.5712, 0.7555)
Qingdao	(0.3430, 0.4520, 0.6635)	(0.5045, 0.6656, 1.0000)
Shenzhen	(0.2696, 0.3452, 0.4827)	(0.4855, 0.6480, 1.0000)
Suzhou	(0.3159, 0.4038, 0.5602)	(0.4545, 0.5741, 0.7922)
Tianjin	(0.2328, 0.2634, 0.3034)	(0.3412, 0.3812, 0.4336)
Yantai	(0.2612, 0.3051, 0.3670)	(0.4072, 0.4724, 0.5660)

implies $\theta_j^{L*} \in \left[\theta_j^{LL*}, \theta_j^{LU*}\right]$, $\theta_j^{M*} \in \left[\theta_j^{ML*}, \theta_j^{MU*}\right]$, and $\theta_j^{U*} \in \left[\theta_j^{UL*}, \theta_j^{UU*}\right]$, representing the variations interval of $\theta_j^{L^*}$, $\theta_j^{M^*}$, and $\theta_j^{U^*}$, respectively. Given the results of the bounded fuzzy efficiency of the eight production facilities reported in Table 5, we obtained the variations interval for each part of the triangular fuzzy efficiencies as presented in Table 6.

Table 6 Interval efficiencies

DMU	$\left[\theta_j^{LL*}, \theta_j^{LU*}\right]$	$\left[\theta_j^{ML*}, \theta_j^{MU*}\right]$	$\left[\theta_j^{UL*}, \theta_j^{UU*}\right]$
Dalian	[0.2493, 0.4630]	[0.3060, 0.5903]	[0.3973, 0.8292]
Guangzhou	[0.2377, 0.4270]	[0.2817, 0.5163]	[0.3460, 0.6599]
Ningbo	[0.3160, 0.4627]	[0.3919, 0.5712]	[0.5164, 0.7555]
Qingdao	[0.3430, 0.5045]	[0.4520, 0.6656]	[0.6635, 1.0000]
Shenzhen	[0.2696, 0.4855]	[0.3452, 0.6480]	[0.4827, 1.0000]
Suzhou	[0.3159, 0.4545]	[0.4038, 0.5741]	[0.5602, 0.7922]
Tianjin	[0.2328, 0.3412]	[0.2634, 0.3812]	[0.3034, 0.4336]
Yantai	[0.2612, 0.4072]	[0.3051, 0.4724]	[0.3670, 0.5660]
Ideal interval	[0.3430, 0.5045]	[0.4520, 0.6656]	[0.6635, 1.0000]

Finally, we used the approach proposed in Sect. 4 to rank the variation intervals of the efficiencies for the eight production facilities. We take three different $\alpha \in \{0.3, 0.6, 0.9\}$ into consideration to apply our ranking method in accordance to the decision maker's preferences. The corresponding index for each facility and the ranking order the facilities are reported in Table 7. As shown in this table, the facilities in Qingdao and Tianjin were ranked first and eight in safety, respectively, in all three levels. However, the facility in Dalian was ranked fifth in the two levels of $\alpha = 0.3$ and $\alpha = 0.6$, and either third or fourth in the $\alpha = 0.9$ level. The safety ranking of the facility in Guangzhou was sixth in the $\alpha = 0.6$ and $\alpha = 0.9$ levels, and either sixth or seventh in the $\alpha = 0.3$ level. The facility in Ningbo was ranked either second or fourth in the $\alpha = 0.3$ and $\alpha = 0.6$ levels and either third or fifth in the $\alpha = 0.9$ level. The facility in Shenzhen was ranked second in the $\alpha = 0.9$ level, either second or fourth in the $\alpha = 0.3$ level, and either second or third in the $\alpha = 0.6$ level. The facility in Suzhou was ranked either second or third in the $\alpha = 0.3$ level, either third or fourth in the $\alpha = 0.6$ level, and either fourth or fifth in the $\alpha = 0.9$ level. Finally, the facility in Yantai was ranked seventh in the $\alpha = 0.6$ and $\alpha = 0.9$ levels and either sixth or seventh in the $\alpha = 0.3$ level. In other words, the facility in Guangzhou was ranked seventh sometimes, and the facility in Yantai was ranked seventh other times.

6 Conclusions

In most real-world performance evaluation problems, some data are either vague or not known precisely. The conventional DEA lacks the flexibility to deal with imprecise or vague data. These imprecise and vague data can be represented by linguistic terms characterized by fuzzy numbers to reflect the decision makers' intuition and subjective judgments. In addition, it is often very expensive to collect precise data for efficiency analysis. Consequently, there is a strong need for developing cost effective methodologies for capturing vagueness and imprecision in performance and efficiency analysis. In this paper, we extend the fuzzy-bounded efficiency approach by using the common set of weights approach. The approach is linked to conventional productivity

Table 7 DMU indicators and rankings

DMU	$\alpha = 0.3$			$\alpha = 0.6$			$\alpha = 0.9$		
	$\left[\theta_j^{LL^*}, \theta_j^{LU^*}\right]$	$\left[\theta_j^{ML^*}, \theta_j^{MU^*}\right]$	$\left[\theta_j^{UL^*}, \theta_j^{UU^*}\right]$	$\left[\theta_j^{LL^*}, \theta_j^{LU^*}\right]$	$\left[\theta_j^{ML^*}, \theta_j^{MU^*}\right]$	$\left[\theta_j^{ML^*}, \theta_j^{MU^*}\right]$	$\left[\theta_j^{LL^*}, \theta_j^{LU^*}\right]$	$\left[\theta_j^{ML^*}, \theta_j^{MU^*}\right]$	$\left[\theta_j^{UL^*}, \theta_j^{UU^*}\right]$
Dalian	0.0781(5)	0.1247(5)	0.2376(5)	0.0624(5)	0.1035(5)	0.2090(5)	0.0468(4)	0.0823(3)	0.1804(3)
Guangzhou	0.0969(7)	0.1640(7)	0.3242(6)	0.0886(6)	0.1577(6)	0.3310(6)	0.0803(6)	0.1514(6)	0.3378(6)
Ningbo	0.0315(2)	0.0703(3)	0.1763(4)	0.0359(2)	0.0806(4)	0.2055(4)	0.0403(3)	0.0909(5)	0.2347(5)
Qingdao	0.0000(1)	0.0000(1)	0.0000(1)	0.0000(1)	0.0000(1)	0.0000(1)	0.0000(1)	0.0000(1)	0.0000(1)
Shenzhen	0.0571(4)	0.0800(4)	0.1266(2)	0.0408(3)	0.0532(2)	0.0723(2)	0.0245(2)	0.0265(2)	0.0181(2)
Suzhou	0.0340(3)	0.0611(2)	0.1347(3)	0.0409(4)	0.0741(3)	0.1660(3)	0.0478(5)	0.0871(4)	0.1974(4)
Tianjin	0.1261(8)	0.2172(8)	0.4220(8)	0.1420(8)	0.2460(8)	0.4839(8)	0.1580(8)	0.2748(8)	0.5458(8)
Yantai	0.0864(6)	0.1607(6)	0.3378(7)	0.0911(7)	0.1747(7)	0.3790(7)	0.0957(7)	0.1886(7)	0.4203(7)

theory through the derivation of an efficient fuzzy frontier, as well as an inefficiency fuzzy frontier. In addition, we rank the DMUs by using a new method for ranking the interval data. We leave for further work the economic interpretation and decomposition of such scores, as well as the definition of various technical assumptions on the fuzzy production possibility set.

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