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Fuzzy stochastic data envelopment analysis with application to base realignment and closure (BRAC)

Madjid Tavana [a,*], Rashed Khanjani Shiraz [b], Adel Hatami-Marbini [c], Per J. Agrell [c], Khalil Paryab [b]

[a] Business Systems and Analytics, Lindback Distinguished Chair of Information Systems and Decision Sciences, La Salle University, Philadelphia, PA 19141, USA
[b] School of Mathematics, Iran University of Science and Technology, Tehran, Iran
[c] Louvain School of Management, Center of Operations Research and Econometrics (CORE), Université catholique de Louvain, L1.03.01, B-1348 Louvain-la-Neuve, Belgium

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ABSTRACT

Data envelopment analysis (DEA) is a non-parametric method for evaluating the relative efficiency of decision-making units (DMUs) on the basis of multiple inputs and outputs. Conventional DEA models assume that inputs and outputs are measured by exact values on a ratio scale. However, the observed values of the input and output data in real-world problems are often vague or random. Indeed, decision makers (DMs) may encounter a hybrid uncertain environment where fuzziness and randomness coexist in a problem. Several researchers have proposed various fuzzy methods for dealing with the ambiguous and random data in DEA. In this paper, we propose three fuzzy DEA models with respect to probability-possibility, probability-necessity and probability-credibility constraints. In addition to addressing the possibility, necessity and credibility constraints in the DEA model we also consider the probability constraints. A case study for the base realignment and closure (BRAC) decision process at the U.S. Department of Defense (DoD) is presented to illustrate the features and the applicability of the proposed models.

1. Introduction

Data envelopment analysis (DEA) is a widely used mathematical programming approach for comparing the inputs and outputs of a set of homogenous decision making units (DMUs) by evaluating their relative efficiency. A DMU is considered efficient when no other DMUs can produce more outputs using an equal or lesser amount of input. DEA generalizes the intuitive single-input single-output ratio efficiency measurement into a multiple-input multiple-output model by using a ratio of the weighted sum of outputs to the weighted sum of inputs. Charnes, Cooper, and Rhodes (1978) introduced the constant returns to scale (CRS) DEA model, also known as the Charnes, Cooper and Rhodes (CCR) model, the origins of which are found already in the work of Farrell (1957) The conventional DEA evaluation process is entirely based on access to well-defined, precise and deterministic data for the production set. However, the input and output data in real-world problems are often fuzzy and random.

In this paper, we propose three new DEA models (i.e., the probability-possiblity, probability-necessity and probability-credibility approaches) for solving CCR models in which the input and output data are assumed to be characterized by fuzzy random variables. We accomplish this task by converting the non-linear models formulated in these approaches to quadratic programming models. A case study for base realignment and closure (BRAC) decision by the U.S. Department of Defense (DoD) is presented to illustrate the features and applicability of the proposed DEA models. The United States Government adopted BRAC to resolve the military, socio-economic and political issue of excess base capacity. The DEA models presented in this serve to inform the members of the BRAC Commission in clarifiying their objectives and reducing the environmental complexities inherent in the base realignment and closure decisions. The BRAC Commission may utilize the proposed models to arrive at a ranking of the military bases on the DoD hit list. Notwithstanding, BRAC is a complex problem requiring communication and negotiation within various branches of government and with the public. Our models are intended to create an even playing field for pursuing consensus and should not be interpreted as a recommendation to adopt a deterministic or mechanistic approach to the BRAC process.

The remainder of the paper is organized as follows. In the next section, we review the relevant literature on fuzzy DEA. We then present some preliminaries in Section 3 and the convetional DEA-CCR model in Section 4. We present the mathematical details of the proposed models in Section 5. In Section 6, we present the results of the case conducted for the BRAC commission to evaluate the efficiency of 40 military bases. Section 7 presents our conclusions and future research directions.

* Corresponding author. Tel.: +1 215 951 1129; fax: +1 267 295 2854.
E-mail addresses: tavana@lasalle.edu (M. Tavana), Khanjani@iust.ac.ir (R. Khanjani Shiraz), adel.hatamimarbini@uclouvain.be (A. Hatami-Marbini), per.agrell@uclouvain.be (P.J. Agrell), Paryab@iust.ac.ir (K. Paryab).
URL: http://tavana.us (M. Tavana).

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2. Literature review

Efficiency and productivity measurement in organizations has received a great deal of attention both in research and in practice. DEA utilizes linear programming (LP) to measure the relative efficiency of the DMUs without requiring a specified functional form. Various fuzzy methods have been proposed to solve DEA problems with fuzzy variables. In a recent study, Hatami-Marbini, Emrouznejad, and Tavana (2011a) classified the fuzzy DEA methods in the literature into five general groups: (1) the tolerance approach (e.g. Sengupta, 1992), (2) the α-level based approach (e.g. Hatami-Marbini, Saati, & Tavana, 2010; Kao & Liu, 2000, 2003), (3) the fuzzy ranking approach (e.g. Guo & Tanaka, 2001; Hatami-Marbini, Tavana, & Ebrahimi, 2011b; León, Liern, Ruiz, & Sirvent, 2003), (4) the possibility approach (e.g. Guo, Tanaka, & Inuiguchi, 2000; Khodabakhshi, Gholami, & Kheirollahi, 2010; Lertworasirikul, Shu-Cherng, Joines, & Nuttle, 2003), and (5) other developments (e.g. Zerafat Angiz, Emrouznejad, Mustafa, & al-Eraqi, 2010; Hougaard, 1999, 2005).

Each of the abovementioned approaches has both advantages and disadvantages in the way it treats uncertain data in DEA models. For example, the tolerance approach fuzzifies the inequality or equality signs but it does not treat fuzzy coefficients directly. However, often it is the input and output data that is imprecise. The α-level based approach provides fuzzy efficiency metrics but requires the ranking of fuzzy efficiency sets. The fuzzy ranking approach provides fuzzy efficiency for an evaluated DMU at a specified level. The possibility approach, finally, requires generally relatively complicated numerical computations compared to other approaches.

The fundamental advantage of DEA is that it does not require a prior weights or explicit specification of functional relationships among the multiple outputs and inputs. However, when evaluating the efficiencies of DMUs, the conventional DEA methods do not allow stochastic variations in the data. Addressing this limitation, stochastic programming has been developed for decision problems where the input data are assumed to be random variables with known probability distributions (Kibzun & Kan, 1996; Prekopa, 1995; Stancu-Minasian, 1984). Zadeh (1978) introduced the so called possibility theory in the context of the fuzzy set theory as a mathematical framework for modeling situations involving uncertainty. He introduced the "fuzzy variable", which is associated with a possibility distribution, similar to a random variable, which is associated with a probability distribution. In a fuzzy LP model, each fuzzy coefficient can be charactertized as a fuzzy variable and each constraint can be viewed as a fuzzy event. Lai and Hwang (1992) have given a systematic classification of all possible problems and existing fuzzy mathematical programming approaches. They also made the distinction between fuzzy LP problems and possibilistic LP problems.

In fuzzy random environments, the crisp inputs and outputs in conventional DEA models become fuzzy random variables (FRVs). Building a DEA model directly with fuzzy random variables is senseless because the meanings of the objective and the constraints are not clear. This problem is apparent in stochastic environment and fuzzy environment, in which decision makers (DMs) are faced with the random data and fuzzy data with probability and credibility, respectively. In this paper, we propose three fuzzy CCR models with respect to probability-possibility, probability-necessity and probability-credibility constraints.

The α-level approach is the most popular fuzzy DEA model (Hatami-Marbini et al., 2011a). In this approach the main idea is to convert the fuzzy DEA model into a pair of parametric programs in order to find the lower and upper bounds of the α-level of the membership functions of the efficiency scores. Using an α-cut method proposed by Sakawa (1993), Lertworasirikul et al. (2003) proposed the possibility and necessity methods for solving a fuzzy DEA-CCR model. They introduced a possibility approach in which constraints were treated as fuzzy events and transformed fuzzy DEA models into possibility DEA models by using possibility measures of the fuzzy events (fuzzy constraints). It is known that possibility theory is based upon two dual fuzzy measures-possibility and necessity measures (Dubois & Prade, 1988; Klir, 1999; Zadeh, 1978).

The credibility theory, founded by Liu (2002, 2004, 2007), is a branch of mathematics for studying the behavior of fuzzy phenomena. Fuzzy DEA models with credibility constraints are very complex and difficult to solve because the proposed model contains the credibility of fuzzy events in the constraints and the expected value of a fuzzy variable in the objective.

One way to manipulate uncertain data in DEA is via probability distributions. Seminal work by Sengupta (1982, 1987) showed how stochastic variables could be included in the non-parametric framework. The work by Land, Lovell, and Thore (1994), Olesen and Petersen (1995) and Cooper, Huang, and Li (1996) developing a stochastic non-parametric efficiency model, later labelled stochastic DEA or SDEA, shows breakthroughs in this aspect. In SDEA, the frontier is no longer a deterministic envelope englobing the production set, but a chance-constrained envelopment based on an *a priori* probability of production space feasibility. Critique against this model argues that the parameters are arbitrary and that the model lacks statistical properties in spite of its ambition to cater for stochastic variables. Moreover, probability distributions in general require either *a priori* predictable regularity or *a posteriori* frequency determinations which are difficult to construct (Dubois & Prade, 1980). The classical probability theory is a popular tool for dealing with randomness and the credibility theory is an appropriate tool for treating fuzziness. However, in many complex real-world problems, DMs may encounter a hybrid uncertain environment where fuzziness and randomness coexist. A fuzzy random variable (FRV) is an effective concept for representing phenomena in which fuzziness and randomness appear simultaneously. FRV was first introduced by Kwakernaak (1978, 1979), and then studied by a number of researchers in the literature (Feng & Liu, 2006; Kruse & Meyer, 1987; Liu & Liu, 2003; Liu, 2004; Puri & Ralescu, 1986). FRV, which is a combination of random variable and fuzzy variable, can characterize both randomness and fuzziness in the real-world problems. The *mean chance* of a fuzzy random event is an important concept in fuzzy random optimization, just like the *probability* of a stochastic event in stochastic optimization and the *credibility* of a fuzzy event in fuzzy optimization.

3. Preliminaries

In this section, we first review some basic definitions of fuzzy sets (Dubois & Prade, 1978; Kaufmann & Gupta, 1985; Klir & Yuan, 1995; Zimmermann, 2001) followed by several definitions associated with fuzzy random variables (FRVs) (Kwakernaak, 1978; Li & Liu, 2006; Liu & Liu, 2002, 2003).

Definition 1. Let U be a universe set. A fuzzy set $\widetilde{A}$ of U is defined by a membership function $\mu_{\widetilde{A}}(x) \to [0,1]$, where $\mu_{\widetilde{A}}(x)$, $\forall x \in U$, indicates the degree of membership of $\widetilde{A}$ to U.

Definition 2. A fuzzy subset $\widetilde{A}$ of real number R is convex if and only if

$\mu_{\widetilde{A}}(\lambda x + (1-\lambda)y) \geqslant (\mu_{\widetilde{A}}(x) \wedge \mu_{\widetilde{A}}(y)), \forall x, y \in R, \forall \lambda \in [0,1]$, where "$\wedge$" denotes the minimum operator.

Definition 3. The α-cut of fuzzy set $\widetilde{A}$, $\widetilde{A}_\alpha$, is the crisp set $\widetilde{A}_\alpha = \{x|\mu_{\widetilde{A}}(x) \geqslant \alpha\}$. The support of $\widetilde{A}$ is the crisp set $Sup(\widetilde{A}) = \{x|\mu_{\widetilde{A}}(x) \geqslant 0\}$.

Definition 4. $\widetilde{A}$ is normal if and only if $Sup_{x\in U}\mu_{\widetilde{A}}(x) = 1$, where U is the universal set.

Definition 5. $\widetilde{A}$ is a fuzzy number if and only if $\widetilde{A}$ is a normal and convex fuzzy subset of the set of real numbers.

Definition 6. A fuzzy number of L–R type is denoted by $\widetilde{A} = (m, \alpha, \beta)_{LR}$ and its membership function can be expressed as

$$\mu_{\widetilde{A}}(t) = \begin{cases} L\left(\frac{m-x}{\alpha}\right), & x \leqslant m, \\ R\left(\frac{x-m}{\beta}\right), & x \geqslant m. \end{cases} \tag{1}$$

where L and R are the left and right functions, respectively, and α and β are the (non-negative) left and right spreads, respectively.

Definition 7. The α-cut, $\alpha \in [0,1]$, of a L–R type fuzzy number $\widetilde{A}$ is a closed interval as follows

$$A_\alpha = \{x|\mu_A(x) \geqslant \alpha\} = \left[A^L_\alpha, A^R_\alpha\right] = [m - L^{-1}(\alpha), m + R^{-1}(\alpha)],$$

where A^L_α and A^R_α are the left and right extreme points, respectively.

Definition 8 (***Fuzzy arithmetic***). Let $\widetilde{A} = (m_1, \alpha_1, \beta_1)_{LR}$ and $\widetilde{B} = (m_2, \alpha_2, \beta_2)_{LR}$ be two fuzzy numbers of L–R type. Then, their arithmetic addition and subtraction operations are defined as

Addition:
$(m_1,\alpha_1,\beta_1)_{LR} + (m_2,\alpha_2,\beta_2)_{LR} = (m_1 + m_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)_{LR}$
Subtraction:
$(m_1,\alpha_1,\beta_1)_{LR} - (m_2,\alpha_2,\beta_2)_{LR} = (m_1 - m_2, \alpha_1 + \beta_2, \beta_1 + \alpha_2)_{LR}$

In addition, if k is non-zero real number, then

$$kA = \begin{cases} (km_1, k\alpha_1, k\beta_1)_{LR}, & k > 0, \\ (km_1, -k\beta_1, -k\alpha_1)_{LR}, & k < 0. \end{cases}$$

Definition 9 (***Extension principle***). This principle allows the generalization of crisp mathematical concepts in fuzzy frameworks. For any function f, mapping points in set X to points in set Y, and any fuzzy set $A \in \widetilde{P}(X)$ where $A = \mu_1(x_1) + \mu_2(x_2) + \cdots + \mu_n(x_n)$, the extension principle expresses:

$$\begin{aligned} f(A) &= f(\mu_1(x_1) + \mu_2(x_2) + \cdots + \mu_n(x_n)) \\ &= f(\mu_1(x_1)) + f(\mu_2(x_2)) + \cdots + f(\mu_n(x_n)). \end{aligned} \tag{2}$$

Definition 10. Let $(\Omega, A, \Pr)$ be a probability space where Ω is a sample space, A is the s-algebra of subsets of Ω (i.e., the set of all possible potentially interesting events), and is a probability measure on, and $F(R)$ be the set of all fuzzy numbers in the set of real numbers R. Generally, F involves the normal convex fuzzy subsets. Thus, a fuzzy random variable (FRV) is a mapping function ξ: $\Omega \to F$ such that for any Borel set B of $\Re$, and $\xi^*(B)(\omega) = Pos\{\xi(\omega) \in B\}$ is a measurable function of ω.

Definition 11. Let $f : \Re^n \to \Re$ be a continuous function and ξ_i be the FRVs defined on $(\Omega_i, A_i, \Pr_i)$, $i = 1,2,\ldots,n$. Then, $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is a FRV on the product probability space $(\Omega_1 \times \Omega_2 \times \cdots \times \Omega_n, A_1 \times A_2 \times \cdots \times A_n, \Pr_1 \times \Pr_2 \times \cdots \times \Pr_n)$ as $\xi(\omega_1, \omega_2, \ldots, \omega_n) = f(\xi_1(\omega_1), \xi_2(\omega_2), \ldots, \xi_n(\omega_n))$ where $(\omega_1, \omega_2, \ldots, \omega_n) \in \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$.

Definition 12 (*Fuzzy random arithmetic*). Let ξ_1 and ξ_2 be two FRVs with the probability spaces $(\Omega_1, A_1, \Pr_1)$ and $(\Omega_2, A_2, \Pr_2)$, respectively. Then $\xi = \xi_1 + \xi_2$ is defined as

$$\xi(\omega_1, \omega_2) = \xi_1(\omega_1) + \xi_2(\omega_2), \quad \forall(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$$

where $(\Omega_1 \times \Omega_2, A_1 \times A_2, \Pr_1 \times \Pr_2)$ is the corresponding probability space.

Definition 13. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ be a fuzzy random vector, and $f : \Re^n \to \Re$ be a continuous function. Then $f(\xi)$ is a FRV.

Definition 14. Given a universe set U, let $P(U)$ be the power set of U. $(U, P(U), Pos)$. The triple $(U, P(U), Pos)$ is called a possibility space where Pos is a possibility measure defined on $P(U)$. For any sets A and B, the properties of the possibility measure are presented as

a. $Pos(\emptyset) = 0$ and $Pos(U) = 1$;
b. Monotonicity: $A \subset B$ implies $Pos(A) \leqslant Pos(B)$ for any $A, B \in P(U)$;
c. Subadditivity: $Pos(A \cup B) + Pos(A \cap B) \leqslant Pos(B) + Pos(A)$ for any $A, B \in P(U)$.

The *necessity measure* of A, denoted by $Nec(A)$, is also defined on $P(U)$ as $Nec\{A\} = 1 - Pos\{A^c\}$ where A^c is the complement set of A. For any sets A and B, the properties of the necessity measure are presented as.

(a) $Nec(\emptyset) = 0$ and $Nec(U) = 1$;
(b) $Pos(A) \geqslant Nec(A)$;
(c) Monotonicity: $A \subset B$ implies $Nec(A) \leqslant Nec(B)$;
(d) Subadditivity: $Nec(A \cup B) + Nec(A \cap B) \geqslant Nec(B) + Nec(A)$.

Definition 15. Let $(U, P(U), Pos)$ be a possibility space, and A a set in $P(U)$. The credibility measure of a fuzzy event A, $Cr(A)$, is defined as the average of its possibility and necessity.

$$Cr(A) = \frac{1}{2}(Pos\{A\} + Nec\{A\}) \tag{3}$$

The credibility measure involves the following properties:

a. $Cr(\emptyset) = 0$, and $Cr\{U\} = 1$;
b. Monotonicity: $A \subset B$ implies $Cr\{A\} \leqslant Cr\{B\}$ for any $A, B \in P(U)$;
c. Self-duality: $Cr\{A\} + Cr\{A^c\} = 1$, for any $A \in P(U)$;
d. $Cr\{\cup_i A_i\} = Sup_i Cr\{A_i\}$ for any collection $\{A_i\}$ in $P(U)$ with $Sup_i Cr\{A_i\} < 0.5$;
e. Subadditivity: $Cr\{A \cup B\} \leqslant Cr\{A\} + Cr\{B\}$ for any $A, B \in P(U)$;
f. $Pos\{A\} \geqslant Cr\{A\} \geqslant Nec\{A\}$

Definition 16. Let ξ be a fuzzy variable on the a possibility space $(U, P(U), Pos)$. The possibility, necessity and credibility of a fuzzy event $\{\xi \geqslant r\}$ are represented by:

$$Pos\{\xi \geqslant r\} = Sup_{t \geqslant r}\mu_\xi(t),$$

$$Nes\{\xi \geqslant r\} = 1 - Sup_{t<r}\mu_\xi(t),$$

$$Cr(\xi \geqslant r) = \frac{1}{2}[Pos\{\xi \geqslant r\} + Nec\{\xi \geqslant r\}].$$

where $\mu_\xi : \Re \to [0,1]$ is the membership function of ξ and r is a real number. Note here that $Cr(\xi \geqslant r) = 1 - Cr(\xi < r)$.

4. Conventional DEA-CCR model with crisp data

Following the Charnes, Cooper, and Rhodes (CCR) model, proposed by Charnes et al. (1978), under a constant returns to scale (CRS) technology, we assume that there are n DMUs to be evaluated where every DMU_j, $j = 1,2,\ldots,n$, produces s outputs, y_{rj} ($r = 1,2,\ldots,s$), using m inputs, $x_{ij}(i = 1,2,\ldots,m)$. The following problem is used to calculate the technical radial input-efficiency of a given DMU_p:

$$\begin{aligned}
&\theta_p^* = \max \sum_{r=1}^{s} u_r y_{rp} \\
&\text{s.t.} \\
&\sum_{i=1}^{m} v_i x_{ip} = 1, \\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leqslant 0, \quad j = 1,\ldots,n, \\
&u_r,\, v_i \geqslant 0, \quad r = 1,\ldots,\ s; i = 1,\ldots,m.
\end{aligned} \tag{4}$$

where the u_r and v_i are the weights assigned to the rth output and the ith input, respectively. The DMU_p is (technically) efficient if $\theta_p^* = 1$, otherwise, DMU_o is (technically) inefficient.

5. Proposed models

In this section, we present the mathematical details of the probability-possibility, probability-necessity and probability-credibility approaches proposed in this study for solving the CCR models in which the input and output data are assumed to be characterized by fuzzy random variables (FRVs).

5.1. Fuzzy probability- possibility CCR model

In this sub-section, we propose a DEA-based method for evaluating the efficiencies of DMUs with fuzzy stochastic inputs and fuzzy stochastic outputs. Consider n DMUs, each of which consumes m fuzzy stochastic inputs, denoted by $\tilde{\bar{x}}_{ij} = \left(x_{ij}^m, x_{ij}^\alpha, x_{ij}^\beta\right)_{LR}$, $i = 1,\ldots,m,\ j = 1,\ldots,n$, and produces s fuzzy stochastic outputs, denoted by $\tilde{\bar{y}}_{rj} = \left(y_{rj}^m, y_{rj}^\alpha, y_{rj}^\beta\right)_{LR}$, $r = 1,\ldots,s,\ j = 1,\ldots,n$. Let x_{ij}^m and y_{rj}^m, denoted by $x_{ij}^m \sim N\left(x_{ij}, \sigma_{ij}^2\right)$ and $y_{rj}^m \sim N\left(y_{rj}, \sigma_{rj}^2\right)$ be normally distributed. Therefore, x_{ij} (y_{rj}) and $\sigma_{ij}^2\left(\sigma_{rj}^2\right)$ are the mean and the variance of x_{ij}^m $\left(y_{rj}^m\right)$ for DMU_j, respectively.

The chance-constrained programming (CCP) developed by Cooper, Deng, Huang, and Li (2002) is a stochastic optimization approach suitable for solving optimization problems with uncertain parameters. Building on CCP and possibility theory as the principal techniques, we propose the following probability-possibility CCR model:

$$\begin{aligned}
&\max \varphi \\
&\text{s.t.} \\
&Pr\left[Pos\left(\varphi \leqslant \sum_{r=1}^{s} u_r \tilde{\bar{y}}_{rp}\right) \geqslant \delta\right] \geqslant \gamma, \quad (i) \\
&Pr\left[Pos\left(\sum_{i=1}^{m} v_i \tilde{\bar{x}}_{ip} = 1\right) \geqslant \delta'\right] \geqslant \gamma', \quad (ii) \\
&Pr\left[Pos\left(\sum_{r=1}^{s} u_r \tilde{\bar{y}}_{rj} - \sum_{i=1}^{m} v_i \tilde{\bar{x}}_{ij} \leqslant 0\right) \geqslant \delta_j\right] \geqslant \gamma_j, \quad j = 1,\ldots,n, \quad (iii) \\
&u_r,\, v_i \geqslant 0, \quad r = 1,\ldots,\ s; i = 1,\ldots,m.
\end{aligned} \tag{5}$$

where δ and $\gamma \in [0,1]$ in constraint (i), δ' and $\gamma' \in [0,1]$ in constraint (ii) and δ_j and $\gamma_j \in [0,1]$ ($j = 1,2,\ldots,n$) in constraint (iii) are the predetermined thresholds defined by the DM. $Pos[\cdot]$ and $Pr[\cdot]$ in model (5) denote the possibility and the probability of $[\cdot]$ event. In this model, the objective function is maximized while satisfying (i) constraint with at least δ and γ levels as well as simultaneously meeting the thresholds of constraints (ii) and (iii). In this paper we assume, without loss of generality, the same *h-thresholds* for all constraints $\delta' = \delta_j = \delta$ and $\gamma' = \gamma_j = \gamma$.

In addition, we presume that the fuzzy stochastic input $\tilde{\bar{x}}_{ij}$ and the fuzzy stochastic output $\tilde{\bar{y}}_{rj}$ are characterized, respectively, by the following two membership functions:

$$\mu_{\tilde{\bar{x}}_{ij}}(t) = \begin{cases} L\left(\frac{x_{ij}^m - t}{x_{ij}^\alpha}\right), & t \leqslant x_{ij}^m, \\ R\left(\frac{t - x_{ij}^m}{x_{ij}^\beta}\right), & t \geqslant x_{ij}^m. \end{cases} \tag{6}$$

and

$$\mu_{\tilde{\bar{y}}_{rj}}(t) = \begin{cases} L\left(\frac{y_{rj}^m - t}{y_{rj}^\alpha}\right), & t \leqslant y_{rj}^m, \\ R\left(\frac{t - y_{rj}^m}{y_{rj}^\beta}\right), & t \geqslant y_{rj}^m. \end{cases} \tag{7}$$

where $x_{ij}^m \sim N(x_{ij}, \sigma_{ij}^2)$ and $y_{rj}^m \sim N(y_{rj}, \sigma_{rj}^2)$.

Using the extension principle (see Definition 9), the fuzzy number $\sum_{r=1}^{s} u_r \tilde{\bar{y}}_{rj}$ and $\sum_{i=1}^{m} v_i \tilde{\bar{x}}_{ij}$ can be converted into the following membership functions:

$$\mu_{\sum_{i=1}^{m} v_i \tilde{\bar{x}}_{ij}}(t) = \begin{cases} L\left(\frac{\sum_{i=1}^{m} v_i x_{ij}^m - t}{\sum_{i=1}^{m} v_i x_{ij}^\alpha}\right), & t \leqslant \sum_{i=1}^{m} v_i x_{ij}^m, \\ R\left(\frac{t - \sum_{i=1}^{m} v_i x_{ij}^m}{\sum_{i=1}^{m} v_i x_{ij}^\beta}\right), & t \geqslant \sum_{i=1}^{m} v_i x_{ij}^m. \end{cases} \tag{8}$$

and

$$\mu_{\sum_{r=1}^{s} u_r y_{rj}^m}(t) = \begin{cases} L\left(\frac{\sum_{r=1}^{s} u_r y_{rj}^m - t}{\sum_{r=1}^{s} u_r y_{rj}^\alpha}\right), & t \leqslant \sum_{r=1}^{s} u_r y_{rj}^m, \\ R\left(\frac{t - \sum_{r=1}^{s} u_r y_{rj}^m}{\sum_{r=1}^{s} u_r y_{rj}^\beta}\right), & t \geqslant \sum_{r=1}^{s} u_r y_{rj}^m. \end{cases} \tag{9}$$

Therefore, $\sum_{r=1}^{s} v_i \tilde{\bar{x}}_{ij}$ and $\sum_{r=1}^{s} u_r \tilde{\bar{y}}_{rj}$ can be denoted as the triple $\left(\sum_{i=1}^{m} v_i x_{ij}^m, \sum_{i=1}^{m} v_i x_{ij}^\alpha, \sum_{i=1}^{m} v_i x_{ij}^\beta\right)_{LR}$ and $\left(\sum_{r=1}^{s} u_r y_{rj}^m, \sum_{r=1}^{s} u_r y_{rj}^\alpha, \sum_{r=1}^{s} u_r y_{rj}^\beta\right)_{LR}$, respectively. In addition, the following corresponding intervals of $\sum_{r=1}^{s} v_i \tilde{\bar{x}}_{ij}$ and $\sum_{r=1}^{s} u_r \tilde{\bar{y}}_{rj}$ infer from the α-cut (see Definition 7):

$$\left[\left(\sum_{i=1}^{m} v_i x_{ij}\right)_\delta^L, \left(\sum_{i=1}^{m} v_i x_{ij}\right)_\delta^R\right] = \left[\sum_{i=1}^{m} v_i x_{ij}^m - L^{-1}(\delta)\sum_{i=1}^{m} v_i x_{ij}^\alpha, \sum_{i=1}^{m} v_i x_{ij}^m + R^{-1}(\delta)\sum_{i=1}^{m} v_i x_{ij}^\beta\right]$$

$$\left[\left(\sum_{r=1}^{s} u_r y_{rj}\right)_\delta^L, \left(\sum_{r=1}^{s} u_r y_{rj}\right)_\delta^R\right] = \left[\sum_{r=1}^{s} u_r y_{rj}^m - L^{-1}(\delta)\sum_{r=1}^{s} u_r y_{rj}^\alpha, \sum_{r=1}^{s} u_r y_{rj}^m + R^{-1}(\delta)\sum_{r=1}^{s} u_r y_{rj}^\beta\right]$$

In order to solve the probability-possibility constrained programming model (5), we convert the constraints in this model into their respective crisp equivalents. Thereby, Theorem 1 and Lemma 1 proposed, respectively, by Liu and Liu (2003) and Sakawa (1993), play a pivotal role in solving the proposed model (5).

Theorem 1. *Assume that ξ is a fuzzy random vector, and g_j are real-valued continuous functions for j = 1,2,…,n. We have:*

a. *The possibility $Pos\{g_j(\xi(\omega)) \leqslant 0,\ j = 1,\ldots,p\}$ is a random variable;*
b. *The necessity $Nec\{g_j(\xi(\omega)) \leqslant 0,\ j = 1,\ldots,p\}$ is a random variable.*
c. *The credibility $Cr\{g_j(\xi(\omega)) \leqslant 0,\ j = 1,\ldots,p\}$ is a random variable.*

Lemma 1. *Let $\bar{\lambda}_1$ and $\bar{\lambda}_2$ be two independent fuzzy numbers with continuous membership functions. For a given confidence level $\alpha \in [0,1]$,*

$$Pos\{\bar{\lambda}_1 \geqslant \bar{\lambda}_2\} \geqslant \alpha \quad \text{if and only if } \lambda^R_{1,\alpha} \geqslant \lambda^L_{2,\alpha},$$
$$Nec\{\bar{\lambda}_1 \geqslant \bar{\lambda}_2\} \geqslant \alpha \quad \text{if and only if } \lambda^L_{1,1-\alpha} \geqslant \lambda^R_{2,\alpha}.$$

where $\lambda^L_{1,\alpha}$, $\lambda^R_{1,\alpha}$ and $\lambda^L_{2,\alpha}$, $\lambda^R_{2,\alpha}$ are the left and the right side extreme points of the α-level sets $\bar{\lambda}_1$ and $\bar{\lambda}_2$, respectively, and $Pos\{\bar{\lambda}_1 \geqslant \bar{\lambda}_2\}$ and $Nec\{\bar{\lambda}_1 \geqslant \bar{\lambda}_2\}$ present the degree of possibility and necessity, respectively.

Based on Lemma 1, the constraint (i) in model (5) is equivalent to the following equations:

$$Pr\left(Pos\left(\varphi \leqslant \sum_{r=1}^{s} u_r \tilde{\bar{y}}_{rp}\right) \geqslant \delta\right) \geqslant \gamma \Longleftrightarrow Pr\left(\varphi \leqslant \left(\sum_{r=1}^{s} u_r \tilde{\bar{y}}_{rp}\right)^R_\delta\right) \geqslant \gamma$$
$$\Longleftrightarrow Pr\left(\varphi \leqslant \sum_{r=1}^{s} u_r y^m_{rp} + R^{-1}(\delta)\sum_{r=1}^{s} u_r y^\beta_{rp}\right) \geqslant \gamma$$

where " $\Longleftrightarrow$ " presents "if and only if" statement.

Analogously, constraint (ii) in model (5), $Pr[Pos(\sum_{i=1}^m v_i \tilde{\bar{x}}_{ip} = 1) \geqslant \delta'] \geqslant \gamma'$, can also be transformed into the following two constraints: $Pr[Pos(\sum_{i=1}^m v_i \tilde{\bar{x}}_{ip} \geqslant 1) \geqslant \delta'] \geqslant \gamma'$, and $Pr[Pos(\sum_{i=1}^m v_i \tilde{\bar{x}}_{ip} \leqslant 1) \geqslant \delta'] \geqslant \gamma'$. These constraints can be rewritten as the following constraints based on Lemma 1:

$$Pr\left[\left(\sum_{i=1}^{m} v_i \tilde{\bar{x}}_{ip}\right)^R_{\delta'} \geqslant 1\right] \geqslant \gamma' \Longleftrightarrow Pr\left[\sum_{i=1}^{m} v_i\left(x^m_{ip} + R^{-1}(\delta')x^\beta_{ip}\right) \geqslant 1\right] \geqslant \gamma', \quad \text{and}$$
$$Pr\left[\left(\sum_{i=1}^{m} v_i \tilde{\bar{x}}_{ip}\right)^L_{\delta'} \leqslant 1\right] \geqslant \gamma' \Longleftrightarrow Pr\left[\sum_{j=1}^{n} v_i\left(x^m_{ip} - L^{-1}(\delta')x^\alpha_{ip}\right) \leqslant 1\right] \geqslant \gamma'.$$

Analogously, the constraints (iii) in model (5) $Pr[Pos(\sum_{r=1}^s u_r \tilde{\bar{y}}_{rj} - \sum_{i=1}^m v_i \tilde{\bar{x}}_{ij} \leqslant 0) \geqslant \delta_j] \geqslant \gamma_j$, $j = 1, \ldots, n$ can be rewritten as the following constraint:

$$Pr\left[\left(\sum_{r=1}^{s} u_r \tilde{\bar{y}}_{rj}\right)^L_{\delta_j} - \left(\sum_{i=1}^{m} v_i \tilde{\bar{x}}_{ij}\right)^R_{\delta_j} \leqslant 0\right] \geqslant \gamma_j, \quad j = 1, \ldots, n \Longleftrightarrow$$
$$Pr\left[\sum_{r=1}^{s} u_r\left(y^m_{rj} - L^{-1}(\delta_j)y^\alpha_{rj}\right) - \sum_{i=1}^{m} v_i\left(x^m_{ij} + R^{-1}(\delta_j)x^\beta_{ij}\right) \leqslant 0\right] \geqslant \gamma_j, \quad j = 1, \ldots, n$$

As a result, model (5) is converted to the following CCP model:

$$\begin{aligned}
&\max \varphi\\
&\text{s.t.}\\
&Pr\left(\varphi \leqslant \sum_{r=1}^{s} u_r y^m_{rp} + R^{-1}(\delta)\sum_{r=1}^{s} u_r y^\beta_{rj}\right) \geqslant \gamma, \quad \text{(i)}\\
&Pr\left[\sum_{i=1}^{m} v_i\left(x^m_{ip} + R^{-1}(\delta')x^\beta_{ip}\right) \geqslant 1\right] \geqslant \gamma', \quad \text{(ii)}\\
&Pr\left[\sum_{i=1}^{m} v_i\left(x^m_{ip} - L^{-1}(\delta')x^\alpha_{ip}\right) \leqslant 1\right] \geqslant \gamma', \quad \text{(iii)}\\
&Pr\left[\sum_{r=1}^{s} u_r\left(y^m_{rj} - L^{-1}(\delta_j)y^\alpha_{rj}\right) - \sum_{i=1}^{m} v_i\left(x^m_{ij} + R^{-1}(\delta_j)x^\beta_{ij}\right) \leqslant 0\right] \geqslant \gamma_j, \; j = 1, \ldots, n, \quad \text{(iv)}\\
&u_r, v_i \geqslant 0, r = 1, \ldots, s; i = 1, \ldots, m.
\end{aligned} \tag{10}$$

The standardized normal distribution, (see, e.g. Cooper et al., 1996, Cooper, Deng, Huang, & Li, 2004), can be used to transform the above CCP model to a deterministic LP model. Consequently, let us consider constraint (i) in model (10) as $Pr(\tilde{h} \geqslant 0) \geqslant \gamma$ where $\tilde{h} = \sum_{r=1}^s u_r y^m_{rp} + R^{-1}(\delta)\sum_{r=1}^s u_r y^\beta_{rp} - \varphi$. Due to the normal distribution of y^m_{rp}, h also has normal distribution with the following mean and variance:

$$E(\tilde{h}) = E\left[\sum_{r=1}^{s} u_r y^m_{rp} + R^{-1}(\delta)\sum_{r=1}^{s} u_r y^\beta_{rp} - \varphi\right] = \sum_{r=1}^{s} u_r y_{rp} + R^{-1}(\delta)\sum_{r=1}^{s} u_r y^\beta_{rp} - \varphi$$
$$Var(\tilde{h}) = Var\left(\sum_{r=1}^{s} u_r y^m_{rp} + R^{-1}(\delta)\sum_{r=1}^{s} u_r y^\beta_{rp} - \varphi\right) = Var\left(\sum_{r=1}^{s} u_r y^m_{rp}\right) = \sum_{r=1}^{s} u_r^2 \sigma^2_{rp}$$

By standardizing the normal distribution, $Pr(\tilde{h} \geqslant 0) \geqslant \gamma$ is converted to $Pr\left(z \geqslant \frac{-E(\tilde{h})}{\sqrt{var(\tilde{h})}}\right) \geqslant \gamma$ where $z = \frac{h - E(\tilde{h})}{\sqrt{var(\tilde{h})}}$ is the standard normal random variable with zero mean and unit variance. The corresponding cumulative distribution function is $\Phi\left(\frac{-E(\tilde{h})}{\sqrt{Var(\tilde{h})}}\right) \leqslant 1 - \gamma$ and it is equal to $\varphi - \sum_{r=1}^s u_r y_{rp} - R^{-1}(\delta) \sum_{r=1}^s u_r y^\beta_{rp} \leqslant \left(\sum_{r=1}^s u_r^2 \sigma^2_{rp}\right)^{1/2} \Phi^{-1}_{1-\gamma}$, where $\Phi^{-1}_{1-\gamma}$ is the inverse of Φ at the level of $1 - \gamma$.

A similar procedure adopted for constraints (ii), (iii) and (iv) in model (10) results in the following equations:

$$(ii): \; \sum_{i=1}^{m} v_i x_{ip} + R^{-1}(\delta')\sum_{i=1}^{m} v_i x^\beta_{ip} + \left(\sum_{i=1}^{m} v_i^2 \sigma^2_{ip}\right)^{1/2} \Phi^{-1}_{1-\gamma'} \geqslant 1,$$
$$(iii): \; \sum_{i=1}^{m} v_i x^m_{ip} - L^{-1}(\delta')\sum_{i=1}^{m} v_i x^\alpha_{ip} - \left(\sum_{i=1}^{m} v_i^2 \sigma^2_{ip}\right)^{1/2} \Phi^{-1}_{1-\gamma'} \leqslant 1$$
$$(iv): \; \sum_{r=1}^{s} u_r y^m_{rj} - \sum_{i=1}^{m} v_i x^m_{ij} - \left(L^{-1}(\delta_j)\sum_{r=1}^{s} u_r y^\alpha_{rj} + R^{-1}(\delta_j)\sum_{i=1}^{m} v_i x^\beta_{ij}\right) - \left(\sum_{r=1}^{s} u_r^2 \sigma^2_{rj} + \sum_{i=1}^{m} v_i^2 \sigma^2_{ij}\right)^{1/2} \Phi^{-1}_{1-\gamma_j} \leqslant 0, \quad j = 1, \ldots, n$$

As a consequence, the deterministic equivalent for model (5) can be set as follows:

$$\begin{aligned}
&\max \varphi\\
&\text{s.t.}\\
&\varphi - \sum_{r=1}^{s} u_r y_{rp} - R^{-1}(\delta)\sum_{r=1}^{s} u_r y^\beta_{rp} \leqslant \left(\sum_{r=1}^{s} u_r^2 \sigma^2_{rp}\right)^{1/2} \Phi^{-1}_{1-\gamma},\\
&\sum_{i=1}^{m} v_i x_{ip} + R^{-1}(\delta')\sum_{i=1}^{m} v_i x^\beta_{ip} + \left(\sum_{i=1}^{m} v_i^2 \sigma^2_{ip}\right)^{1/2} \Phi^{-1}_{1-\gamma'} \geqslant 1,\\
&\sum_{i=1}^{m} v_i x_{ip} - L^{-1}(\delta')\sum_{i=1}^{m} v_i x^\alpha_{ip} - \left(\sum_{i=1}^{m} v_i^2 \sigma^2_{ip}\right)^{1/2} \Phi^{-1}_{1-\gamma'} \leqslant 1,\\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - \left(L^{-1}(\delta_j)\sum_{r=1}^{s} u_r y^\alpha_{rj} + R^{-1}(\delta_j)\sum_{i=1}^{m} v_i x^\beta_{ij}\right)\\
&- \left(\sum_{r=1}^{s} u_r^2 \sigma^2_{rj} + \sum_{i=1}^{m} v_i^2 \sigma^2_{rj}\right)^{1/2} \Phi^{-1}_{1-\gamma_j} \leqslant 0, \quad j = 1, \ldots, n,\\
&u_r, v_i \geqslant 0, \quad r = 1, \ldots, s; i = 1, \ldots, m.
\end{aligned} \tag{11}$$

The above model is obviously a non-linear program. We hence use the following alteration variables to transform the non-linear model (11) to a quadratic program.

$$\bar{\sigma}^C_p = \left(\sum_{r=1}^{s} u_r^2 \sigma^2_{rp}\right)^{1/2}$$
$$\bar{\sigma}^I_p = \left(\sum_{i=1}^{m} v_i^2 \sigma^2_{ip}\right)^{1/2}$$
$$\bar{\sigma}^A_j = \left(\sum_{r=1}^{s} u_r^2 \sigma^2_{rj} + \sum_{i=1}^{m} v_i^2 \sigma^2_{ij}\right)^{1/2}, \quad j = 1, \ldots, n.$$

Thus, the following quadratic program is formulated:

$$
\begin{aligned}
&\max \varphi\\
&\text{s.t.}\\
&\varphi - \sum_{r=1}^{s} u_r y_{rp} - R^{-1}(\delta)\sum_{r=1}^{s} u_r y_{rp}^{\beta} \leqslant \bar{\sigma}_p^C \Phi_{1-\gamma}^{-1},\\
&\sum_{i=1}^{m} v_i x_{ip} + R^{-1}(\delta')\sum_{i=1}^{m} v_i x_{ip}^{\beta} + \bar{\sigma}_p^I \Phi_{1-\gamma'}^{-1} \geqslant 1,\\
&\sum_{i=1}^{m} v_i x_{ip} - L^{-1}(\delta')\sum_{i=1}^{m} v_i x_{ip}^{\alpha} - \bar{\sigma}_p^I \Phi_{1-\gamma'}^{-1} \leqslant 1,\\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - \left(L^{-1}(\delta_j)\sum_{r=1}^{s} u_r y_{rj}^{\alpha} + R^{-1}(\delta_j)\sum_{i=1}^{m} v_i x_{ij}^{\beta}\right)\\
&\quad - \bar{\sigma}_j^A \Phi_{1-\gamma_j}^{-1} \leqslant 0, \quad j=1,\ldots,n,\\
&\left(\bar{\sigma}_p^C\right)^2 = \left(\sum_{r=1}^{s} u_r^2 \sigma_{rp}^2\right),\\
&\left(\bar{\sigma}_p^I\right)^2 = \left(\sum_{i=1}^{m} v_i^2 \sigma_{ip}^2\right),\\
&\left(\bar{\sigma}_j^A\right)^2 = \left(\sum_{r=1}^{s} u_r^2 \sigma_{rj}^2 + \sum_{i=1}^{m} v_i^2 \sigma_{ij}^2\right), \quad j=1,\ldots,n,\\
&u_r, v_i, \bar{\sigma}_p^C, \bar{\sigma}_p^I, \bar{\sigma}_j^A \geqslant 0, r=1,\ldots,s; i=1,\ldots m; j=1,\ldots,n.
\end{aligned}
\tag{12}
$$

We present the following definition to define the efficiency of a DMU.

Definition 17. A DMU is called probabilistic-possibilisticly γ-efficient if the objective function of model (12), φ, is greater than or equal to one at the probability-possibility level $1-\gamma$; otherwise, it is called probabilistic-possibilisticly γ-inefficient.

5.2. Fuzzy probability-necessity CCR model

Let us continue with the earlier assumptions regarding n DMUs with m fuzzy stochastic inputs $\tilde{\bar{x}}_{ij} = \left(x_{ij}^m, x_{ij}^{\alpha}, x_{ij}^{\beta}\right)_{LR}$, $i=1,\ldots,m$, $j=1,\ldots,n$ and s fuzzy stochastic outputs $\tilde{\bar{y}}_{rj} = \left(y_{rj}^m, y_{rj}^{\alpha}, y_{rj}^{\beta}\right)_{LR}$, $r=1,\ldots,s$, $j=1,\ldots,n$ where $x_{ij}^m \sim N\left(x_{ij}, \sigma_{ij}^2\right)$ and $y_{rj}^m \sim N\left(y_{rj}, \sigma_{rj}^2\right)$.

The CCR DEA model is transformed into the following Model (13) in the presence of fuzzy probability necessity constraints.

$$
\begin{aligned}
&\max \bar{\varphi}\\
&s.t.\\
&Pr\left[Nec\left(\bar{\varphi} \leqslant \sum_{r=1}^{s} u_r \tilde{\bar{y}}_{rp}\right) \geqslant \delta\right] \geqslant \gamma,\\
&Pr\left[Nec\left(\sum_{i=1}^{m} v_i \tilde{\bar{x}}_{ip} = 1\right) \geqslant \delta'\right] \geqslant \gamma',\\
&Pr\left[Nec\left(\sum_{r=1}^{s} u_r \tilde{\bar{y}}_{rj} - \sum_{i=1}^{m} v_i \tilde{\bar{x}}_{ij} \leqslant 0\right) \geqslant \delta_j\right] \geqslant \gamma_j, \quad j=1,\ldots,n,\\
&u_r, v_i \geqslant 0, \quad r=1,\ldots,s; i=1,\ldots,m.
\end{aligned}
\tag{13}
$$

Consider the last set of constraints in model (13) for the deterministic equivalent. Using Lemma 1, these constraints can be written as follows:

$$
\begin{aligned}
&Pr\left[Nec\left(\sum_{r=1}^{s} u_r \tilde{\bar{y}}_{rj} - \sum_{i=1}^{m} v_i \tilde{\bar{x}}_{ij} \leqslant 0\right) \geqslant \delta_j\right] \geqslant \gamma_j\\
&\Longleftrightarrow Pr\left[\left(\sum_{r=1}^{s} u_r \tilde{\bar{y}}_{rj}\right)_{\delta_j}^{R} - \left(\sum_{i=1}^{m} v_i \tilde{\bar{x}}_{ij}\right)_{1-\delta_j}^{L} \leqslant 0\right] \geqslant \gamma_j\\
&\Longleftrightarrow Pr\left[\sum_{r=1}^{s} u_r \left(y_{rj}^m + R^{-1}(\delta_j) y_{rj}^{\beta}\right) - \sum_{i=1}^{m} v_i \left(x_{ij}^m - L^{-1}(1-\delta_j) x_{ij}^{\alpha}\right) \leqslant 0\right] \geqslant \gamma_j
\end{aligned}
$$

Similarly, Lemma 1 can be applied to the remaining constraints and ultimately model (13) is transformed into model (14).

$$
\begin{aligned}
&\max \bar{\varphi}\\
&\text{s.t.}\\
&Pr\left(\bar{\varphi} \leqslant \sum_{r=1}^{s} u_r y_{rp}^m - L^{-1}(1-\delta)\sum_{r=1}^{s} u_r y_{rp}^{\alpha}\right) \geqslant \gamma,\\
&Pr\left[\sum_{i=1}^{m} v_i \left(x_{ip}^m - L^{-1}(1-\delta') x_{ip}^{\alpha}\right) \geqslant 1\right] \geqslant \gamma',\\
&Pr\left[\sum_{i=1}^{m} v_i \left(x_{ip}^m + R^{-1}(\delta') x_{ip}^{\beta}\right) \leqslant 1\right] \geqslant \gamma',\\
&Pr\left[\sum_{r=1}^{s} u_r \left(y_{rj}^m + R^{-1}(\delta_j) y_{rj}^{\beta}\right) - \sum_{i=1}^{m} v_i \left(x_{ij}^m - L^{-1}(1-\delta_j) x_{ij}^{\alpha}\right) \leqslant 0\right] \geqslant \gamma_j,\\
&j=1,\ldots,n,\\
&u_r, v_i \geqslant 0, \quad r=1,\ldots, s; i=1,\ldots,m.
\end{aligned}
\tag{14}
$$

We can identify model (15), the deterministic equivalent of model (14), by following procedures similar to those that were used to determine the deterministic equivalent of model (11).

$$
\begin{aligned}
&\max \bar{\varphi}\\
&\text{s.t.}\\
&\sum_{r=1}^{s} u_r y_{rp} - L^{-1}(1-\delta)\sum_{r=1}^{s} u_r y_{rj}^{\alpha} + \bar{\sigma}_p^C \Phi_{1-\gamma}^{-1} - \bar{\varphi} \geqslant 0,\\
&\sum_{i=1}^{m} v_i x_{ip} - L^{-1}(1-\delta')\sum_{i=1}^{m} v_i x_{ip}^{\alpha} + \bar{\sigma}_p^I \Phi_{1-\gamma'}^{-1} \geqslant 1,\\
&\sum_{i=1}^{m} v_i x_{ip} + R^{-1}(\delta')\sum_{i=1}^{m} v_i x_{ip}^{\beta} - \bar{\sigma}_p^I \Phi_{1-\gamma'}^{-1} \leqslant 1\\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + R^{-1}(\delta_j)\sum_{i=1}^{m} u_r y_{rj}^{\beta} + L^{-1}(1-\delta_j)\sum_{i=1}^{m} v_i x_{ij}^{\alpha} - \bar{\sigma}_j^A \Phi_{1-\gamma_j}^{-1} \leqslant 0, \ j=1,\ldots,n,\\
&\left(\bar{\sigma}_p^C\right)^2 = \left(\sum_{r=1}^{s} u_r^2 \sigma_{rp}^2\right),\\
&\left(\bar{\sigma}_p^I\right)^2 = \left(\sum_{i=1}^{m} v_i^2 \sigma_{ip}^2\right),\\
&\left(\bar{\sigma}_j^A\right)^2 = \left(\sum_{r=1}^{s} u_r^2 \sigma_{rj}^2 + \sum_{i=1}^{m} v_i^2 \sigma_{ij}^2\right), \ j=1,\ldots,n,\\
&u_r, v_i, \bar{\sigma}_p^C, \bar{\sigma}_p^I, \bar{\sigma}_j^A \geqslant 0, \ r=1,\ldots,s; i=1,\ldots,m; j=1,\ldots,n.
\end{aligned}
\tag{15}
$$

We present the following definition to define the relevant efficiency criterion.

Definition 18. A DMU is called probabilistic-necessity γ-efficient if the objective function in model (15), φ, is greater than or equal to one at the probability-possibility level $1-\gamma$; otherwise, it is called to be probabilistic-necessity γ-inefficient.

5.3. Fuzzy probability-credibility CCR model

Analogously to the models above, the corresponding fuzzy probability-credibility CCR DEA model is given as:

$$
\begin{aligned}
&\max \bar{\varphi}\\
&\text{s.t.}\\
&Pr\left[Cr\left(\bar{\varphi} \leqslant \sum_{r=1}^{s} u_r \tilde{\bar{y}}_{rp}\right) \geqslant \delta\right] \geqslant \gamma,\\
&Pr\left[Cr\left(\sum_{i=1}^{m} v_i \tilde{\bar{x}}_{ip} = 1\right) \geqslant \delta'\right] \geqslant \gamma',\\
&Pr\left[Cr\left(\sum_{r=1}^{s} u_r \tilde{\bar{y}}_{rj} - \sum_{i=1}^{m} v_i \tilde{\bar{x}}_{ij} \leqslant 0\right) \geqslant \delta_j\right] \geqslant \gamma_j, \quad j=1,\ldots,n,\\
&u_r, v_i \geqslant 0, \quad r=1,\ldots, s; i=1,\ldots,m.
\end{aligned}
\tag{16}
$$

We present the Theorem 2 below to explain the deterministic equivalent of model (16).

Theorem 2. *Let $\bar{\lambda}_1 = (m_1, \alpha_1, \beta_1)_{LR}$ and $\bar{\lambda}_2 = (m_2, \alpha_2, \beta_2)_{LR}$ be two independent L–R type fuzzy numbers with continuous membership functions. For a given confidence level $\alpha \in [0,1]$,*

(a) *When $\alpha \leqslant 0.5$, $Cr\{\bar{\lambda}_1 \geqslant \bar{\lambda}_2\} \geqslant \alpha$ if and only if $m_1 + \beta_1 R^{-1}(2\alpha) \geqslant m_2 - \alpha_2 R^{-1}(2\alpha)$, and*
(b) *When $\alpha > 0.5$, $Cr\{\bar{\lambda}_1 \geqslant \bar{\lambda}_2\} \geqslant \alpha$ if and only if $m_1 - \alpha_1 L^{-1}(2(1-\alpha)) \geqslant m_2 + \beta_2 L^{-1}(2(1-\alpha))$.*

Proof. Suppose that $\bar{\lambda}_1 = (m_1, \alpha_1, \beta_1)_{LR}$ and $\bar{\lambda}_2 = (m_2, \alpha_2, \beta_2)_{LR}$ are the two L–R type fuzzy variables (see Eq. (1)). On the basis of the fuzzy arithmetic presented in Definition 8, $\bar{\lambda} = \bar{\lambda}_1 - \bar{\lambda}_2$ is equal to $(m_1 - m_2, \alpha_1 + \beta_2, \alpha_2 + \beta_1)_{LR}$ that $\bar{\lambda}$ is a L–R type fuzzy number and accordingly with respect to Definition 15, the credibility of the fuzzy event $Cr\{\bar{\lambda} \geqslant 0\}$ is expressed as follows:

$$Cr\{\bar{\lambda} \geqslant 0\} = \begin{cases} 1, & 0 \leqslant \bar{m} - \bar{\alpha}, \\ 1 - \frac{1}{2}L\left(\frac{\bar{m}}{\bar{\alpha}}\right), & \bar{m} - \bar{\alpha} \leqslant 0 \leqslant \bar{m}, \\ \frac{1}{2}R\left(\frac{-\bar{m}}{\bar{\beta}}\right), & \bar{m} \leqslant 0 \leqslant \bar{m} + \bar{\beta}, \\ 0, & 0 > \bar{m} + \bar{\beta}. \end{cases}$$

where $\bar{\alpha} = \alpha_1 + \beta_2$, $\bar{\beta} = \alpha_2 + \beta_1$ and $\bar{m} = m_1 - m_2$. Let us consider $Cr\{\bar{\lambda} \geqslant 0\} \geqslant \alpha$. If $\alpha \leqslant 0.5$, then

$$\alpha \leqslant \frac{1}{2}R\left(\frac{-\bar{m}}{\bar{\beta}}\right) \Longleftrightarrow R^{-1}(2\alpha) \geqslant \frac{-\bar{m}}{\bar{\beta}} \Longleftrightarrow (\alpha_2 + \beta_1)R^{-1}(2\alpha) \geqslant -(m_1 - m_2) \Longleftrightarrow m_1 + \beta_1 R^{-1}(2\alpha) \geqslant m_2 - \alpha_2 R^{-1}(2\alpha)$$

and if $0.5 < \alpha \leqslant 1$, then

$$\alpha \leqslant 1 - \frac{1}{2}L\left(\frac{\bar{m}}{\bar{\alpha}}\right) \Longleftrightarrow 2(1-\alpha) \geqslant L\left(\frac{\bar{m}}{\bar{\alpha}}\right) \Longleftrightarrow L^{-1}(2(1-\alpha)) \leqslant \frac{\bar{m}}{\bar{\alpha}} \Longleftrightarrow (\alpha_1 + \beta_2)L^{-1}(2(1-\alpha)) \leqslant (m_1 - m_2) \Longleftrightarrow m_1 - \alpha_1 L^{-1}(2(1-\alpha)) \geqslant m_2 + \beta_2 L^{-1}(2(1-\alpha))$$

□

Therefore, the proof is accomplished.

Let us take into account the third set of constraints in Model (16). Using Theorem 2, this constraint can be rewritten as follows:

(a) If $\delta_j \leqslant 0.5$, then

$$Pr\left[Cr\left(\sum_{r=1}^{s} u_r \tilde{\bar{y}}_{rj} - \sum_{i=1}^{m} v_i \tilde{\bar{x}}_{ij} \leqslant 0\right) \geqslant \delta_j\right] \geqslant \gamma_j \Longleftrightarrow Pr\left[\sum_{r=1}^{s} u_r\left(y_{rj}^m - R^{-1}(2\delta_j)y_{rj}^{\alpha}\right) - \sum_{i=1}^{m} v_i\left(x_{ij}^m + R^{-1}(2\delta_j)x_{ij}^{\beta}\right) \leqslant 0\right] \geqslant \gamma_j$$

(b) If $\delta_j > 0.5$, then

$$Pr\left[Cr\left(\sum_{r=1}^{s} u_r \tilde{\bar{y}}_{rj} - \sum_{i=1}^{m} v_i \tilde{\bar{x}}_{ij} \leqslant 0\right) \geqslant \delta_j\right] \geqslant \gamma_j \Longleftrightarrow Pr\left[\sum_{r=1}^{s} u_r\left(y_{rj}^m + L^{-1}(2(1-\delta_j))y_{rj}^{\beta}\right) - \sum_{i=1}^{m} v_i\left(x_{ij}^m - L^{-1}(2(1-\delta_j))x_{ij}^{\alpha}\right) \leqslant 0\right] \geqslant \gamma_j$$

We can apply a similar process on the remaining constraints in model (16). Thus, model (16) for $\delta_j, \delta', \delta \leqslant 0.5$ and $\delta_j, \delta', \delta > 0.5$ can be transformed into the following two models:

$$\begin{aligned}
&\max_{\delta_j, \delta', \delta \leqslant 0.5} \bar{\varphi} \\
&\text{s.t.} \\
&Pr\left(\bar{\varphi} \leqslant \sum_{r=1}^{s} u_r y_{rp}^m + R^{-1}(2\delta)\sum_{r=1}^{s} u_r y_{rp}^{\beta}\right) \geqslant \gamma, \\
&Pr\left[\sum_{i=1}^{m} v_i\left(x_{ip}^m + R^{-1}(2\delta')x_{ip}^{\beta}\right) \geqslant 1\right] \geqslant \gamma', \\
&Pr\left[\sum_{i=1}^{m} v_i\left(x_{ip}^m - R^{-1}(2\delta')x_{ip}^{\alpha}\right) \leqslant 1\right] \geqslant \gamma', \\
&Pr\left[\sum_{r=1}^{s} u_r\left(y_{rj}^m - R^{-1}(2\delta_j)y_{rj}^{\alpha}\right) - \sum_{i=1}^{m} v_i\left(x_{ij}^m + R^{-1}(2\delta_j)x_{ij}^{\beta}\right) \leqslant 0\right] \geqslant \gamma_j, \\
&j = 1, \ldots, n, \\
&u_r, v_i \geqslant 0, \quad r = 1, \ldots, s; i = 1, \ldots, m.
\end{aligned} \tag{17}$$

$$\begin{aligned}
&\max_{\delta_j, \delta', \delta > 0.5} \bar{\varphi} \\
&\text{s.t.} \\
&Pr\left(\bar{\varphi} \leqslant \sum_{r=1}^{s} u_r y_{rp}^m - L^{-1}(2(1-\delta))\sum_{r=1}^{s} u_r y_{rp}^{\alpha}\right) \geqslant \gamma, \\
&Pr\left[\sum_{i=1}^{m} v_i\left(x_{ip}^m - L^{-1}(2(1-\delta'))x_{ip}^{\alpha}\right) \geqslant 1\right] \geqslant \gamma', \\
&Pr\left[\sum_{i=1}^{m} v_i\left(x_{ip}^m + L^{-1}(2(1-\delta'))x_{ip}^{\beta}\right) \leqslant 1\right] \geqslant \gamma', \\
&Pr\left[\sum_{r=1}^{s} u_r\left(y_{rj}^m + L^{-1}(2(1-\delta_j))y_{rj}^{\beta}\right) - \sum_{i=1}^{m} v_i\left(x_{ij}^m - L^{-1}(2(1-\delta_j))x_{ij}^{\alpha}\right) \leqslant 0\right] \geqslant \gamma_j, \\
&j = 1, \ldots, n, \\
&u_r, v_i \geqslant 0, r = 1, \ldots, s; i = 1, \ldots, m.
\end{aligned} \tag{18}$$

These models can be similarly transformed into the following two deterministic models:

$$\begin{aligned}
&\max_{\delta_j, \delta', \delta \leqslant 0.5} \bar{\varphi} \\
&\text{s.t.} \\
&\bar{\varphi} - \sum_{r=1}^{s} u_r y_{rp} - R^{-1}(2\delta)\sum_{r=1}^{s} u_r y_{rp}^{\beta} - \bar{\theta}_p^O \Phi_{1-\gamma}^{-1} \leqslant 0, \\
&\sum_{i=1}^{m} v_i\left(x_{ip} + R^{-1}(2\delta')x_{ip}^{\beta}\right) + \Phi_{1-\gamma'}^{-1}\bar{\theta}_p^I \geqslant 1, \\
&\sum_{i=1}^{m} v_i\left(x_{ip} - R^{-1}(2\delta')x_{ip}^{\alpha}\right) - \Phi_{1-\gamma'}^{-1}\bar{\theta}_p^I \leqslant 1, \\
&\sum_{r=1}^{s} u_r\left(y_{rj} - R^{-1}(2\delta_j)y_{rj}^{\alpha}\right) - \sum_{i=1}^{m} v_i\left(x_{ij} + R^{-1}(2\delta_j)x_{ij}^{\beta}\right) - \Phi_{1-\gamma_j}^{-1}\bar{\lambda}_j \leqslant 0, j = 1, \ldots, n, \\
&\left(\theta_p^O\right)^2 = \sum_{r=1}^{s} u_r^2\, var(y_{rp}^m), \\
&\left(\theta_p^I\right)^2 = \sum_{i=1}^{m} v_i^2\, var(x_{ip}^m), \\
&(\bar{\lambda}_j)^2 = \sum_{r=1}^{s} u_r^2\, var(y_{rj}^m) + \sum_{i=1}^{m} v_i^2\, var(x_{ij}^m), j = 1, \ldots, n \\
&u_r, v_i, \bar{\theta}_p^O, \bar{\theta}_p^I, \bar{\lambda}_j \geqslant 0, r = 1, \ldots, s; i = 1, \ldots, m; j = 1, \ldots, n.
\end{aligned} \tag{19}$$

$$\max_{\delta_j,\delta',\delta>0.5} \bar{\varphi}$$

s.t.

$$\bar{\varphi}-\sum_{r=1}^{s}u_r y_{rp}+L^{-1}(2(1-\delta))\sum_{r=1}^{s}u_r y_{rp}^{\alpha}-\bar{\theta}_p^O \Phi_{1-\gamma}^{-1}\leqslant 0,$$

$$\sum_{i=1}^{m}v_i\left(x_{ip}-L^{-1}(2(1-\delta'))x_{ip}^{\alpha}\right)+\Phi_{1-\gamma'}^{-1}\bar{\theta}_p^I\geqslant 1,$$

$$\sum_{i=1}^{m}v_i\left(x_{ip}+L^{-1}(2(1-\delta'))x_{ip}^{\beta}\right)-\Phi_{1-\gamma'}^{-1}\bar{\theta}_p^I\leqslant 1,$$

$$\sum_{r=1}^{s}u_r(y_{rj}+L^{-1}(2(1-\delta_j))y_{rj}^{\beta})-\sum_{i=1}^{m}v_i(x_{ij}-L^{-1}(2(1-\delta_j))x_{ij}^{\alpha})-\Phi_{1-\gamma}^{-1}\bar{\lambda}_j\leqslant 0, j=1,\ldots,n,$$

$$\left(\theta_p^O\right)^2=\sum_{r=1}^{s}u_r^2\, var(y_{rp}^m),$$

$$\left(\theta_p^I\right)^2=\sum_{i=1}^{m}v_i^2\, var(x_{ip}^m),$$

$$(\bar{\lambda}_j)^2=\sum_{r=1}^{s}u_r^2\, var(y_{rj}^m)+\sum_{i=1}^{m}v_i^2\, var(x_{ij}^m), j=1,\ldots,n,$$

$$u_r, v_i, \bar{\theta}_p^O, \bar{\theta}_p^I, \bar{\lambda}_j\geqslant 0, r=1,\ldots,s; i=1,\ldots,m; j=1,\ldots,n. \quad (20)$$

Definition 19. A DMU is said to be probabilistic-credibility γ-efficient if the objective function of models (19) and (20), φ, is greater than or equal to unity at the threshold level $1-\gamma$; otherwise, it is said to be probabilistic-credibility γ-inefficient.

6. Case study

The U.S. Department of Defense (DoD) adopted the base realignment and closure (BRAC) process as a national strategy to resolve the military, economic and political issue of excess base capacity created by the collapse of the former Soviet Union. As the military forces were reduced, excess base capacity was created. BRAC was designed to evaluate the overall efficiency of military bases in the United States based on certain criteria and set forth a recommendation to the Secretary of Defense to retain, close or realign military bases. The strategic and financial impacts of BRAC are immense. When bases are closed or realigned, socio-economic and environmental effects on the local community are often dramatic. Economic outcomes in terms of costs and savings are of great importance in BRAC. Social, community and environmental impacts are other direct consequences of the closure and realignment efforts. The most widespread socio-economic consequences of base closure are local unemployment, falls in land value, and depopulation. Beginning in 1988, Congress authorized the DoD to conduct five rounds of BRAC. At the completion of five rounds, the latest in 2005, the DoD had 130 fewer major bases, 84 major realignments and hundreds of other smaller facilities realigned (United States Government Accountability Office, 2012). Table 1 provides a general overview of BRAC Activities since its initiation in 1988.

Table 1
The history of BRAC rounds.

BRAC	Major base closures	Major base Realignments	Minor closures and realignments	Costs ($Billions)	Annual recurring savings ($Billion)
1988	16	4	23	$2.7	$0.9
1991	26	17	32	$5.2	$2.0
1993	28	12	123	$7.7	$2.6
1995	27	22	57	$6.5	$1.7
2005	33	29	775	$31.0	$4.0

Legislation authorizing BRAC stipulates that closure and realignment decisions must be based upon selection criteria, a current force structure plan and infrastructure inventory developed by the Secretary of Defense. The criteria historically included employment, environmental, financial, strategic and tactical impacts. BRAC is essentially a multi-criteria capital budgeting problem where the Commission is charged to determine whether the military bases on the hit list should be left alone, realigned or closed. Ideally, the Commission should retain those military bases that enhance the American welfare. A large body of performance evaluation models has evolved over the last three decades to assist DMs in strategic decision making. While these models have made great strides, the intuitive models lack a structured framework and the analytical models do not capture intuitive preferences. In this section we demonstrate the method on partial data from the BRAC project to evaluate forty military bases. A complete listing of the military bases selected for this study is presented in Appendix A. Four independent input variables and two independent output variables were used to compare the efficiency of the forty military bases selected for this study. The input data includes employment and financial impacts. The employment impact factors include reduction in military employment (Input 1) and reduction in civilian employment (Input 2). The financial impact factors include the initial closure cost (Input 3) and the environmental restoration costs (Input 4). The output data includes annual recurring cost-savings (Output 1) and the 20-year net present value savings (Output 2). The lower values for employment impacts (Input 1 and Input 2) and financial impacts (Input 3 and Input 4) meant higher efficiency; similarly, higher values for annual recurring savings and 20-year net present value savings (Output 1 and Output 2) meant higher efficiency.

In Table 2, we present the four fuzzy random inputs and the two fuzzy random outputs, where all the inputs and outputs data are triangular fuzzy random numbers. These data are denoted by (m,α,β) where m is the center value with normal distribution and α and β are the left and right tails, respectively.

Three different γ-threshold levels of $(\gamma = 0.95, \delta = 0.6)$, $(\gamma = 0.75, \delta = 0.6)$, and $(\gamma = 0.95, \delta = 0.4)$ were considered based on the DMs' previous performance evaluation studies in military bases for the three models (i.e., the probability-possibility, probability-necessity and probability-credibility models) proposed in this study. The overall results from these three models are intended to assist and inform the decision makers by identifying the military bases most suited for realignment or closure.

In Table 3, we present the efficiency values associated with the military bases for three specified threshold levels of $(\gamma = 0.95, \delta = 0.6)$, $(\gamma = 0.75, \delta = 0.6)$, and $(\gamma = 0.95, \delta = 0.4)$. The results of model (12) for three different probability-possibility levels are reported first in Table 3. As shown in this table, DMU 3 is probabilistic-possibilistic γ-efficient at three given levels, whereas DMUs 11, 26 and 28 are probabilistic-possibilistic γ-efficient only at some probability-possibility levels. Consider $(\gamma = 0.95, \delta = 0.6)$ and $(\gamma = 0.75, \delta = 0.6)$ levels under the probability-possibility method in Table 3. Aside from DMUs 1, 12 and 22, the efficiency of the DMUs at $(\gamma = 0.75, \delta = 0.6)$ level is higher than or equal to $(\gamma = 0.95, \delta = 0.6)$ level. When $\gamma = 0.95$ was kept unchanged and $\delta = 0.6$ was reduced to 0.4, the efficiency score of 93% for the DMUs was augmented.

The results of model (15) for three different probability-necessity levels of $(\gamma = 0.95, \delta = 0.6)$, $(\gamma = 0.75, \delta = 0.6)$, and $(\gamma = 0.95, \delta = 0.4)$ are also reported in Table 3. As shown in this table, the probabilistic-possibilistic efficient DMUs identified earlier were also probabilistic-necessity efficient. In addition to those units, DMUs 26 and 14 turned out to be probabilistic-necessity effi-

Table 2
The fuzzy random input and output data.

DMU	Input 1	Input 2	Input 3	Input 4	Output 1	Output 2
1	(N(709,1),28.36,28.36)	(N(1234,1),61.7,61.7)	(N(53.72,1),2.686,2.686)	(N(0,1),0,0)	(N(27.88,1),1.9516,1.9516)	(N(305.4,1),24.432,24.432)
2	(N(1297,1),25.94,25.94)	(N(1268,1),63.4,63.4)	(N(325.3,1),13.012,13.012)	(N(46.05,1),0.4605,0.4605)	(N(102.06,1),5.103,5.103)	(N(940.7,1),75.256,75.256)
3	(N(2388,1),143.28,143.28)	(N(381,1),26.67,26.67)	(N(108.17,1),6.4902,6.4902)	(N(13.7,1),0.548,0.548)	(N(206.5,1),6.195,6.195)	(N(2647.46,132.373,132.373))
4	(N(1,1),0.04,0.04)	(N(469,1),4.69,4.69)	(N(18.26,1),0.3652,0.3652)	(N(90.15,1),3.606,3.606)	(N(37.95,1),0.759,0.759)	(N(407.45,1),24.447,24.447)
5	(N(691,1),13.82,13.82)	(N(26,1),0.78,0.78)	(N(32.87,1),1.6435,1.6435)	(N(0,1),0,0)	(N(12.11,1),0.6055,0.6055)	N(126.94,1.2694,1.2694)
6	(N(511,1),20.44,20.44)	(N(570,1),39.9,39.9)	(N(56.8,1),2.272,2.272)	(N(45.1,1),2.255,2.255)	(N(35.3,1),1.059,1.059)	(N(421.5,1),33.72,33.72)
7	(N(2260,1),113,113)	(N(1881,1),112.86,112.86)	(N(214.54,1),2.1454,2.1454)	(N(17.52,1),0.7008)	(N(82.09,1),5.7463,5.7463)	(N(878.65,70.292,70.292))
8	(N(620,1),24.8,24.8)	(N(4652,1),372.16,372.16)	(N(780.43,1),31.2172,31.2172)	(N(13.9,0.139,0.139)	(N(146,1),11.68,11.68)	(N(1093.41,1),98.4069,98.4069)
9	(N(1393,1),27.86,27.86)	(N(1948,1),136.36,136.36)	(N(72.4,1),5.068,5.068)	(N(1.8,1),0.072,0.072)	(N(56.85,1),3.9795,3.9795)	(N(686.6,1),48.062,48.062)
10	(N(66,1),3.96,3.96)	(N(370,1),18.5,18.5)	(N(49.4,1),2.964,2.964)	(N(2.86,1),0.0572,0.0572)	(N(6.47,1),0.3882,0.3882)	(N(38.49,1.1547,1.1547))
11	(N(1434,1),71.7,71.7)	(N(70,1),2.8,2.8)	(N(104.18,1),1.0418,1.0418)	(N(0,1),0,0)	(N(66.69,1),5.3352,5.3352)	(N(637.08,1),19.1124,19.1124)
12	(N(0,1),0,0)	(N(8,1),0.48,0.48)	(N(25.2,1),0.756,0.756)	(N(63.88,1),2.5552,2.5552)	(N(10.28,0.3084,0.3084))	(N(101.4,5.07,5.07))
13	(N(218,1),10.9,10.9)	(N(241,1),7.23,7.23)	(N(147.37,2.9474,2.9474))	(N(0.75,1),0.03,0.03)	(N(16.42,1),0.821,0.821)	(N(70.63,1),4.2378,4.2378)
14	(N(2,1),0.14,0.14)	(N(18,1),0.9,0.9)	(N(29,1),1.45,1.45)	(N(24.04,1),0.7212,0.7212)	(N(17.31,1),1.2117,1.2117)	(N(164.2,1),8.21,8.21)
15	(N(424,1),25.44,25.44)	(N(136,1),2.72,2.72)	(N(1.1,1),0.066,0.066)	(N(0,1),0,0)	(N(11.64,0.582,0.582))	(N(164.4,1),6.576,6.576)
16	(N(0,1),0,0)	(N(4,1),0.32,0.32)	(N(32.4,1),1.944,1.944)	(N(2.3,1),0.092,0.092)	(N(5.09,0.1018,0.1018))	(N(38.6,1),2.702,2.702)
17	(N(640,1),19.2,19.2)	(N(36,1),1.08,1.08)	(N(91.38,1),3.6552,3.6552)	(N(0,1),0,0)	(N(23.82,1.9056,1.9056))	(N(189.33,1),13.2531,13.2531)
18	(N(926,1),37.04,37.04)	(N(89,1),4.45,4.45)	(N(177.05,1),5.3115,5.315)	(N(0,1),0,0)	(N(59.52,1),1.7856,1.7856)	(N(614.2,42.994,42.994))
19	(N(9580,1),670.6,670.6)	(N(1593,1),143.37,143.37)	(N(146.75,1),1.4675,1.4675)	(N(0,1),0,0)	(N(14.67,1),0.2934,0.2934)	(N(123.82,1),4.9528,4.9528)
20	(N(392,1),3.92,3.92)	(N(699,1),55.92,55.92)	(N(4.1,1),0.123,0.123)	(N(0,1),0,0)	(N(0.79,1),0.0316,0.0316)	(N(7.61,1),0.5327,0.5327)
21	(N(1274,1),50.96,50.96)	(N(156,1),7.8,7.8)	(N(40.41,1),2.4246,2.4246)	(N(0,1),0,0))	(N(33.72,1),0.3372,0.3372)	(N(445.98,1),35.6784,35.6784)
22	(N(2880,1),230.4,230.4)	(N(395,1),23.7,23.7)	(N(193.12,1),3.8624,3.8624)	(N(80.5,1),1.61,1.61)	(N(88.68,1),4.434,4.434)	(N(797.86,63.8288,63.8288))
23	(N(968,1),87.12,87.12)	(N(679,1),61.11,61.11)	(N(239.5,1),7.185,7.185)	(N(16.61,0.1661,0.1661))	(N(73.9,1),4.434,4.434)	(N(757.8,1),53.046,53.046)
24	(N(0,1),0,0)	(N(92,1),7.36,7.36)	(N(14.3,1),1.001,1.001)	(N(0,1),0,0)	(N(3.48,1),0.174,0.174)	(N(17.69,1),1.0614,1.0614)
25	(N(221,1),17.68,17.68)	(N(1421,1),113.68,113.68)	(N(25.2,1),1.26,1.26)	(N(0,1),0,0)	(N(12.42,1),0.1242,0.1242)	(N(38.45,1),1.1535,1.1535)
26	(N(463,1),18.52,18.52)	(N(25,1),1,1)	(N(448.4,1),31.388,31.388)	(N(0,1),0,0)	(N(128.57,1),7.7142,7.7142)	(N(1262.4,1),37.872,37.872)
27	(N(1726,1),34.52,34.52)	(N(254,1),10.16,10.16)	(N(177.05,1),3.541,3.541)	(N(0,1),0,0)	(N(59.52,1),2.976,2.976)	(N(614.2,6.142,6.142))
28	(N(844,1),33.76,33.76)	(N(112,1),4.48,4.48)	(N(17.9,1),0.179,0.179)	(N(0,1),0,0)	(N(47.43,1),1.8972,1.8972)	(N(665.7,1),33.285,33.285)
29	(N(1270,1),101.6,101.6)	(N(603,1),54.27,54.27)	(N(55.85,1),3.351,3.351)	(N(0.3,0.012,0.012)	(N(8.33,1),0.1666,0.1666)	(N(41.54,1),2.4924,2.4924)
30	(N(0,1),0,0)	(N(463,1),18.52,18.52)	(N(13.45,1),0.9415,0.9415)	(N(0,1),0,0)	(N(21.02,1),0.6306,0.6306)	(N(180.78,1),3.6156,3.6156)
31	(N(0,1),0,0)	(N(69,1),2.07,2.07)	(N 14,1),0.14,0.14)	(N(95.2,1),3.808,3.808)	(N(16.39,1),0.3278,0.3278)	(N(199.7,1),5.991,5.991)
32	(N(7,1),0.56,0.56)	(N(19,1),1.14,1.14)	(N(2.28,1),0.0228,0.0228)	(N(17.52,1),0.7008,0.7008)	(N(10.92,1),0.546,0.546)	(N(132.61,1),10.6088,10.6088)
33	(N(107,1),4.28,4.28)	(N(171,1),3.42,3.42)	(N(123.73,1),4.9492,4.9492)	(N(0,1),0,0)	(N(25.9,1),0.518,0.518)	(N(211,1),4.22,4.22)
34	(N(62,1),3.72,3.72)	(N(443,1),26.58,26.58)	(N(53.72,1),2.1488,2.1488)	(N(230.23,1),4.6046,4.6046)	(N(27.88,1),1.6728,1.6728)	(N(305.4,1),18.324,18.324)
35	(N(0,1),0,0)	(N(257,1),12.85,12.85)	(N(150.89,1),1.5089,1.5089)	(N(80.46,1),3.2184,3.2184)	(N(22.54,1),0.6762,0.6762)	(N(187.65,1),1.8765,1.8765)
36	(N(0,1),0,0)	(N(4,1),0.12,0.12)	(N(25.2,1),0.504,0.504)	(N(60.7,1),2.428,2.428)	(N(6.54,1),0.3924,0.3924)	(N(53.3,1),4.264,4.264)
37	(N(2464,1),197.12,197.12)	(N(156,1),9.36,9.36)	(N(80.41,1),4.8246,4.8246)	(N(0,1),0,0)	(N(17.35,1),0.1735,0.1735)	(N(101.62,1),3.0486,3.0486)
38	(N(1,1),0.08,0.08)	(N(348,1),20.88,20.88)	(N(13.62,1),0.4086,0.4086)	(N(63.59,1),1.2718,1.2718)	(N(34.69,1),2.4283,2.4283)	(N(347.88,1),24.3516,24.3516)
39	(N(6,1),0.49,0.49)	(N(155,1),12.4,12.4)	(N(8.27,1),0.2481,0.2481)	(N(7.89,1),0.3156,0.3156)	(N(1.48,1),0.0444,0.0444)	(N(11.16,1),0.2232,0.2232)
40	(N(2668,1),106.72,106.72)	(N(2373,1),189.84,189.84)	(N(988.76,1),59.3256,59.3256)	(N(0,1),0,0)	(N(145.27,1),2.9054,2.9054)	(N(830.56,1),41.528,41.528)

Table 3
The stochastic fuzzy efficiency scores.

DMUS	Probability-possibility			Probability-necessity			Probability-credibility		
	($\gamma = 0.95, \delta = 0.6$)	($\gamma = 0.75, \delta = 0.6$)	($\gamma = 0.95, \delta = 0.4$)	($\gamma = 0.95, \delta = 0.6$)	($\gamma = 0.75, \delta = 0.6$)	($\gamma = 0.95, \delta = 0.4$)	($\gamma = 0.95, \delta = 0.6$)	($\gamma = 0.75, \delta = 0.6$)	($\gamma = 0.95, \delta = 0.4$)
1	0.2835	0.2762	0.2884	0.287	0.3024	0.2839	0.2640	0.2773	0.2517
2	0.4019	0.4452	0.4555	0.4453	0.4389	0.4518	0.4060	0.3168	0.4177
3	1.0998	1.0998	1.1535	1.1146	1.1146	1.1382	0.8932	0.8789	1.0487
4	0.3041	0.7950	0.6088	0.3446	0.8038	0.5858	0.8237	0.8695	0.1448
5	0.1235	0.4721	0.1787	0.1232	0.4777	0.1762	1.0930	0.9981	0.0758
6	0.4074	0.4238	0.4251	0.4124	0.4279	0.3823	0.3633	0.2930	0.3699
7	0.2887	0.2895	0.3022	0.2927	0.309	0.304	0.2662	0.2486	0.2676
8	0.4056	0.4056	0.4246	0.4093	0.4093	0.4207	0.2644	0.2065	0.3874
9	0.3994	0.4263	0.379	0.4011	0.4309	0.3865	0.3626	0.3695	0.3833
10	0.0843	0.1955	0.1247	0.0851	0.1975	0.1235	0.2523	0.2171	0.0587
11	0.8939	1.0636	0.9942	0.9161	1.0492	0.9994	0.9848	0.9766	0.6896
12	0.8939	0.5182	0.9942	0.9161	0.498	0.9994	1.1280	1.0282	0.6896
13	0.1697	0.2892	0.2369	0.1716	0.3002	0.2341	0.3141	0.2916	0.1448
14	0.1697	0.9930	0.2369	0.1716	1.0021	0.2341	0.9746	0.9830	0.1448
15	0.2740	0.5143	0.3843	0.2753	0.5154	0.3852	1.1105	1.0181	0.1664
16	0.2740	0.2740	0.3843	0.2753	0.2767	0.3852	1.2599	1.0791	0.1664
17	0.3011	0.5470	0.4887	0.3042	0.5551	0.4703	0.7661	0.7282	0.1716
18	0.6610	0.6751	0.6402	0.684	0.7286	0.6344	0.7051	0.6958	0.5713
19	0.0298	0.0307	0.0268	0.0259	0.0321	0.0265	0.0328	0.0289	0.0282
20	0.0097	0.0168	0.013	0.0093	0.0158	0.0122	0.1596	0.0887	0.0073
21	0.4682	0.4770	0.4923	0.4721	0.4464	0.4869	0.4685	0.4615	0.4363
22	0.3674	0.3561	0.3739	0.3572	0.3572	0.3717	0.2980	0.1177	0.3392
23	0.4651	0.5027	0.4714	0.4983	0.4968	0.5089	0.4499	0.3589	0.4607
24	0.0580	0.2618	0.0846	0.0586	0.264	0.0837	1.3173	1.0848	0.4607
25	0.1497	0.2437	0.1754	0.1494	0.2543	0.1735	0.2726	0.2603	0.1423
26	0.9728	1.0920	1.096	1.0700	1.0402	1.1011	0.9735	0.9728	0.9404
27	0.3561	0.4199	0.3811	0.3681	0.4033	0.397	0.4239	0.4326	0.3651
28	0.9324	1.0205	1.046	0.9428	1.0046	1.0271	1.0122	0.9899	0.6159
29	0.059	0.0716	0.0612	0.0588	0.0741	0.0582	0.0932	0.0868	0.0598
30	0.2972	0.9450	0.4281	0.3001	0.9259	0.4249	1.0416	0.9967	0.1948
31	0.2972	0.5348	0.4028	0.3001	0.5411	0.3983	1.0939	1.0242	0.1948
32	0.2972	0.5348	0.4028	0.3001	0.5411	0.3983	1.1261	1.0301	0.1948
33	0.2951	0.6700	0.4411	0.298	0.6769	0.4369	0.6899	0.6864	0.1800
34	0.1776	0.1878	0.1897	0.1733	0.2088	0.1881	0.1728	0.1301	0.1727
35	0.2362	0.3287	0.2821	0.2389	0.3316	0.2581	0.3008	0.3770	0.2089
36	0.2362	0.2817	0.2821	0.2389	0.2842	0.2581	1.2142	1.0531	0.2089
37	0.1702	0.1947	0.1742	0.1671	0.1961	0.1743	0.2059	0.1958	0.1677
38	0.4995	0.9604	0.6128	0.508	0.954	0.6197	1.0056	0.9756	0.3731
39	0.0272	0.0626	0.0342	0.0267	0.0668	0.0349	0.2496	0.1651	0.0176
40	0.3041	0.3227	0.3089	0.3054	0.3264	0.3056	0.2602	0.2564	0.2862

cient at ($\gamma = 0.95, \delta = 0.6$) and ($\gamma = 0.75$, $\delta = 0.6$) levels, respectively. In comparison with the ($\gamma = 0.95, \delta = 0.6$) level of the probabilistic-necessity case, the efficiency score for 88% of the DMUs increased at ($\gamma = 0.75, \delta = 0.6$) level. In addition, when we compared the efficiency at ($\gamma = 0.95, \delta = 0.6$) and ($\gamma = 0.95, \delta = 0.4$) levels in the probabilistic-necessity model, the latter efficiency was often larger, similar to the probabilistic-possibilistic model. In the probability-credibility model, DMUs 5, 28, 30, and 38 were probabilistic-credibility efficient at ($\gamma = 0.95, \delta = 0.6$) while DMUs 12, 15, 16, 24, 31, 32 and 36 were probabilistic-credibility efficient at both ($\gamma = 0.95, \delta = 0.6$) and ($\gamma = 0.75, \delta = 0.6$) levels. As for ($\gamma = 0.95, \delta = 0.4$) level of the probability-credibility model, DMU 3 was considered probabilistic-credibility efficient compared with other units. When we compared the efficiency score of the probabilistic-credibility model at ($\gamma = 0.95, \delta = 0.6$) and ($\gamma = 0.75, \delta = 0.6$) levels, six DMUs at ($\gamma = 0.75, \delta = 0.6$) had higher efficiency while the efficiencies of nine DMUs at ($\gamma = 0.95, \delta = 0.4$) level were higher than the efficiencies for ($\gamma = 0.75, \delta = 0.6$) level.

The average efficiency scores and the final rankings of the 40 military bases are presented in Table 4. After a series of discussions with the top military officials in the 40 bases, the DMs used a 10% cutoff rule for identifying the military bases for closure and a 20% cutoff rule for identifying the military bases for realignment. As shown in Fig. 1 and Table 4, three military bases of Sheppard AFB (DMU 37), Otis Air National Guard Base (DMU 34), and Gen Mitchell International Airport ARS (DMU 10) where put on the military realignment list and four military bases of W.K. Kellogg Air Force Guard Station (DMU 39), Naval Support Activity, New Orleans (DMU 29), NAS Pensacola (DMU 20), and NAS Oceana (DMU 19) were placed on the military closure list in 2005.

Table 4
The overall rankings and final recommendations.

DMU	Military base	Average efficiency score	Overall ranking
3	Cannon Air Force Base	1.0601	1
26	Naval Medical Center Portsmouth	1.0288	2
28	Naval Station Pascagoula	0.9546	3
11	Grand Forks AFB	0.9519	4
12	Kansas Ammunition Plant	0.8517	5
38	Umatilla Army Depot	0.7232	6
18	NAS Corpus Chisti	0.6662	7
30	Naval Support Activity Crane	0.6171	8
4	Desecret Chemical Depot	0.5867	9
14	Lone Star Army Ammunition Plant	0.5455	10
32	Newport Chemical Depot	0.5361	11
31	Naval Weapons Stations Seal Beach	0.5319	12
15	McChord AFB	0.5159	13
16	Mississippi Army Ammunition Plant	0.4861	14
33	Onizuka Air Force Station	0.4860	15
17	Mountain Home AFB	0.4814	16
23	Naval Air Station Williow Grove	0.4681	17
21	Naval Air Station Atlanta	0.4677	18
36	Riverbank Army Ammunition Plant	0.4508	19
2	Brooks City Base	0.4199	20
5	Eielson, AFB	0.4131	21
24	Naval Base Coranado	0.4082	22
27	Naval Station Ingleside	0.3941	23
9	Fort Monroe	0.3932	24
6	Fort Gillem	0.3895	25
8	Fort Monmouth	0.3704	26
22	Naval Air Station Brunswick	0.3265	27
40	Walter Reed National Military Medical Center	0.2973	28
7	Fort McPherson	0.2854	29
35	Red River Army Depot	0.2847	30
1	Army Reserve Personnel Center St. Louis	0.2794	31
13	Kulis Air Guard Station	0.2391	32
25	Naval Base Ventura City	0.2024	33
37	Sheppard AFB	0.1829[a]	34
34	Otis Air National Guard Base	0.1779[a]	35
10	Gen Mitchell International Airport ARS	0.1487[a]	36
39	W.K. Kellogg Air Force Guard Station	0.0761[b]	37
29	Naval Support Activity, New Orleans	0.0692[b]	38
20	NAS Pensacola	0.0369[b]	39
19	NAS Oceana	0.0291[b]	40

[a] Realignment decision.
[b] Closure decision.

7. Conclusions and future research directions

Conventional DEA is based on classic production theory, where identified resources are transformed into desired products and services with unobserved technologies. All the quanta are deterministic, as is the resulting production frontier in itself. Radial or additive distance measures naturally provide valuable insights for predicting individual and sectorial changes of processes, product profiles and management skills. However, following the emergence of DEA as a general performance assessment method, with applications beyond the classical settings of neoclassical production theory, new challenges arise. Evaluation of organizational behavior, medical and political options and socio-economic instruments may also benefit from non-parametric distance functions, but the underlying assumptions are no longer true. The inputs of such processes are frequently intangible, vague or uncertain, based on partial or estimated data, opinions and distributions. Likewise, the outcomes of these complex processes are often uncertain, random and only partially preferentially defined. Far from being a dichotomous choice between a deterministic and stochastic world, the real decision makers face hybrid situations where fuzziness, uncertainty and randomness coincide in the same problems.

In this paper, we proposed three fuzzy DEA models with respect to probability-possibility, probability-necessity and probability-credibility constraints. We consolidate earlier work on integration of the possibility, necessity and credibility constraints in the DEA model with taking into account also the probability constraints.

A case study for BRAC decision at the DoD illustrates how a complex socio-economic problem with multiple stakeholders, multiple resources and multiple fuzzy desirable or undesirable consequences can be addressed using the three models to inform decision makers about the relative merits of candidates for restructuring. The added nuance of the probability-uncertainty triptych provides the essential information needed to clarify critical discussions, without resorting to the scalar simplifications of conventional DEA. That being said, we repeat our earlier disclaimer about the normative value of non-parametric models in general for this kind of complex decision analysis.

Future work is needed to investigate the respective properties and relations of the three elements in the triptych, as well as the interpretations of the derived performance metrics. In particular, the relationships to the stochastic DEA models could be an interesting path of further work..

8. Disclaimer

The views expressed in this paper are those of the authors and do not reflect the official policy or position of the United States Department of Defense.

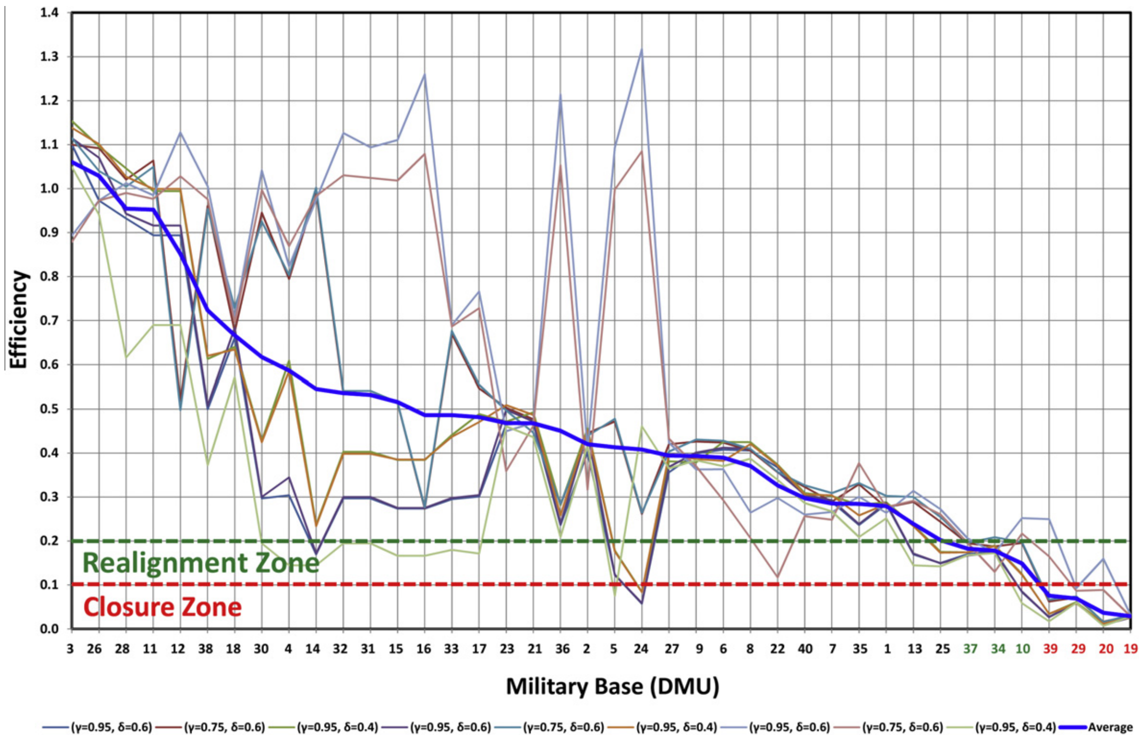

Fig. 1. The final recommendation.

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Appendix A. Military bases selected for this study

DMU	Military base	Branch	State
1	Army Reserve Personnel Center St. Louis	Army	Missouri
2	Brooks City Base	Air Force	Texas
3	Cannon Air Force Base	Air Force	New Mexico
4	Desecret Chemical Depot	Army	Utah
5	Eielson, AFB	Air Force	Alaska
6	Fort Gillem	Army	Georgia
7	Fort McPherson	Army	Georgia
8	Fort Monmouth	Army	New Jersey
9	Fort Monroe	Army	Virginia
10	Gen Mitchell International Airport ARS	Air Force	Wisconsin
11	Grand Forks AFB	Air Force	North Dakota
12	Kansas Ammunition Plant	Army	Kansas
13	Kulis Air Guard Station	Air Force	Alaska
14	Lone Star Army Ammunition Plant	Army	Texas
15	McChord AFB	Air Force	Washington

Military bases selected for this study (*continued*)

DMU	Military base	Branch	State
16	Mississippi Army Ammunition Plant	Army	Mississippi
17	Mountain Home AFB	Air Force	Idaho
18	NAS Corpus Chisti	Navy	Texas
19	NAS Oceana	Navy	Virginia
20	NAS Pensacola	Navy	Florida
21	Naval Air Station Atlanta	Navy	Georgia
22	Naval Air Station Brunswick	Navy	Maine
23	Naval Air Station Williow Grove	Navy	Pennsylvania
24	Naval Base Coranado	Navy	California
25	Naval Base Ventura City	Navy	California
26	Naval Medical Center Portsmouth	Navy	Virginia
27	Naval Station Ingleside	Navy	Texas
28	Naval Station Pascagoula	Navy	Mississippi
29	Naval Support Activity, New Orleans	Navy	Louisiana
30	Naval Support Activity Crane	Navy	Indiana
31	Naval Weapons Stations Seal Beach	Navy	California
32	Newport Chemical Depot	Army	Indiana
33	Onizuka Air Force Station	Air Force	California
34	Otis Air National Guard Base	Air Force	Massachusetts
35	Red River Army Depot	Army	Texas
36	Riverbank Army Ammunition Plant	Army	California
37	Sheppard AFB	Air	Texas

Military bases selected for this study (*continued*)

DMU	Military base	Branch	State
		Force	
38	Umatilla Army Depot	Army	Oregon
39	W.K. Kellogg Air Force Guard Station	Air Force	Michigan
40	Walter Reed National Military Medical Center	Army	DC

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