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A robust optimization approach for imprecise data envelopment analysis ☆

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ABSTRACT

Crisp input and output data are fundamentally indispensable in traditional data envelopment analysis (DEA). However, the input and output data in real-world problems are often imprecise or ambiguous. Some researchers have proposed interval DEA (IDEA) and fuzzy DEA (FDEA) to deal with imprecise and ambiguous data in DEA. Nevertheless, many real-life problems use linguistic data that cannot be used as interval data and a large number of input variables in fuzzy logic could result in a significant number of rules that are needed to specify a dynamic model. In this paper, we propose an adaptation of the standard DEA under conditions of uncertainty. The proposed approach is based on a *robust optimization* model in which the input and output parameters are constrained to be within an *uncertainty set* with additional constraints based on the worst case solution with respect to the uncertainty set. Our robust DEA (RDEA) model seeks to maximize efficiency (similar to standard DEA) but under the assumption of a worst case efficiency defied by the uncertainty set and it's supporting constraint. A Monte-Carlo simulation is used to compute the conformity of the rankings in the RDEA model. The contribution of this paper is fourfold: (1) we consider ambiguous, uncertain and imprecise input and output data in DEA; (2) we address the gap in the imprecise DEA literature for problems not suitable or difficult to model with interval or fuzzy representations; (3) we propose a robust optimization model in which the input and output parameters are constrained to be within an uncertainty set with additional constraints based on the worst case solution with respect to the uncertainty set; and (4) we use Monte-Carlo simulation to specify a range of Gamma in which the rankings of the DMUs occur with high probability.

1. Introduction

DEA is a methodology for evaluating and measuring the relative efficiencies of a set of decision making units (DMUs) that use multiple inputs to produce multiple outputs. The DEA method is based on the economic notion of Pareto optimality, which states that a DMU is considered to be inefficient if some other DMUs can produce at least the same amount of output with less of the same input and not more of any other inputs. Otherwise, a DMU is considered to be Pareto efficient. Due to its solid underlying mathematical basis and wide applications to real-world problems, much effort has been devoted to the DEA models since the pioneering work of Charnes, Cooper, and Rhodes (1978).

In the conventional DEA, all the data assume the form of specific numerical values. However, the observed values of the input and output data in real-life problems are sometimes imprecise or vague. The imprecise or vague data in the DEA models have been examined in the literature in different ways. Some DEA applications propose the exclusion of the units that have imprecise or vague values from the analysis (O'Neal, Ozcan, & Yanqiang, 2002). This approach is not suitable for DEA as it affects the efficiency of the other DMUs due to the comparative evaluation which may possibly disturb the statistical properties of the relative efficiencies of the DMUs (Simar & Wilson, 2000). Other approaches use imputation techniques to estimate the exact approximations of the imprecise or vague values. The imputation techniques used in DEA may lead to misleading efficiency results because of the stability problems where a unit accepting an infinitesimal perturbation may change its classification from an efficient to an inefficient status or vice versa (Cooper, Seiford, & Tone, 1999).

The stochastic approach is also used to model uncertainty in the DEA literature. This approach involves specifying a probability distribution function (e.g., normal) for the error process (Sengupta,

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1992). However, as pointed out by Sengupta (1992), the stochastic approach has two drawbacks:

(a) Small sample sizes in DEA make it difficult to use stochastic models.
(b) In stochastic approaches, the decision maker is required to assume a specific error distribution (e.g., normal or exponential) to derive specific results. However, this assumption may not be realistic because on an *a priori* basis there is very little empirical evidence to choose one type of distribution over another.

More recently, the imprecise or vague data are expressed by two approaches; the interval DEA first proposed by Cooper, Park, and Yu (1999) and the fuzzy DEA first proposed by Sengupta (1992). Cooper, Park, et al. (1999) has developed an interval approach that permits mixtures of imprecise and precise data by transforming the DEA model into an ordinary linear programming (LP) form. One of the difficulties in the interval approach is the evaluation of the lower and upper bounds of the relative efficiencies of the DMUs. In spite of this difficulty, several researchers have proposed different variations of the interval approach (Despotis & Smirlis, 2002; Entani, Maeda, & Tanaka, 2002; Kao, 2006; Kao & Liu, 2000; Wang, Greatbanks, & Yang, 2005). Despotis and Smirlis (2002) have developed an interval approach for dealing with imprecise data in DEA by transforming a non-linear DEA model to an LP equivalent. The upper and lower bounds for the efficiency scores of the DMUS are defined. They use a post-DEA model and the endurance indices to discriminate among the efficient DMUs. They further formulate another post-DEA model to determine input thresholds that turn an inefficient DMU into an efficient one.

The concerns related to the lack of robustness of the efficiency frontier and the probabilistic feasibility of the inequality constraints in DEA motivated Sengupta (1992) to propose a fuzzy approach and use a fuzzy linear programming transformation as a viable approach in such situations. In the fuzzy approach, several fuzzy mathematical programming approaches are proposed such as possibilistic programming and α-cut approaches to assess the relative efficiency of the DMUs (Guo & Tanaka, 2001; Lertworasirikul, Fang, Joines, & Nuttle, 2003; León, Liern, Ruiz, & Sirvent, 2003; Saati, Memariani, & Jahanshahloo, 2002). However, sometimes the complexity of the fuzzy approach can grow exponentially. Soleimani-damaneh, Jahanshahloo, and Abbasbandy (2006) have addressed the pitfalls of some fuzzy DEA models in the literature.

Lertworasirikul et al. (2003) have proposed a possibility approach to the treatment of various fuzzy DEA models. However, Soleimani-damaneh et al. (2006) showed that their model results in unbounded optimal values and has limited applicability in real-world problems. In another paper, Guo and Tanaka (2001) introduced an α-cut based approach that changed a fuzzy DEA model to a bi-level LP model. Soleimani-damaneh et al. (2006) showed that their model cannot be generalized as the provided model has an optimal solution under a specific restrictive condition. In spite of the concerns raised by Guo and Tanaka (2008) and Soleimani-damaneh et al. (2006) used the fuzzy DEA model proposed by Guo and Tanaka (2001) and introduced a fuzzy aggregation framework for integrating multiple attribute fuzzy values. Furthermore, Guo (2009) used the model proposed by Guo and Tanaka (2001, 2008) in a case study for a restaurant location problem in China. Kao and Liu (2000) proposed a technique which transforms a fuzzy DEA model into a family of crisp DEA models by applying the α-cut approach. Their technique requires solving multiple LP problems to approximate the membership function of the efficiency measure and to assess a DMU. Soleimani-damaneh et al. (2006) show their model is computationally expensive. This considerable shortcoming holds for some other fuzzy DEA models (Guo & Tanaka, 2001; Jahanshahloo, Soleimani-damaneh, & Nasrabadi, 2004; León et al., 2003).

Liu (2008) developed a fuzzy DEA model to find the efficiency measures embedded with the assurance region (AR) concept. He applied an alpha-cut approach and Zadeh's extension principle to transform the fuzzy DEA/AR model into a pair of parametric mathematical programs in order to work out the lower and upper bounds of the efficiency scores of the DMUs. The membership function of efficiency was approximated by using different possibility levels. Jahanshahloo, Sanei, Rostamy-Malkhalifeh, and Saleh (2009) commented on the fuzzy DEA model proposed by Liu (2008) and corrected the proof of his theorem. Liu and Chuang (2009) further used the fuzzy DEA/AR model suggested by Liu (2008) to evaluate the performance of 24 university libraries in Taiwan. Soleimani Damaneh (2008) used a fuzzy signed distance and fuzzy upper bound concepts to formulate a fuzzy additive model in DEA with fuzzy input–output data. Soleimani-damaneh (2009) put forward a theorem on the fuzzy DEA model proposed by Soleimani Damaneh (2008) to show the existence of a distance-based upper bound for the objective function of the model. Hatami-Marbini and Saati (2009) proposed a fuzzy DEA model to assess the efficiency scores in fuzzy environments. They applied the proposed fuzzy number ranking method proposed by Asady and Zendehnam (2007) and obtained the precise efficiency scores at sixteen bank branches in Iran. Wang, Luo, and Liang (2009) proposed two fuzzy DEA models with fuzzy inputs and outputs by means of fuzzy arithmetic. They converted each proposed fuzzy model into three linear programming models in order to calculate the efficiencies of the DMUs as fuzzy numbers and rank them. Although these studies have made great strides in DEA research, none of them address the gap in the imprecise DEA literature for problems not suitable or difficult to model with interval or fuzzy representations.

In this paper, we propose a robust optimization method for dealing with data uncertainties that covers the interval approach results with less complexity than the fuzzy approach. This method is based on the adaptation of recently developed robust optimization approaches proposed by Ben-Tal and Nemirovski (2000) and Bertsimas, Pachamanova, and Sim (2004). Robust optimization was first introduced by Soyster (1973) who discussed, in a very specific setting, uncertain hard constraints in linear programming models. This topic was widely discussed successively by Ben-Tal and Nemirovski (1998, 1999) who proved, in relation to some specified uncertain data sets, that the robust counterpart of convex programming is a computationally solvable optimization problem. An additional attempt was taken by Bertsimas and Sim (2004), El-Ghaoui and Lebret (1997) and El-Ghaoui, Oustry, and Lebret (1998) to further develop a theory for robust optimization. Sadjadi and Omrani (2008) have proposed a robust DEA model with consideration of uncertainty on output parameters for the performance assessment of electricity distribution companies. They show their robust DEA approach is a more usable method for ranking alternative strategies compared to the existing DEA methods.

This paper is organized into five sections. In Section 2, we present the fundamentals of the DEA model with precise and interval data. In Section 3, we illustrate the mathematical details of the proposed robust DEA framework. In Section 4, we demonstrate some attractive features of the proposed model with experimental results. Finally, in Section 5, we sum up our conclusions and future research directions.

2. The DEA model with precise and interval data

In this section, we present the basic concepts of the DEA model with precise and interval data. Let us assume that n DMUs convert m inputs into s outputs. The following procedure is used to obtain the relative efficiency of each DMU. Suppose x_{ij} ($i = 1, \dots, m, j = 1, \dots, n$) and y_{rj} ($r = 1, \dots, s, j = 1, \dots, n$) are the ith input ant the rth output of DMU$_j$, respectively. The relative efficiency of DMU$_p$, $p \in \{1, \dots, n\}$, is defined as the maximum value of θ_p and can be obtained by using the following linear programming (LP) model (called CCR) proposed by Charnes et al. (1978):

$$\begin{aligned} \max \quad & \theta_p = \sum_{r=1}^{s} u_r y_{rp}, \\ \text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{ip} = 1, \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leqslant 0, \quad \forall j, \\ & u_r, v_i \geqslant \varepsilon, \quad \forall r, i. \end{aligned} \tag{1}$$

where v_i and u_r are the input and output weights assigned to the ith input and rth output and $\varepsilon > 0$ is a non-Archimedean element smaller than any positive real number.

Several researchers have attempted to determine the exact or an interval value for the non-Archimedean ε (e.g., Ali & Seiford, 1993; Alirezaee, 2005; Alirezaee & Khalili, 2006; Amin & Toloo, 2004, 2007; Jahanshahloo & Khodabakhshi, 2004; Mehrabian, Jahanshahloo, Alirezaee, & Amin, 2000; Mirhassani & Alirezaee, 2005; Shang & Sueyoshi, 1995; Tone, 1993). Amin and Toloo (2004) have presented an algorithm for computing the Archimedian infinitesimal ε in DEA models. Alirezaee (2005) have proposed an algorithm for determining the overall assurance interval of the non-Archimedean ε in the DEA models. Ali and Seiford (1993) have proposed an upper bound on ε for the feasibility of the multiplier side and boundedness of the envelopment side in the CCR and BCC models. Mehrabian et al. (2000) have shown that Ali and Seiford's (1993) bound of ε is invalid for feasibility (for the multiplier model) and invalid for boundedness (for the envelopment model) of the respective linear programming. They proposed a method for determining an assurance interval for the non-Archimedean ε. In order to obtain the value of ε, we apply Mehrabian et al.'s (2000) model to this paper, which is expressed as follows:

$$\begin{aligned} \max \quad & \varepsilon_p, \\ \text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{ip} = 1, \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leqslant 0, \quad \forall j, \\ & u_r, v_i \geqslant \varepsilon_p, \quad \forall r, i. \end{aligned} \tag{2}$$

where ε_p is a variable and ε_p^* is an optimal solution for DMU$_p$. $\varepsilon^* = \min\{\varepsilon_1^*, \dots, \varepsilon_n^*\}$ is an intersection of all ε_p^* for the assessment of all DMUs. The interval $[0, \varepsilon^*]$ is feasible and bounded for the multiplier and envelopment CCR model, respectively (see Mehrabian et al. (2000) for more details).

Crisp input and output data are essentially indispensable in the aforementioned DEA model. However, the input and output data in real-world problems are often imprecise or ambiguous and cannot be estimated exactly. Consider imprecise input and output levels, x_{ij} and y_{rj}, and are known to lie within the upper and lower bounds represented by the intervals $x_{ij} \in \left[x_{ij}^L, x_{ij}^U\right]$ and $y_{rj} \in \left[y_{rj}^L, y_{rj}^U\right]$, where $x_{ij}^L > 0$ and $y_{rj}^L > 0$. In this case, the relative efficiency of DMU$_p$ is an interval represented by an upper limit obtained from the optimistic viewpoint and a lower limit obtained from the pessimistic viewpoint. Despotis and Smirlis (2002) have proposed the following two LP formulations to generate a bounded interval $\left[\theta_p^L, \theta_p^U\right]$:

$$\begin{aligned} \max \quad & \theta_p^U = \sum_{r=1}^{s} u_r y_{rp}^U, \\ \text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{ip}^L = 1, \\ & \sum_{r=1}^{s} u_r y_{rp}^U - \sum_{i=1}^{m} v_i x_{ip}^L \leqslant 0, \\ & \sum_{r=1}^{s} u_r y_{rj}^L - \sum_{i=1}^{m} v_i x_{ij}^U \leqslant 0, \quad \forall j \neq p, \\ & u_r, v_i \geqslant \varepsilon, \quad \forall r, i. \end{aligned} \tag{3}$$

and

$$\begin{aligned} \max \quad & \theta_p^L = \sum_{r=1}^{s} u_r y_{rp}^L, \\ \text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{ip}^U = 1, \\ & \sum_{r=1}^{s} u_r y_{rp}^L - \sum_{i=1}^{m} v_i x_{ip}^U \leqslant 0, \\ & \sum_{r=1}^{s} u_r y_{rj}^U - \sum_{i=1}^{m} v_i x_{ij}^L \leqslant 0, \quad \forall j \neq p, \\ & u_r, v_i \geqslant \varepsilon, \quad \forall r, i. \end{aligned} \tag{4}$$

A careful examination of models (3) and (4) shows that $\theta_p^L \leqslant \theta_p^U$. Furthermore, we should note that analyzing the efficiency changes in the obtained interval $\left[\theta_p^L, \theta_p^U\right]$ is a complex process requiring sophisticated analysis. The method proposed by Despotis and Smirlis (2002) and other similar DEA methods with interval data have used θ_p^L and θ_p^U for the final decision. However, we have shown that the maximum conformity may not always occur in optimistic and pessimistic cases, and the analyst can search the maximum conformity in the rankings between θ_p^L and θ_p^U.

3. The robust DEA model

In this section, we present the mathematical details of the robust DEA model proposed in this paper. Let us consider the DMU$_j$ and assume that J_j^x and J_j^y are the index sets of the imprecise input and output values, respectively. Let us further consider parameters γ_j^x and γ_j^y, not necessarily integer, that assume values in the bounded intervals $\left[0, |J_j^x|\right]$ and $\left[0, |J_j^y|\right]$, where, $|\cdot|$ is the cardinal of a set. The role of the parameters γ_j^x and γ_j^y is to adjust the robustness of the proposed model against the conservatism level of the solution. In reality, it is unlikely that all of the imprecise or uncertain inputs and outputs ($i \in J_j^x$ and $r \in J_j^y$) will change simultaneously. Our intention is protection against changes in all $\left\lfloor\gamma_j^x\right\rfloor$ and $\left\lfloor\gamma_j^y\right\rfloor$ combinations, and changes in $x_{t^x j}$ and $y_{t^y j}$ by $\left(\gamma_j^x - \left\lfloor\gamma_j^x\right\rfloor\right)\left(x_{t_j^x j}^U - x_{t_j^x j}^L\right)$ and $\left(\gamma_j^y - \left\lfloor\gamma_j^y\right\rfloor\right)\left(y_{t_j^y j}^U - y_{t_j^y j}^L\right)$, respectively. In other words, we stipulate that only a subset of the input and output data should change to affect the solution that was defined by Bertsimas and Sim (2004). Then, considering the robust optimization approach in (3), we change it as follows:

$$\begin{aligned}
\max \quad & \theta_p = \sum_{r=1}^{s} u_r y_{rp}^U - \beta_p^y, \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{ip}^L + \beta_p^x = 1, \\
& \sum_{r=1}^{s} u_r y_{rj}^L - \sum_{i=1}^{m} v_i x_{ij}^U + \beta_j^y + \beta_j^x \leqslant 0, \quad \forall j \neq p, \\
& \theta_p \leqslant 1, \\
& u_r, v_i \geqslant \varepsilon, \quad \forall r, i.
\end{aligned} \tag{5}$$

We use the robust optimization approach to introduce $\beta_j^y\left(y, \gamma_j^y\right)$ and $\beta_j^x\left(x, \gamma_j^x\right)$ to move from the optimistic to the pessimistic viewpoint. In other words, these variables protect the constraints against data uncertainty and keep them feasible. If:

$$C_j^y = \left\{S_j^y \cup \left\{t_j^y\right\} \middle| S_j^y \subseteq J_j^y, \left|S_j^y\right| = \left\lfloor \gamma_j^y \right\rfloor, t_j^y \in J_j^y \setminus S_j^y\right\}$$

then:

$$\beta_j^y\left(y, \gamma_j^y\right) = \max_{C_j^y}\left\{\sum_{r \in S_j^y} u_r\left(y_{rj}^U - y_{rj}^L\right) + \left(\gamma_j^y - \left\lfloor \gamma_j^y \right\rfloor\right) u_{t_j^y}\left(y_{t_j^y j}^U - y_{t_j^y j}^L\right)\right\} \tag{6}$$

and, if:

$$C_j^x = \left\{S_j^x \cup \left\{t_j^x\right\} \middle| S_j^x \subseteq J_j^x, \left|S_j^x\right| = \left\lfloor \gamma_j^x \right\rfloor, t_j^x \in J_j^x \setminus S_j^x\right\},$$

then:

$$\beta_j^x\left(x, \gamma_j^x\right) = \max_{C_j^x}\left\{\sum_{i \in S_j^x} v_i\left(x_{ij}^U - x_{ij}^L\right) + \left(\gamma_j^x - \left\lfloor \gamma_j^x \right\rfloor\right) v_{t_j^x}\left(x_{t_j^x j}^U - x_{t_j^x j}^L\right)\right\} \tag{7}$$

Therefore, we may consider the non-linear effect of this approach in (5).

If γ_j^x and γ_j^y are chosen as an integer, this restriction is protected by:

$$\beta_j^y(y, \gamma_j^y) = \max_{\left\{S_j^y \middle| S_j^y \subseteq J_j^y, \left|S_j^y\right| = \left\lfloor \gamma_j^y \right\rfloor\right\}} \left\{\sum_{r \in S_j^y} u_r\left(y_{rj}^U - y_{rj}^L\right)\right\} \tag{8}$$

and

$$\beta_j^x\left(x, \gamma_j^x\right) = \max_{\left\{S_j^x \middle| S_j^x \subseteq J_j^x, \left|S_j^x\right| = \left\lfloor \gamma_j^x \right\rfloor\right\}} \left\{\sum_{i \in S_j^x} v_i\left(x_{ij}^U - x_{ij}^L\right)\right\} \tag{9}$$

If we consider $\gamma_j^y = 0$ and $\gamma_j^x = 0$ then $\beta_j^y\left(y, \gamma_j^y\right) = 0$ and $\beta_j^x\left(x, \gamma_j^x\right) = 0$. Hence, the constraints of (5) are equivalent to the constraints of (3) (optimistic viewpoint) and similarly, when $\gamma_j^x = \left|J_j^x\right|$ and $\gamma_j^y = \left|J_j^y\right|$ then $\beta_j^y\left(y, \gamma_j^y\right) = \sum_{r=1}^{s} u_r\left(y_{rj}^U - y_{rj}^L\right)$ and $\beta_j^x\left(x, \gamma_j^x\right) = \sum_{i=1}^{m} v_i\left(x_{ij}^U - x_{ij}^L\right)$, so that the constraints are equivalent to (4) (pessimistic viewpoint).

Therefore, by varying $\gamma_j^x \in \left[0, |J_j^x|\right]$ and $\gamma_j^y \in \left[0, |J_j^y|\right]$, we have the flexibility of adjusting the robustness of the method against the level of conservatism of the solution. In this paper, we use the proposition of Bertsimas and Sim (2004) to transform the non-linear effect into the following linear constraints:

$$\begin{aligned}
& \sum_{r=1}^{s} u_r y_{rj}^L - \sum_{i=1}^{m} v_i x_{ij}^U + z_j^y \gamma_j^y + z_j^x \gamma_j^x + \sum_{r=1}^{s} p_{rj} + \sum_{i=1}^{m} q_{ij} \leqslant 0 \\
& z_j^y + p_{rj} \geqslant u_r\left(y_{rj}^U - y_{rj}^L\right) \\
& z_j^x + q_{ij} \geqslant v_i\left(x_{ij}^U - x_{ij}^L\right) \\
& z_j^x, z_j^y, q_{ij}, p_{rj} \geqslant 0
\end{aligned} \tag{10}$$

Similarly, the other constraints and the objective function in (5) are changed. Finally, the robust DEA model (RDEA) is mathematically formulated as follows:

$$\begin{aligned}
\max \quad & \theta_p = \sum_{r=1}^{s} u_r y_{rp}^U - z_p^y \gamma_p^y - \sum_{r=1}^{s} p_{rp}, \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{ip}^L + z_p^x \gamma_p^x + \sum_{i=1}^{m} q_{ip} = 1, \\
& \sum_{r=1}^{s} u_r y_{rj}^L - \sum_{i=1}^{m} v_i x_{ij}^U + z_j^y \gamma_j^y + z_j^x \gamma_j^x \\
& \quad + \sum_{r=1}^{s} p_{rj} + \sum_{i=1}^{m} q_{ij} \leqslant 0, \quad \forall j \neq p, \ \text{(I)} \\
& z_j^y + p_{rj} \geqslant u_r\left(y_{rj}^U - y_{rj}^L\right), \quad \forall r, j, \ \text{(II)} \\
& z_j^x + q_{ij} \geqslant v_i\left(x_{ij}^U - x_{ij}^L\right), \quad \forall i, j, \ \text{(III)} \\
& \theta_p \leqslant 1, \\
& v_i, u_r \geqslant \varepsilon, \quad \forall i, r, \\
& z_j^x, z_j^y, q_{ij}, p_{rj} \geqslant 0, \quad \forall i, j, r.
\end{aligned} \tag{11}$$

To reduce the complexity, one can consider the simple substitution $z_j^x = z_j^y = z_j (j = 1, \ldots, n)$ to calculate the $\gamma_j (j = 1, \ldots, n)$ values. Therefore, by defining $\Gamma_j = \gamma_j^x + \gamma_j^y (j = 1, \ldots, n)$, (11) will become as follows:

$$\begin{aligned}
\max \quad & \theta_p = \sum_{r=1}^{s} u_r y_{rp}^U - z_p \gamma_p^y - \sum_{r=1}^{s} p_{rp}, \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{ip}^L + z_p \gamma_p^x + \sum_{i=1}^{m} q_{ip} = 1, \\
& \sum_{r=1}^{s} u_r y_{rj}^L - \sum_{i=1}^{m} v_i x_{ij}^U + z_j \Gamma_j \\
& \quad + \sum_{r=1}^{s} p_{rj} + \sum_{i=1}^{m} q_{ij} \leqslant 0, \quad \forall j \neq p, \\
& z_j + p_{rj} \geqslant u_r\left(y_{rj}^U - y_{rj}^L\right), \quad \forall r, j, \\
& z_j + q_{ij} \geqslant v_i\left(x_{ij}^U - x_{ij}^L\right), \quad \forall i, j, \\
& \theta_p \leqslant 1, \\
& v_i, u_r \geqslant \varepsilon, \quad \forall i, r, \\
& z_j, q_{ij}, p_{rj} \geqslant 0, \quad \forall i, j, r.
\end{aligned} \tag{12}$$

The obtained value for Γ_j represents a set of uncertain inputs and outputs which have the most effect in maximizing the objective function and it is obvious $\Gamma_j \in \left[0, \left|J_j^x\right| + \left|J_j^y\right|\right]$.

Model (12) is solved for different combinations of Γs, and values of weights and rankings for each DMU are saved by using the obtained values for θ. The DMU efficiencies obtained for each Gamma through the RDEA model may result in different rankings of the DMUs. In such cases, many analysts allow a decision maker to use his or her preferences in selecting a suitable Gamma. We utilize a graphical presentation of the results enhanced with a Monte-Carlo simulation to provide additional insight for making a final decision about Gamma and the overall rankings of the DMUs. For each input and output of DMUs, a number is randomly generated by using Monte-Carlo simulation, and by the resulting weights from $\sum_{r=1}^{s} u_r y_{rp} / \sum_{i=1}^{m} v_i x_{ip}$, efficiencies of DMUs are recalculated and ranked again. The conformity value between the rankings resulting from the mathematical model and the simulations is computed and analyzed.

We should note that, the complexity in model (11) arises from the level of conservatism in the input and output data which have to be discretely determined by the decision maker (or expert). In order to reduce this complexity, we assume that all Γ_j are equal to Γ. Consequently, the decision maker's intervention is not required to determine the level of conservatism (Γ_j). Furthermore, we add the constraint $\gamma_p^x + \gamma_p^y = \Gamma$ to model (13) because in model (12), γ_p^x and γ_p^y are the parameters indicating the level of conservatism of the inputs and outputs for the DMUs. However, γ_p^x and γ_p^y are decision variables in model (13) and according to the parameter Γ, their values indicate the total level of conservatism (both inputs and outputs). Consequently, the total level of conservatism (Γ) has to equal the sum of γ_p^x and γ_p^y. The constraint $\gamma_p^x \leqslant m$ is added because γ_p^x cannot be larger than the number of imprecise inputs (assuming m interval inputs). Similarly, the constraint $\gamma_p^y \leqslant s$ is added to control the number of imprecise outputs. As a result, we formulate the following non-linear model:

$$
\begin{aligned}
\max \quad & \theta_p = \sum_{r=1}^{s} u_r y_{rp}^U - z_p \gamma_p^y - \sum_{r=1}^{s} p_{rp}, \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{ip}^L + z_p \gamma_p^x + \sum_{i=1}^{m} q_{ip} = 1, \\
& \sum_{r=1}^{s} u_r y_{rj}^L - \sum_{i=1}^{m} v_i x_{ij}^U + z_j \Gamma \\
& \quad + \sum_{r=1}^{s} p_{rj} + \sum_{i=1}^{m} q_{ij} \leqslant 0, \quad \forall j \neq p, \\
& z_j + p_{rj} \geqslant u_r (y_{rj}^U - y_{rj}^L), \quad \forall r, j, \\
& z_j + q_{ij} \geqslant v_i (x_{ij}^U - x_{ij}^L), \quad \forall i, j, \\
& \gamma_p^x + \gamma_p^y = \Gamma, \\
& \theta_p \leqslant 1, \\
& \gamma_p^x \leqslant m, \\
& \gamma_p^y \leqslant s, \\
& \gamma_p^x, \gamma_p^y \geqslant 0, \\
& v_i, u_r \geqslant \varepsilon, \quad \forall i, r, \\
& z_j, q_{ij}, p_{rj} \geqslant 0, \quad \forall i, j, r.
\end{aligned}
\tag{13}
$$

Although, the performance of the DMUs could theoretically be determined with model (11), model (13) is more applicable to the real-world problems. Furthermore, although p_{rj} and q_{ij} may take different values in models (11) and (12), the final results remain without change. When the level of conservatism takes the value 0 (the minimum value), the values of z_j^x and z_j^y are equivalent for the right hand side in constraints (II) and (III) in model (11), respectively. Therefore, $z_j = \max\left(z_j^y, z_j^x\right)$ and also $\Gamma = \gamma_j^y + \gamma_j^x = 0 + 0 = 0$, so that constraint (I) in model (11) and its counterpart in model (12) remains unchanged. However, if the conservatism level takes the maximum value ($z_j^x = z_j^y = z_j$), according to its objective function, p_{rj} and q_{ij} will take different values so that constraints (II) and (III) in model (11) become $z_j + p_{rj} = u_r\left(y_{rj}^U - y_{rj}^L\right)$ and $z_j + q_{ij} = v_i\left(x_{ij}^U - x_{ij}^L\right)$. Note that we used the model (2) for determining the value of ε^* in model (13). Thus, at first the value of $\varepsilon_p^{*(\Gamma)}$, $p \in \{1, \ldots, n\}$, is computed for each Γ and then $\varepsilon^* = \min\left\{\varepsilon_1^{*(\Gamma)}, \ldots, \varepsilon_n^{*(\Gamma)}\right\}$.

4. The experimental results

In this section, we will explain some attractive aspects of our framework through two examples. In the first example, we present a problem with five DMUs, one interval input and one interval output. Initially, we utilize a pictorial view of the problem followed by a solution based on (13) using the generalized algebraic modeling system (GAMS) for different combinations of Γ s. Then, we determine the optimal weight of each DMU and the overall ranking order of the DMUs based on the obtained θ_j values. Next, for each input and output of the DMUs, a random number is generated in its interval uniformly using a Monte-Carlo simulation as depicted in Fig. 1.

In the first phase, for each Gamma, we run model (13) and obtain the optimal weights and the overall rankings of the DMUs. In the second phase, we generate 10,000 random numbers between $\left[x_{ij}^L, x_{ij}^U\right]$ and $\left[y_{rj}^L, y_{rj}^U\right]$ for each Gamma. Following the random number generation, in the third phase, we first recalculate the efficiencies of the DMUs by using the optimal weights obtained in the first phase and the random numbers generated in the second phase from $\sum_{r=1}^{s} u_r y_{rp} / \sum_{i=1}^{m} v_i x_{ip}$. Next, we determine a new overall ranking based on the efficiencies of the DMUs obtained in the third phase. This results in two overall rankings for the DMUs, one from the model in phase 1 and another from the simulation in phase 3. For example, if DMU 1 is ranked first in both rankings from phase 1 and phase 3, we assign a 1 to this simulation run, otherwise, we assign a 0. We repeat this procedure for each Gamma 10,000 times and calculate the percent of conformity for each DMU and for each Gamma. For example, the conformity of DMU 1 is 0.9 (90%) for a given Gamma if in 9000 out of the 10,000 simulation runs DMU 1 is placed first in both rankings.

In the second example, a general (extended) problem is presented as given in Despotis and Smirlis (2002) and the above procedure is repeated to demonstrate some attractive aspects of our framework. Let us consider the first numerical example. Each input and output data are a variable which lies in the related interval presented in Table 1.

The feasible region of all data for each DMU is depicted in Fig. 2 as a square. The circular points (•) and the triangular points (▲) denote the best and worst situation for each DMU, respectively. Each DMU under consideration is analyzed with regards to an optimistic scenario and a pessimistic scenario. In the optimistic scenario, lines 1 and 2 are obtained as the frontiers when calculating the efficiency of the DMUs with model (3). Moreover, in the pessimistic scenario, lines 1 and 2 are obtained as the frontiers by calculating the efficiency of the DMUs with model (4).

Table 2 presents the best and the worst situation values for each DMU denoted by $\left(x_{1j}^L, y_{1j}^U\right)$ and $\left(x_{1j}^U, y_{1j}^L\right)$, respectively.

Consider the DMU_p evaluation. In the optimistic scenario, the best situation (circle point in Fig. 2) for DMU_p and the worst situation (triangle points in Fig. 2) for $\text{DMU}_j (j \neq p)$ are used by model (3) to form the production frontier for DMU_p. On the other hand, in the pessimistic scenario, the worst situation for DMU_p and the best situation for DMU_j $(j \neq p)$ are used by model (4) to form the production frontier for DMU_p.

In other words, in the optimistic scenario, the data sets {(20, 110), (70, 28), (10, 21), (19, 21), (6, 33)}, {(47, 25.5), (10, 55), (10, 21), (19, 21), (6, 33)}, {(47, 25.5), (70, 28), (6, 29), (19, 21), (6, 33)} and {(47, 25.5), (70, 28), (10, 21), (12, 25), (6, 33)} are used by model (3) to form the production frontier for DMU 1, DMU 2, DMU 3 and DMU 4, respectively, denoted by the dotted line 2 in Fig. 2. In addition, the data set {(47, 25.5), (70, 28), (10, 21), (19, 21), (5, 40)} is utilized by model (3) to form the production frontier for DMU 5 denoted by the dotted line 1 in Fig. 2.

On the other hand, in the pessimistic scenario, the data sets {(47, 25.5), (10, 55), (6, 29), (12, 25), (5, 40)}, {(20, 110), (70, 28), (6, 29), (12, 25), (5, 40)}, {(20, 110), (10, 55), (10, 21), (12, 25), (5, 40)} and {(20, 110), (10, 55), (6, 29), (19, 21), (5, 40)} are used by model

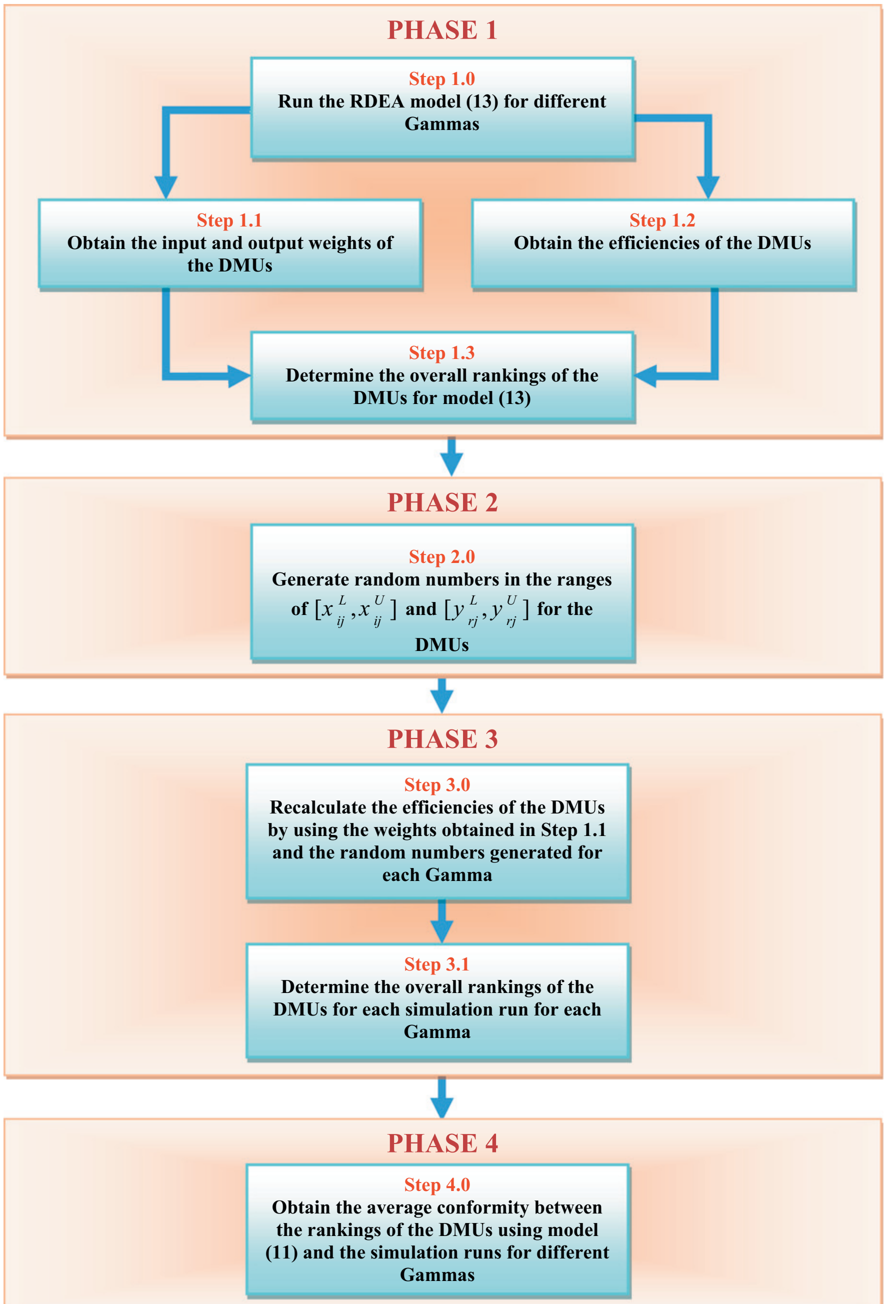

Fig. 1. The Monte-Carlo simulation model.

Table 1
The input and output values in example 1.

DMU$_j$	Inputs	Outputs
	X	Y
1	[20, 47]	[25.5, 110]
2	[10, 70]	[28, 55]
3	[6, 10]	[21, 29]
4	[12, 19]	[21, 25]
5	[5, 6]	[33, 40]

(4) to form the production frontier for DMU 1, DMU 2, DMU 3 and DMU 4, respectively, denoted by the dotted line 1 in Fig. 2. In addition, the data set {(20, 110), (10, 55), (6, 29), (12, 25), (6, 33)} is utilized by model (4) to form the production frontier for DMU 5 denoted by the dotted line 2 in Fig. 2. The results are summarized in Table 3.

Note that, as shown in Fig. 2, DMU 5 lies between the two aforementioned dotted lines. In other words, if we consider the worst situation for DMU 5 and the best situation for the other DMUs, DMU 5 is efficient. This can be also observed in Fig. 3 as the efficiency of DMU 5 is equal to unity for all the Γ value. The input and output levels of each DMU are uncertain so Γ can vary between 0 and 2. We increase Γ by 0.01 for each iteration in model (13) and achieve the θ values given in Fig. 3. Note that the value of ε in model (13) is 17×10^{-4}, which is obtained by using the Mehrabian et al.'s (2000) model.

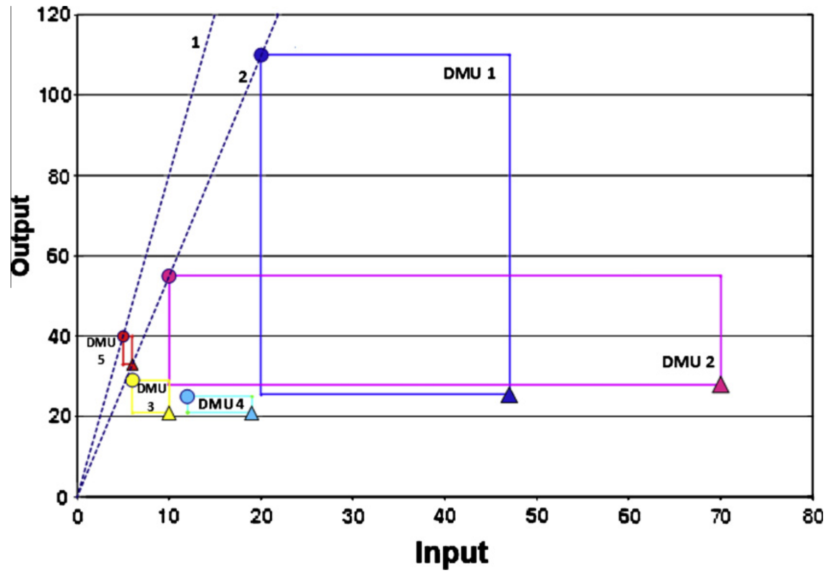

Fig. 2. The feasible regions for the four DMUs in example 1.

Table 2
The best and the worst situation values for each DMU.

DMU	The best situation	The worst situation
1	(20, 110)	(47, 25.5)
2	(10, 55)	(70, 28)
3	(6, 29)	(10, 21)
4	(12, 25)	(19, 21)
5	(5, 40)	(6, 33)

Table 3
The production frontiers in example 1.

DMU	1	2	3	4	5
Optimistic scenario	Line 2	Line 2	Line 2	Line 2	Line 1
Pessimistic scenario	Line 1	Line 1	Line 1	Line 1	Line 2

Now, we calculate the conformity between the results of the mathematical model (13) and the results of Monte-Carlo simulation for 10,000 simulation runs. The conformities values are presented in Fig. 4. As is shown in Fig. 4, the maximum conformity occurs in [0.37, 1.64] for Γ with the maximum average conformity of 69%. Therefore, we can conclude that the results based on the upper and lower bounds of efficiency do not necessarily have maximum conformity with reality; but specific values of Γ can maximize conformity and thus more authentic final rankings for the DMUs in this interval of Γ may be expected.

As was mentioned in the previous section, the proposed robust DEA model allows the input and output values to diverge in the related intervals with the least additional complexity in the model.

In the second example, we consider the problem introduced by Despotis and Smirlis (2002). The input and output values for this example are presented in Table 4.

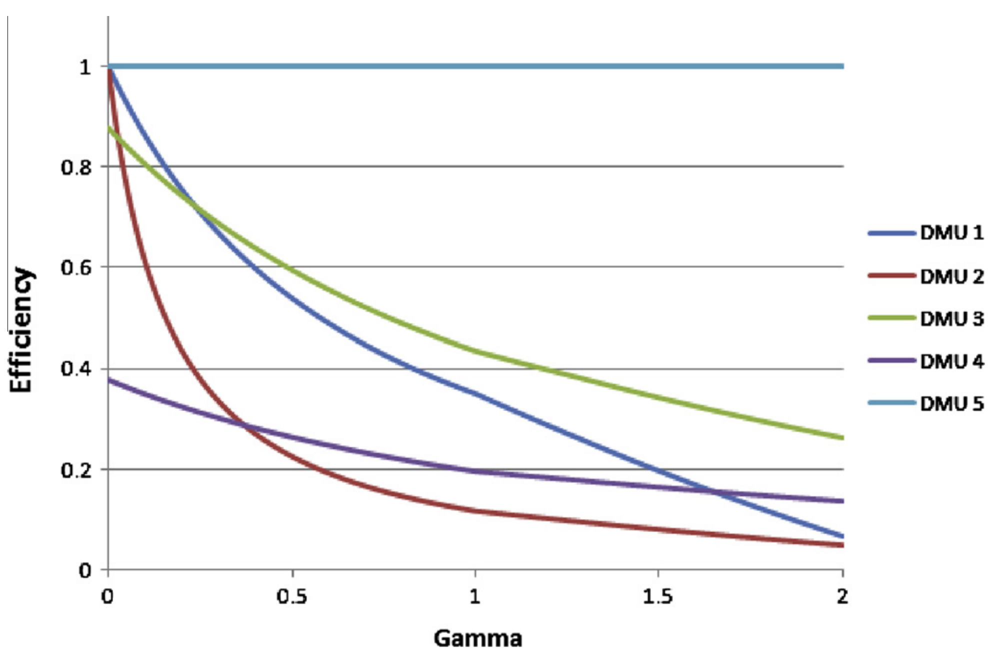

Fig. 3. The efficiencies of the five DMUs for different Γ in example 1.

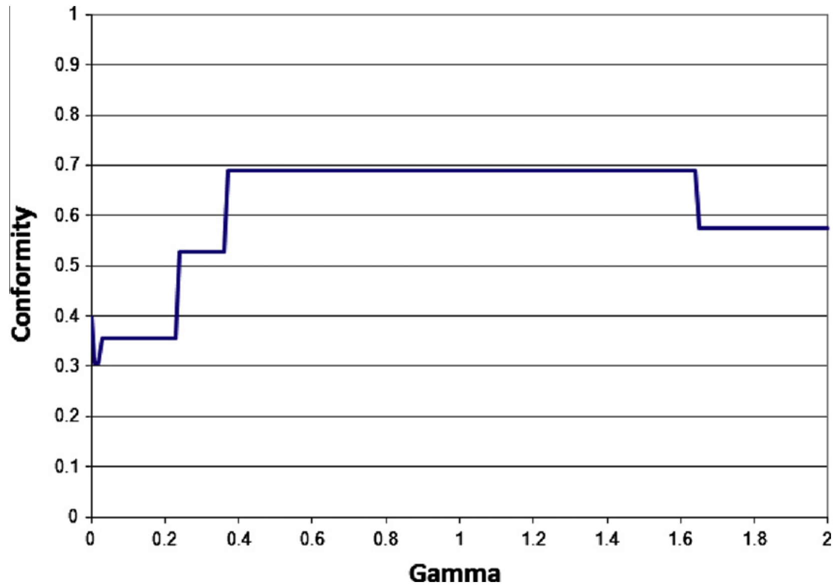

Fig. 4. The average conformity between the ranking of the DMUs using model (13) and the simulation runs for different Γ in example 1.

Table 4
The input and output values in example 2.

DMU_j	Inputs		Outputs	
	X_1	X_2	Y_1	Y_2
1	[14, 15]	[0.06, 0.09]	[157, 161]	[28, 40]
2	[4, 12]	[0.16, 0.35]	[157, 198]	[21, 29]
3	[10, 17]	[0.10, 0.70]	[143, 159]	[28, 35]
4	[12, 15]	[0.21, 0.48]	[138, 144]	[21, 22]
5	[19, 22]	[0.12, 0.19]	[158, 181]	[21, 25]

The Γ can vary in [0, 4] because each DMU has 2 uncertain inputs and 2 uncertain outputs. We solve model (13) for different Γ s, by increasing it by 0.01 in each step. The efficiencies of the DMUs are depicted in Fig. 5.

Next, we provide a comparison between the proposed RDEA and the method proposed by Despotis and Smirlis (2002). Let us consider models (2) and (3) proposed by Despotis and Smirlis (2002) and calculate the bounded interval $\left[\theta_p^L, \theta_p^U\right]$. The bounded interval results are presented in Table 5.

Despotis and Smirlis (2002) have proposed three classifications (E^{++}, E^{+} and E^{-}) with respect to the interval efficiencies of the DMUs:

I. If the lower efficiency score of a DMU is equal to unity, then, the DMU is classified as an under E^{++} DMU.
II. If the lower efficiency score of a DMU is smaller than unity and its upper efficiency score is equal to unity, then, the DMU is classified as an under E^{+} DMU.
III. If the lower efficiency score of a DMU is smaller than unity, then, the DMU is classified as an under E^{-} DMU.

Accordingly, DMU 1 is placed on the first position in the ranking order because this DMU is classified as an E^{++} DMU and it is efficient for all cases (any combination of the input/output data). In addition, they introduced two indices for ranking the DMUs in the E^{+} class. This results in the following ranking of the DMUs: DMU 1, DMU 3, DMU 2, DMU 4 and DMU 5.

Next, we applied the proposed approach to example 2. In addition, we used the Mehrabian et al.'s (2000) method to find the value 10^{-4} for ε used in this example. As shown in Table 6, when $\Gamma = 0$, the efficiencies of the DMUs are almost identical to the lower efficiency scores of Despotis and Smirlis (2002). On the other hand, when $\Gamma = 4$, the obtained efficiencies from the proposed approach are almost identical to the upper efficiencies scores of Despotis and Smirlis (2002). In our framework, the decision maker can examine the ranking of the DMUs for each Gamma whereas the method proposed by Despotis and Smirlis (2002) uses the lower efficiency score $\left(\theta_p^L\right)$ and the upper efficiency score $\left(\theta_p^U\right)$. In our approach, we use Monte-Carlo simulation to provide additional insight for making the final decision about Gamma and the overall rankings of the DMUs.

The conformity between the mathematical model and the Monte-Carlo simulation runs is shown in Fig. 6. The average ranking conformity in this figure show that the maximum conformity occurs while Γ is 1.65 and the maximum average conformity is 75% while for the optimistic and pessimistic approaches these values are 24% and 32%, respectively.

Table 7 presents the efficiencies of the DMUs for three cases: the optimistic case ($\Gamma = 0$), the maximum conformity case ($\Gamma = 1.65$), and the pessimistic case ($\Gamma = 4$). Furthermore, this table shows the average percentages of conformity for each case. The average conformity percent for $\Gamma = 1.65$ is more than twice that of the optimistic and pessimistic values. We should note that, in solving this non-linear model we did not encounter any difficulties. Moreover, we can separately enumerate γ_p^x and γ_p^y in large problems by solving the linear model.

5. Conclusions and future research directions

In the conventional DEA, all the data assume the form of specific numerical values. However, the observed values of the input and output data in real-life problems are sometimes imprecise or vague. The imprecise or vague data in the DEA models have been examined in the literature in different ways. The exclusion of the units with imprecise or vague values from the analysis, the imputation techniques to estimate the exact approximations of the imprecise or vague values, and the stochastic approach are among the methods most commonly used to model uncertainty in the DEA literature. More recently, the interval DEA and the fuzzy DEA are used to deal with the imprecise, vague or incomplete information in DEA. As we discussed in Section 1, each method has certain pitfalls that can present practical difficulties in applying DEA.

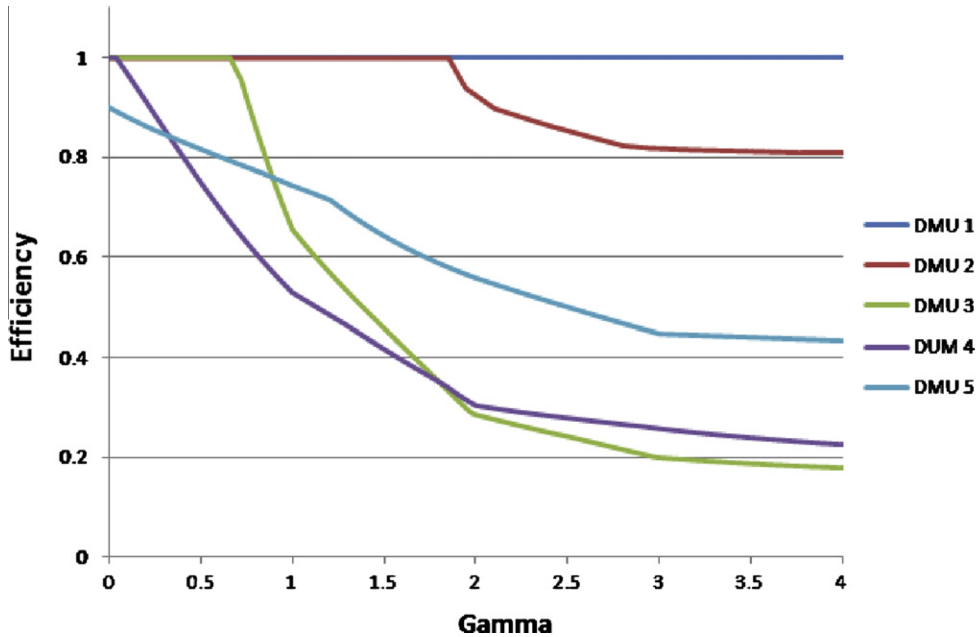

Fig. 5. The efficiencies of the five DMUs for different Γ in example 2.

Table 5
The results of Despotis and Smirlis (2002).

DMU	Lower efficiency score $\left(\theta_p^L\right)$	Upper efficiency score $\left(\theta_p^U\right)$	Classification	Ranking
1	1	1	E^{++}	1
2	0.823	1	E^{+}	3
3	0.227	1	E^{+}	2
4	0.224	1	E^{+}	4
5	0.445	0.907	E^{-}	5

Table 6
The efficiency of the five DMUs for different Gamma levels in example 2.

Gamma	DMU				
	1	2	3	4	5
0	1.000	1.000	1.000	1.000	0.900
0.1	1.000	1.000	1.000	0.967	0.881
0.2	1.000	1.000	1.000	0.912	0.862
0.3	1.000	1.000	1.000	0.857	0.846
0.4	1.000	1.000	1.000	0.802	0.831
0.5	1.000	1.000	1.000	0.749	0.816
0.6	1.000	1.000	1.000	0.699	0.801
0.7	1.000	1.000	0.970	0.651	0.787
0.8	1.000	1.000	0.859	0.607	0.773
0.9	1.000	1.000	0.750	0.566	0.758
1	1.000	1.000	0.655	0.528	0.743
1.1	1.000	1.000	0.611	0.506	0.729
1.2	1.000	1.000	0.569	0.484	0.715
1.3	1.000	1.000	0.530	0.462	0.689
1.4	1.000	1.000	0.493	0.438	0.664
1.5	1.000	1.000	0.457	0.416	0.642
1.6	1.000	1.000	0.421	0.393	0.622
1.7	1.000	1.000	0.386	0.371	0.604
1.8	1.000	1.000	0.350	0.350	0.587
1.9	1.000	0.968	0.314	0.325	0.572
2	1.000	0.924	0.284	0.303	0.558
2.1	1.000	0.898	0.275	0.297	0.545
2.2	1.000	0.886	0.266	0.292	0.534
2.3	1.000	0.875	0.257	0.287	0.522
2.4	1.000	0.863	0.248	0.282	0.510
2.5	1.000	0.853	0.240	0.278	0.499
2.6	1.000	0.843	0.232	0.273	0.488
2.7	1.000	0.833	0.223	0.269	0.477
2.8	1.000	0.823	0.214	0.265	0.466
2.9	1.000	0.819	0.205	0.261	0.456
3	1.000	0.817	0.198	0.256	0.446
3.1	1.000	0.815	0.195	0.252	0.444
3.2	1.000	0.814	0.193	0.248	0.443
3.3	1.000	0.813	0.190	0.245	0.442
3.4	1.000	0.812	0.188	0.241	0.440
3.5	1.000	0.811	0.186	0.238	0.439
3.6	1.000	0.810	0.184	0.235	0.438
3.7	1.000	0.809	0.182	0.232	0.437
3.8	1.000	0.809	0.180	0.229	0.435
3.9	1.000	0.809	0.179	0.227	0.434
4	1.000	0.809	0.177	0.224	0.433

The contribution of this paper is fourfold: (1) we consider ambiguous, uncertain or imprecise input and output data in DEA; (2) we address the gap in the imprecise DEA literature for problems not suitable or difficult to model with interval or fuzzy representations; (3) we propose a robust optimization model in which the input and output parameters are constrained to be within an uncertainty set with additional constraints based on the worst case solution with respect to the uncertainty set; and (4) we use Monte-Carlo simulation to specify a range of Gamma in which the rankings of the DMUs occur with high probability.

In this paper, we presented a robust optimization method for encountering data uncertainties that covers the interval approach and results in less complexity than the fuzzy approach. In this model, by using the protection level parameters for each DMU, we can adjust the conservativeness of the model and let their efficiencies vary between optimistic and pessimistic bounds. To illustrate the importance of varying efficiencies for different levels of Γ, we used Monte-Carlo simulation and computed the conformity of the rankings resulting from the mathematical model with reality. It was also shown that the maximum conformity may not occur in optimistic and pessimistic cases, and the analyst can search the maximum conformity in rankings by varying the level of Γ .

The robust optimization method proposed in this study provides an alternative approach to interval and fuzzy DEA. Most interval DEA approaches evaluate the performance of the DMUs based on the lower and upper bounds of the efficiency. However, we demonstrated that the ranking of the DMUs may not always develop according to the lower and upper bound efficiencies. Even

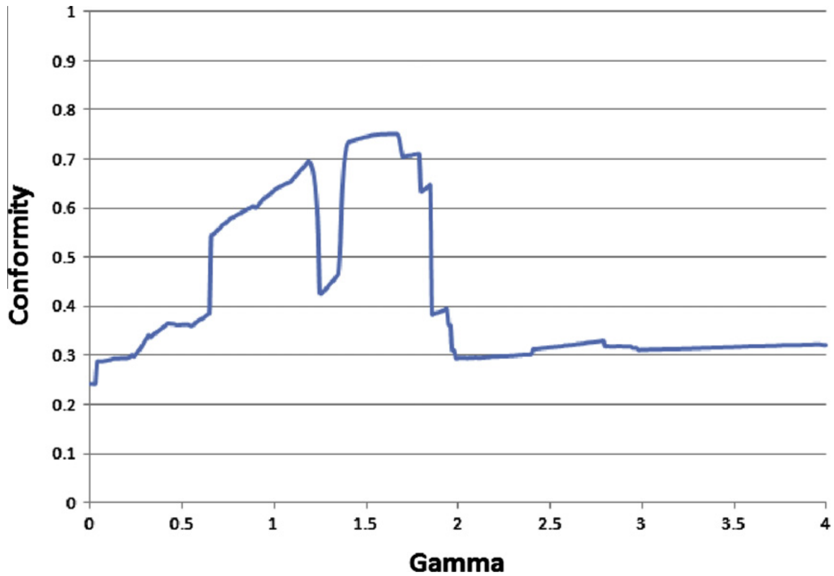

Fig. 6. The average conformity between the ranking of the DMUs using model (13) and the simulation runs for different Γ in example 2.

Table 7
The efficiency values of the five DMUs for the three special cases of Γ in example 2.

Γ	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5	Fitness (%)
0	1	1	1	1	0.90	24
1.65	1	1	0.40	0.38	0.61	75
4	1	0.81	0.18	0.22	0.43	32

though the fuzzy DEA deals with data expressed in linguistic form, most of them use the alpha-cut approach to transform the fuzzy DEA model into a binary model. Throughout this transformation, lower and upper bound efficiencies are calculated for each alpha and the fuzzy DEA approach has the same problem as the interval DEA approaches. However, this work does not imply that interval and fuzzy DEA approaches are not useful in uncertain environments.

The proposed robust optimization method enables analysts to assimilate the imprecise and vague data in a formal systematic approach. However, there are a number of challenges involved in the proposed research that provide a great deal of fruitful scope for future research. The current paper discusses and develops a robust optimization method based on the CCR model. Similar discussions can be developed for other DEA models. The issue of dealing with imprecise and vague data in DEA is an important topic. Modeling other types of imprecise, vague or incomplete data and developing associated robust optimization methods are interesting topics for future research. Finally, we plan to implement the proposed framework in the real-world and write a follow-up paper demonstrating the practical implications of our model in real-life problems.

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References

Ali, A. I., & Seiford, L. M. (1993). Computational accuracy and infinitesimals in data envelopment analysis. *INFOR, 31*, 290–297.

Alirezaee, M. R. (2005). The overall assurance interval for the non-Archimedean Epsilon in DEA models; a partition base algorithm. *Applied Mathematics and Computation, 164*, 667–674.

Alirezaee, M. R., & Khalili, M. (2006). Recognizing the efficiency, weak efficiency and inefficiency of DMUs with an epsilon independent linear program. *Applied Mathematics and Computation, 183*, 1323–1327.

Amin, G. R., & Toloo, M. (2004). A polynomial-time algorithm for finding Epsilon in DEA models. *Computers & Operations Research, 31*, 803–805.

Amin, G. R., & Toloo, M. (2007). Finding the most efficient DMUs in DEA: An improved integrated model. *Computers & Industrial Engineering, 52*, 71–77.

Asady, B., & Zendehnam, A. (2007). Ranking fuzzy numbers by distance minimization. *Applied Mathematical Modeling, 31*, 2589–2598.

Ben-Tal, A., & Nemirovski, A. (1998). Robust convex optimization. *Mathematics of Operations Research, 23*, 769–805.

Ben-Tal, A., & Nemirovski, A. (1999). Robust solutions of uncertain linear programs. *Operations Research Letters, 25*, 1–13.

Ben-Tal, A., & Nemirovski, A. (2000). Robust solutions of linear programming problems contaminated with uncertain data. *Mathematical Programming, 88*, 411–424.

Bertsimas, D., Pachamanova, D., & Sim, M. (2004). Robust linear optimization under general norms. *Operations Research Letters, 32*, 510–516.

Bertsimas, D., & Sim, M. (2004). The price of robustness. *Operations Research, 52*, 35–53.

Charnes, A., Cooper, W. W., & Rhodes, E. L. (1978). Measuring the efficiency of decision making units. *European Journal of Operational Research, 2*, 429–444.

Cooper, W. W., Park, K. S., & Yu, G. (1999). IDEA and AR-IDEA: Models for dealing with imprecise data in DEA. *Management Science, 45*, 597–607.

Cooper, W. W., Seiford, L., & Tone, K. (1999). *Data envelopment analysis: A comprehensive text with models, applications, references and DEA-Solver software*. London: Kluwer Academic Publishers.

Despotis, D. K., & Smirlis, Y. G. (2002). Data envelopment analysis with imprecise data. *European Journal of Operational Research, 140*, 24–36.

El-Ghaoui, L., & Lebret, H. (1997). Robust solutions to least-squares problems with uncertain data. *SIAM Journal on Matrix Analysis and Applications, 18*, 1035–1064.

El-Ghaoui, L., Oustry, F., & Lebret, H. (1998). Robust solutions to uncertain semidefinite programs. *SIAM Journal on Optimization, 9*, 33–52.

Entani, T., Maeda, Y., & Tanaka, H. (2002). Dual models of interval DEA and its extension to interval data. *European Journal of Operational Research, 136*, 32–45.

Guo, P. (2009). Fuzzy data envelopment analysis and its application to location problems. *Information Sciences, 179*, 820–829.

Guo, P., & Tanaka, H. (2001). Fuzzy DEA: A perceptual evaluation method. *Fuzzy Sets and Systems, 119*, 149–160.

Guo, P., & Tanaka, H. (2008). Decision making based on fuzzy data envelopment analysis. In D. Ruan & K. Meer (Eds.), *Intelligent decision and policy making support systems* (pp. 39–54). Berlin: Springer.

Hatami-Marbini, A., & Saati, S. (2009). Stability of RTS of efficient DMUs in DEA with fuzzy under fuzzy data. *Applied Mathematical Sciences, 3*, 2157–2166.

Jahanshahloo, G. R., & Khodabakhshi, M. (2004). Determining assurance interval for non-Archimedean element in the improving outputs model in DEA. *Applied Mathematics and Computation, 151*, 501–506.

Jahanshahloo, G. R., Sanei, M., Rostamy-Malkhalifeh, M., & Saleh, H. (2009). A comment on "A fuzzy DEA/AR approach to the selection of flexible manufacturing systems". *Computers and Industrial Engineering, 56*, 1713–1714.

Jahanshahloo, G. R., Soleimani-damaneh, M., & Nasrabadi, E. (2004). Measure of efficiency in DEA with fuzzy input–output levels: A methodology for assessing, ranking and imposing of weights restrictions. *Applied Mathematics and Computation, 156*, 175–187.

Kao, C. (2006). Interval efficiency measures in data envelopment analysis with imprecise data. *European Journal of Operational Research, 174*, 1087–1099.

Kao, C., & Liu, S. T. (2000). Fuzzy efficiency measures in data envelopment analysis. *Fuzzy Sets and Systems, 113*, 427–437.

León, T., Liern, V., Ruiz, J. L., & Sirvent, I. (2003). A fuzzy mathematical programming approach to the assessment of efficiency with DEA Models. *Fuzzy Sets and Systems, 139*, 407–419.

Lertworasirikul, S., Fang, S. C., Joines, J. A., & Nuttle, H. L. W. (2003). Fuzzy data envelopment analysis (DEA): A possibility approach. *Fuzzy Sets and Systems, 139*, 379–394.

Liu, S. T. (2008). A fuzzy DEA/AR approach to the selection of flexible manufacturing system. *Computer and Industrial Engineering, 54*, 66–76.

Liu, S. T., & Chuang, M. (2009). Fuzzy efficiency measures in fuzzy DEA/AR with application to university libraries. *Expert Systems with Applications, 36*, 1105–1113.

Mehrabian, S., Jahanshahloo, G. R., Alirezaee, M. R., & Amin, G. R. (2000). An assurance interval of the non-Archimedean Epsilon in DEA models. *Operations Research, 48*, 344–347.

Mirhassani, S. A., & Alirezaee, M. R. (2005). An efficient approach for computing non-Archimedean in DEA based on integrated models. *Applied Mathematics and Computation, 166*, 449–456.

O'Neal, P. V., Ozcan, Y. A., & Yanqiang, M. (2002). Benchmarking mechanical ventilation services in teaching hospitals. *Journal of Medical Systems, 26*, 227–240.

Saati, S., Memariani, A., & Jahanshahloo, G. R. (2002). Efficiency analysis and ranking of DMUs with fuzzy data. *Fuzzy Optimization and Decision Making, 1*, 255–267.

Sadjadi, S. J., & Omrani, H. (2008). Data envelopment analysis with uncertain data: An application for Iranian electricity distribution companies. *Energy Policy, 36*, 4247–4254.

Sengupta, J. K. (1992). A fuzzy systems approach in data envelopment analysis. *Computers and Mathematics with Applications, 24*, 259–266.

Shang, J., & Sueyoshi, T. (1995). A unified frame work for the selection of a flexible manufacturing system. *European Journal of Operational Research, 85*, 297–315.

Simar, L., & Wilson, P. (2000). Statistical inference in non-parametric frontier models: The state of the art. *Journal of Productivity Analysis, 13*, 49–78.

Soleimani Damaneh, M. (2008). Fuzzy upper bounds and their applications. *Chaos, Solitons and Fractals, 36*, 217–225.

Soleimani-damaneh, M. (2009). Establishing the existence of a distance-based upper bound for a fuzzy DEA model using duality. *Chaos, Solitons and Fractals, 41*, 485–490.

Soleimani-damaneh, M., Jahanshahloo, G. R., & Abbasbandy, S. (2006). Computational and theoretical pitfalls in some current performance measurement techniques and a new approach. *Applied Mathematics and Computation, 181*, 1199–1207.

Soyster, A. L. (1973). Convex programming with set-inclusive constraints and applications to inexact linear programming. *Operations Research, 21*, 1154–1157.

Tone, K. (1993). An Epsilon-free DEA and a new measure of efficiency. *Journal of Operations Research Society of Japan, 36*, 167–174.

Wang, Y. M., Greatbanks, R., & Yang, J. B. (2005). Interval efficiency assessment using data envelopment analysis. *Fuzzy Sets and Systems, 153*, 347–370.

Wang, Y. M., Luo, Y., & Liang, L. (2009). Fuzzy data envelopment analysis based upon fuzzy arithmetic with an application to performance assessment of manufacturing enterprises. *Expert Systems with Applications, 36*, 5205–5211.