Convex and non-convex approaches for cost efficiency models with fuzzy data

Khalil Paryab

School of Mathematics,
Iran University of Science and Technology,
Tehran 13114-16846, Iran
Email: Paryab@iust.ac.ir

Madjid Tavana*

Lindback Distinguished Chair of
Information Systems and Decision Sciences,
La Salle University,
Philadelphia, PA 19141, USA
Fax: +1-267-295-2854
Email: tavana@lasalle.edu
*Corresponding author

Rashed Khanjani Shiraz

School of Mathematics,
Iran University of Science and Technology,
Tehran 13114-16846, Iran
Email: Khanjani@iust.ac.ir

Abstract: Classical cost efficiency (CE) measurement models require exact and accurate knowledge of the input and output values for each decision making unit (DMU). However, the observed values of the input and output data in real-world problems are often imprecise or vague. In recent years, fuzzy data envelopment analysis (DEA) has been successfully used to deal with imprecise or vague data in efficiency measurement. In this paper, we incorporate fuzzy set theory into the traditional CE measurement. We propose two approaches based on the convex DEA and non-convex free disposable hull (FDH) approach with fuzzy variables. The purpose of this paper is two-fold: 1) we develop a CE analysis for non-parametric convex methods based on fuzzy set theory; 2) we further develop a non-convex CE analysis model where the non-convexity is formulated based on the FDH approach. We also present a numerical example to demonstrate the applicability of the proposed models and exhibit the efficacy of the procedures and algorithms.

Keywords: data envelopment analysis; DEA; cost efficiency; fuzzy set theory; free disposable hull; non-parametric convex; non-convex.

Reference to this paper should be made as follows: Paryab, K., Tavana, M. and Shiraz, R.K. (2015) 'Convex and non-convex approaches for cost efficiency models with fuzzy data', *Int. J. Data Mining, Modelling and Management*, Vol. 7, No. 3, pp.213–238.

Biographical notes: Khalil Paryab is an Assistant Professor of Mathematics at Iran University of Science and Technology in Tehran, Iran. He received his BSc and PhD in Pure Mathematic, from Tabriz University. He received his second PhD in Applied Mathematics from Iran University of Science and Technology. He has published several papers and books in pure and applied mathematics. His research interests are in the application of graph theory, numerical analysis, and mathematical programming.

Madjid Tavana is a Professor of Business Systems and Analytics and the Lindback Distinguished Chair of Information Systems and Decision Sciences at La Salle University, where he served as Chairman of the Management Department and Director of the Center for Technology and Management. He is a Distinguished Research Fellow at Kennedy Space Center, Johnson Space Center, Naval Research Laboratory at Stennis Space Center, and Air Force Research Laboratory. He was recently honoured with the prestigious Space Act Award by NASA. He holds a MBA, PMIS, and PhD in Management Information Systems and received his Post-Doctoral Diploma in Strategic Information Systems from the Wharton School at the University of Pennsylvania. He is the Editor-in-Chief of *Decision Analytics*, *International Journal of Applied Decision Sciences*, *International Journal of Management and Decision Making*, *International Journal of Strategic Decision Sciences*, and *International Journal of Enterprise Information Systems*. He has published several books and over 100 research papers in academic journals such as *Information Sciences*, *Decision Sciences*, *Information Systems*, *Interfaces*, *Annals of Operations Research*, *Advances in Space Research*, *Omega*, *Information and Management*, *Knowledge-Based Systems*, *Expert Systems with Applications*, *European Journal of Operational Research*, *Journal of the Operational Research Society*, *Computers and Operations Research*, *Energy Economics*, *Applied Soft Computing*, and *Energy Policy*.

Rashed Khanjani Shiraz is a doctoral student in Applied Mathematics and Operations Research at the Iran University of Science and Technology in Tehran, Iran. He received his BS in Pure Mathematics from the University of Mohaghegh Ardabili in Ardabil and his MS in Applied Mathematics and Operations Research from the Iran University of Science and Technology. He has published several papers in international journals including *Knowledge-Based Systems*, *Expert Systems with Application International Journal of Data Analysis Techniques* and *Strategies Asia-Pacific Journal of Operational Research*. His research interests are in stochastic programming, fuzzy programming, mathematical programming and data envelopment analysis.

1 Introduction

Data envelopment analysis (DEA) is a non-parametric mathematical programming approach that evaluates a group of decision making units (DMUs) with comparative efficiencies. The approach was proposed by Charnes et al. (1978) and extended by Banker et al. (1984). Cost efficiency (CE), as a DEA model, is used to evaluate the ability of a DMU to produce the current outputs at a minimal cost, given the input price that is paid at each DMU. The CE model, originated by Farrell (1957), can be used to investigate the optimal input-mix that produces the current outputs at minimum cost. The measure of CE is the ratio of the minimum cost to the actual observed cost. See, e.g., Färe et al. (1985) and Jahanshahloo et al. (2007a, 2008) for more details concerning the CE analysis with precise data).

The conventional DEA methods such as CCR (Charnes et al., 1978) and BCC (Banker et al., 1984) require precise measurement of both the inputs and outputs. One of the main challenges associated with the application of DEA is the difficulty in quantifying some of the input and output data in real-world problems where the observed values are often imprecise. One way to manipulate uncertain data in DEA is to use probability distributions. However, probability distributions require either *a priori* predictable regularity or *a posteriori* frequency determinations which are difficult to construct. An alternative approach is to represent the imprecise values by membership functions of the fuzzy set theory.

In this study, we propose the concept of fuzzy CE use fuzzy set theory to handle CE models with imprecise data. A few researchers have considered uncertainty in the CE context See, e.g., Thompson et al. (1996), Schaffnit et al. (1997), Kuosmanen and Post (2001), Kuosmanen (2003), Camanho and Dyson (2005), Mostafaee and Saljooghi (2010), Fang and Li (2013), Jahanshahloo et al. (2007b), and Bagherzadeh Valami (2009).

The CE model with incomplete price information was introduced by Thompson et al. (1996) and Schaffnit et al. (1997). Kuosmanen and Post (2001) and Kuosmanen (2003) extended a DEA-based method to derive both upper and lower bounds for Farrell's (1957) CE with incomplete price data in the form of a convex polyhedral cone. Camanho and Dyson (2005) proposed an estimation of the upper and lower bonds for the CE measure in situations where input prices appeared in the form of ranges. They applied standard weight restriction techniques in the form of input cone assurance regions in a standard DEA model. Jahanshahloo et al. (2007b) provided some models for treating ordinal data in CE analysis. Bagherzadeh Valami (2009) extended the classical CE analysis to a framework with fuzzy input prices treated as triangular fuzzy numbers. Mostafaee and Saljooghi (2010) considered the situation in which the input data, the output data, and the input prices were imprecise and took the form of ranges. They proposed a pair of two-level mathematical programming problems to obtain the upper and lower bounds of the CE. Fang and Li (2013) discussed the theoretical properties of the efficiency solutions for the cone-ratio DEA models and the CE models under uncertain prices.

More recently, DEA based on fuzzy data for the inputs and outputs has received some attention. Fuzzy set algebra developed by Zadeh (1965) is the formal body of theory that allows the treatment of imprecise estimates in uncertain environments. Sengupta (1992) incorporated fuzziness into DEA by defining tolerance levels for both the objective function and the constraint violations and proposed a fuzzy mathematical programming approach. Triantis and Girod (1998) modified the radial DEA model and the free disposal hull (FDH) approach to incorporate imprecision in the measurement of the input and output data. Guo and Tanaka (2001) and Lertworasirikul et al. (2003a) applied the possibility measure proposed by Zadeh (1978) to the fuzzy DEA model. Lertworasirikul et al. (2003b) extended the possibility and credibility approaches to fuzzy BCC model. León et al. (2003) proposed a fuzzy BCC model based on Guo and Tanaka's (2001) model. Kao and Liu (2003, 2005) transformed fuzzy input and fuzzy output data into intervals by using α-level sets, and built a family of crisp DEA models for the intervals. Tavana et al. (2012) proposed both stochastic and fuzzy DEA models, and obtained their crisp equivalent models. Wang and Chin (2011) presented a fuzzy expected value approach for fuzzy DEA frontiers.

In this paper, we develop two CE models based on the convex DEA and non-convex FDH approach. First, we develop a fuzzy efficiency model based on the convex DEA approach and then extend the fuzzy efficiency to Farrell's (1957) CE and a standard DEA formulation with the addition of multiplier restrictions (Camanho and Dyson, 2005). Second, we extend the fuzzy CE models into a non-convex FDH approach with various returns to scale assumptions, including the variable returns to scale (VRS), non-increasing returns to scale, non-decreasing returns to scale, and constant returns to scale (CRS).

The remainder of this paper is organised as follows. Section 2 describes the preliminaries on DEA models. In Section 3, we introduce some definitions for fuzzy sets. Section 4 presents CE models with fuzzy variables. In Section 5, we introduce a chance constraint CE model. In Section 6, we propose the FDH-CE models. In Section 7, we present a numerical example to demonstrate the applicability of the proposed models and exhibit the efficacy of the procedures and algorithms. Section 8 draws the conclusive remarks and future research directions.

2 Preliminaries

In this section, we first review the basic DEA models for measuring the technical efficiency and then present the non-parametric CE models. Let us assume that n DMUs consume varying amounts of m different inputs to produce s different outputs. Assume that x_{ij} ($i = 1,\dots,m$) and y_{rj} ($r = 1,\dots,m$) represent, respectively, the input and output of the DNU_j ($j = 1,\dots,n$). The input-oriented version of the primal and its dual CCR models can be formulated as follows:

$$
\begin{array}{ll}
\text{Primal CCR model} & \text{Dual CCR model} \\
\min \theta & \max \sum_{r=1}^{s} u_r y_{rk} \\
\text{subject to:} & \text{subject to:} \\
\sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{ik} \quad i = 1,\dots,m & \sum_{i=1}^{m} v_i x_{ik} = 1, \\
\sum_{j=1}^{J} \lambda_j y_{rj} \ge y_{rk} \quad r = 1,\dots,s & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j = 1,\dots,n, \\
\lambda_j \ge 0, \quad j = 1,\dots,n. & u_r, v_i \ge 0, \quad r = 1,\dots,s; i = 1,\dots,m.
\end{array}
\tag{1}
$$

The concept of CE underlying a DEA assessment was first introduced by Farrell (1957). In order to obtain a measure of CE for the DMUs with multiple inputs and outputs, the minimum cost for the production of DMUs current outputs with existing input prices is obtained solving the following linear programming problem, as first formulated by Färe et al. (1985):

$$
\begin{aligned}
&\min \sum_{i=1}^{m} p_{ik} x_i \\
&\text{subject to:} \\
&\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{rk}, \quad r = 1, \ldots, s \\
&\sum_{j=1}^{n} \lambda_j x_{ij} \le x_i, \quad i = 1, \ldots, m \\
&\lambda_j \ge 0, \quad j = 1, \ldots, n \\
&x_i \ge 0, \quad i = 1, \ldots, m.
\end{aligned}
\tag{2}
$$

In (3), p_{ik} ($i = 1,\ldots,m$; $k = 1,\ldots,n$) is the price of input i for DMU$_k$, and x_i ($i = 1,\ldots,m$) and λ_j ($j = 1,\ldots,n$) are the decision variables. CE is then obtained as the ratio of the minimum cost with current prices to the current cost of DMU$_k$ as follows:

$$
\text{cost efficiency} = \frac{\sum_{i=1}^{m} p_{ik} x_i^*}{\sum_{i=1}^{m} p_{ik} x_{ik}} \tag{3}
$$

A DMU with an efficiency score of one is cost efficient; otherwise, it is cost inefficient. Alternatively, the measure of CE can be obtained with the inclusion of the weight restrictions in the standard DEA model proposed by Camanho and Dyson (2005) as follows:

$$
\begin{aligned}
&\max \sum_{r=1}^{s} u_r y_{rk} \\
&\text{subject to:} \\
&\sum_{i=1}^{m} v_i x_{ik} = 1, \\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j = 1, \ldots, n, \\
&v_{i^a} - \frac{p_{i^a k}}{p_{i^b k}} v_{i^b} = 0, \quad i^a < i^b, i^a, i^b = 1, \ldots, m, \\
&u_r \ge \varepsilon, \quad r = 1, \ldots, s.
\end{aligned}
\tag{4}
$$

As a result, model (2) and model (4) are the alternative version of the CE model proposed by Farrell (1957) and their optimal values are the Farrell's CE of DMU$_k$. In the next section, we provide some background on fuzzy set theory.

3 Background on fuzzy set theory

Fuzzy set theory, which was introduced by Zadeh (1965), has been well developed and applied in a wide variety of real-world problems.

Definition 1: A fuzzy number $\tilde{A}$ is called positive (negative), if it membership function is such that $\mu_{\tilde{A}}(x) = 0, \forall x < 0 (x > 0)$.

Definition 2 (Zimmermann, 1996): A fuzzy subset $\tilde{A}$ of real number R is convex if and only if $\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min(\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)), \forall x, y \in R, \forall \lambda \in [0, 1]$. Alternatively, a fuzzy set is convex if all α-cuts are convex.

Definition 3 (Zadeh, 1978; Zimmermann, 1996): Let $(\Theta, P(\Theta), Pos)$ be a possibility space where Θ is a non-empty set involving all possible potentially events and $P(\Theta)$ is the power set of Θ. For each $A \subseteq P(\Theta)$, there is a non-negative number $Pos(A)$, so-called a *possibility measure*, with the following properties

1 $Pos\{\emptyset\} = 1, Pos\{\Theta\} = 1$

2 $A \subseteq B$ implies $Pos(A) \leq Pos(B)$ for any $A, B \in (\Theta)$

3 $Pos\{\cup_k A_k\} = Sup_k Pos\{A_k\}$.

The triplet (Θ, P(Θ), *Pos*) is called a possibility space, and the function *Pos* is referred to as a possibility measure.

Definition 4 (Liu and Liu, 2002): Let ξ be a fuzzy variable on the possibility space $(U, P(U), Pos)$. The possibility of a fuzzy event $\{\xi \leq r\}$ is represented by:

$$Pos\{\xi \geq r\} = \underset{t \geq r}{Sup}\ \mu_\xi(t)$$

where $\mu_\xi: \Re \rightarrow [0, 1]$ is the membership function of ξ and r is a real number.

Definition 5 (Dubois and Prade, 1980): A fuzzy interval of LR-type is denoted by $\tilde{A} = (\alpha, m_1, m_2, \beta)_{LR}$ where α and β are the (non-negative) left and right spreads, respectively, and m_1 and m_2 are the mean values of $\tilde{A}$. The membership function of $\tilde{A}$ can be expressed as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\dfrac{m_1 - x}{\alpha}\right), & x \leq m_1, \\ 1, & m_1 \leq x \leq m_2 \\ R\left(\dfrac{x - m_2}{\beta}\right), & x \geq m_2. \end{cases}$$

where L and R are the left and right functions, respectively. Particularly, suppose that

$$L(x) = R(x) = \begin{cases} 1 - x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$\tilde{A} = (\alpha, m_1, m_2, \beta)_{LR} = (\alpha, m_1, m_2, \beta)$ is called a trapezoidal fuzzy number. Also, if m_1 = m_2 = m, $\tilde{A} = (\alpha, m, \beta)_{LR} = (\alpha, m, \beta)$ is called a triangular fuzzy number. Alternatively, the trapezoidal and triangular fuzzy numbers can be indicated as ($m_1 - \alpha$, m_1, m_2, $m_2 + \beta$) and ($m - \alpha$, m, $m + \beta$), respectively.

Definition 6 (fuzzy arithmetic) (Dubois and Prade, 1980): Let $\tilde{A} = (\alpha, m_1, m_2, \beta)_{LR}$ and $\tilde{B} = (\bar{\alpha}, \bar{m}_1, \bar{m}_2, \bar{\beta})_{LR}$ be two positive fuzzy numbers. Then, the fuzzy arithmetic of $\tilde{A}$ and $\tilde{B}$ can be defined as follows:

- addition:

$$\left(\alpha, m_1, m_2, \beta\right)_{LR} + \left(\bar{\alpha}, \bar{m}_1, \bar{m}_2, \bar{\beta}\right)_{LR} = \left(\alpha + \bar{\alpha}, m_1 + \bar{m}_1, m_2 + \bar{m}_2, \beta + \bar{\beta}\right)_{LR}$$

- subtraction:

$$\left(\alpha, m_1, m_2, \beta\right)_{LR} - \left(\bar{\alpha}, \bar{m}_1, \bar{m}_2, \bar{\beta}\right)_{LR} = \left(\alpha + \bar{\beta}, m_1 - \bar{m}_2, m_2 - \bar{m}_1, \beta + \bar{\alpha}\right)_{LR}$$

- multiplication (approximation):

$$\begin{aligned} \tilde{A} \otimes \tilde{B} &= \left(\alpha, m_1, m_2, \beta\right)_{LR} \otimes \left(\bar{\alpha}, \bar{m}_1, \bar{m}_2, \bar{\beta}\right)_{LR} \\ &\simeq \left(m_1\bar{\alpha} + \bar{m}_1\alpha - \alpha\bar{\alpha}, m_1\bar{m}_1, m_2\bar{m}_2, m_2\bar{\beta} + \bar{m}_2\beta + \beta\bar{\beta}\right)_{LR} \end{aligned}$$

  and if k is the non-zero real number, then

$$k\left(\alpha, m_1, m_2, \beta\right)_{LR} = \begin{cases} \left(k\alpha, km_1, km_2, k\beta\right)_{LR}, & \text{if } k > 0 \\ \left(-k\beta, km_2, km_1, -k\alpha\right)_{LR}, & \text{if } k < 0 \end{cases}$$

- division (approximation):

$$\begin{aligned} \tilde{A}(\div)\tilde{B} &= \left(\alpha, m_1, m_2, \beta\right)_{LR} (\div) \left(\bar{\alpha}, \bar{m}_1, \bar{m}_2, \bar{\beta}\right)_{LR} \\ &\simeq \left(\frac{m_1\bar{\beta} + \bar{m}_2\alpha}{\bar{m}_2\left(\bar{m}_2 + \bar{\beta}\right)}, \frac{m_1}{\bar{m}_2}, \frac{m_2}{\bar{m}_1}, \frac{m_2\bar{\alpha} + \bar{m}_1\beta}{\bar{m}_1\left(\bar{m}_1 - \bar{\alpha}\right)}\right)_{LR} \end{aligned}$$

- inverse (approximation):

$$\left(\alpha, m_1, m_2, \beta\right)_{LR}^{-1} \simeq \left(\frac{\beta}{m_2\left(m_2 + \alpha\right)}, \frac{1}{m_2}, \frac{1}{m_1}, \frac{\alpha}{m_1\left(m_1 - \alpha\right)}\right)_{LR}$$

Definition 7: For some threshold $\alpha \in [0,1]$, the α-cut of a LR-type fuzzy number $\tilde{A}$ is a closed interval:

$$A_\alpha = \left\{x \middle| \mu_A(x) \geq \alpha\right\} = \left[A_\alpha^L, A_\alpha^R\right] = \left[m - L^{-1}(\alpha), m + R^{-1}(\alpha)\right]$$

where A_α^L and A_α^R are the left and the right extreme points, respectively.

4 CE models with fuzzy variables

4.1 CE with fuzzy input price variables

The estimation of CE in DEA requires complete and accurate information of the input prices for each DMU. However, in many real-world problems the exact and precise values of the relevant prices is often incomplete, vague, ambiguous, linguistic, or imprecise. In order to deal with the uncertain prices, we first present a CE approach with fuzzy input prices and exact input-output data.

Let $\tilde{P}_j = (\tilde{p}_{1j}, \ldots, \tilde{p}_{mj}) \in \Re_+^m$ be the fuzzy input prices of DMU$_j$ and each of their elements is shown as follows:

$$\tilde{p}_{ij} = \left(p_{ij}^{\alpha}, p_{ij}^{m_1}, p_{ij}^{m_2}, p_{ij}^{\beta} \right), \quad i = 1, \ldots, m, \quad j = 1, \ldots, n.$$

The weight restriction (multiplier) CE model with fuzzy input prices is given in equation (5).

$$\begin{aligned}
&\max \sum_{r=1}^{s} u_r y_{rk} \\
&\text{subject to:} \\
&\sum_{i=1}^{m} v_i x_{ik} = 1, \\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j = 1, \ldots, n, \\
&v_{i^a} - \frac{\tilde{p}_{i^a k}}{\tilde{p}_{i^b k}} v_{i^b} = 0, \quad i^a < i^b, \quad i^a, i^b = 1, \ldots, m, \\
&u_r \ge \varepsilon, \quad r = 1, \ldots, s.
\end{aligned} \tag{5}$$

4.2 CE with fuzzy input-output variables

In this subsection, we present a CE model with fuzzy input-output data and exact input price data. Denote that $\tilde{X}_j = (\tilde{x}_{1j}, \ldots, \tilde{x}_{mj})^{\mathrm{T}} \in \Re^m$ and $Y_j = (\tilde{y}_{1j}, \ldots, \tilde{y}_{sj})^{\mathrm{T}} \in \Re^s$ are the fuzzy input-output vectors of DMU$_j$ (j = 1,…,n), and each of their elements are denoted as follows:

$$\begin{aligned}
\tilde{x}_{ij} &= \left(x_{ij}^{\alpha}, x_{ij}^{m_1}, x_{ij}^{m_2}, x_{ij}^{\beta} \right), \\
\tilde{y}_{rj} &= \left(y_{rj}^{\alpha}, y_{rj}^{m_1}, y_{rj}^{m_2}, y_{rj}^{\beta} \right)
\end{aligned}$$

Also, let $P_j = (p_{1j}, \ldots, p_{mj}) \in \Re_+^m$ be the fixed input price vector of DMU$_j$, i.e., input prices are known exactly at each DMU. Now we consider the following CE model with fuzzy inputs and fuzzy outputs:

$$\max \sum_{r=1}^{s} u_r \tilde{y}_{rk}$$

subject to:

$$\begin{aligned}
&\sum_{i=1}^{m} v_i \tilde{x}_{ik} = 1, \\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \le 0, \quad j = 1, \ldots, n, \\
&v_{i^a} - \frac{p_{i^a k}}{p_{i^b k}} v_{i^b} = 0, \quad i^a < i^b, \quad i^a, i^b = 1, \ldots, m, \\
&u_r \ge \varepsilon, \quad r = 1, \ldots, s.
\end{aligned} \tag{6}$$

4.3 CE with fuzzy input-output and fuzzy input price variables

In the current subsection, we generalise the theory of CE to situations in which input and output data as well as input prices appear in the form of fuzzy variables. In this case, we have following model:

$$\max \sum_{r=1}^{s} u_r \tilde{y}_{rk}$$

subject to:

$$\begin{aligned}
&\sum_{i=1}^{m} v_i \tilde{x}_{ik} = 1, \\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \le 0, \quad j = 1, \ldots, n, \\
&v_{i^a} - \frac{\tilde{p}_{i^a k}}{\tilde{p}_{i^b k}} v_{i^b} = 0, \quad i^a < i^b, \quad i^a, i^b = 1, \ldots, m, \\
&u_r \ge \varepsilon, \quad r = 1, \ldots, s.
\end{aligned} \tag{7}$$

5 The chance-constrained fuzzy CE models

In this section we develop an imprecise DEA-based formulation for dealing with the fuzzy parameters on a possibility space (Θ, *P*(Θ), *Pos*). Charnes and Cooper (1959) introduced chance-constrained programming (CCP). In this paper the chance-constrained concept is used to solve a fuzzy CE model. More details about the CCP can be found in (Charnes and Cooper, 1959; Liu, 2002).

5.1 Chance-constrained CE with fuzzy input price variables

Using the concepts of chance constrained programming and the possibility of fuzzy events, the weight restriction (multiplier) of the FCE models with fuzzy input price becomes the following weight restriction possibility CE model:

$$\begin{aligned}
&\max \sum_{r=1}^{s} u_r y_{rk} \\
&\text{subject to:} \\
&\sum_{i=1}^{m} v_i x_{ik} = 1, \\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j = 1, \ldots, n, \\
&Pos\left(v_{i^a} = \frac{\tilde{P}_{i^a k}}{\tilde{P}_{i^b k}} v_{i^b} \right) \ge \delta, \quad i^a < i^b, \quad i^a, i^b = 1, \ldots, m, \\
&u_r, v_i \ge 0, \quad r = 1, \ldots, s; i = 1, \ldots, m.
\end{aligned} \tag{8}$$

where $\delta \in [0, 1]$ is the pre-specified acceptable level of possibility defined by the decision maker (DM). This parameter, assumed known a priori, is also called the threshold (or aspiration) level.

5.2 Chance-constrained CE with fuzzy input-output variables

The weight restriction (multiplier) of the fuzzy CE model with fuzzy input-output data and exact input price becomes the following weight restriction possibility CE model by using the concepts of chance constrained programming and possibility of fuzzy events:

$$\begin{aligned}
&\max \varphi \\
&\text{subject to:} \\
&Pos\left(\varphi \le \sum_{r=1}^{s} u_r \tilde{y}_{rk} \right) \ge \delta, \\
&Pos\left(\sum_{i=1}^{m} v_i \tilde{x}_{ik} = 1 \right) \ge \delta, \\
&Pos\left(\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \le 0 \right) \ge \delta, \quad j = 1, \ldots, n, \\
&v_{i^a} - \frac{P_{i^a k}}{P_{i^b k}} v_{i^b} = 0, \quad i^a < i^b, \quad i^a, i^b = 1, \ldots, m, \\
&u_r, v_i \ge \varepsilon, \quad r = 1, \ldots, s; i = 1, \ldots, m.
\end{aligned} \tag{9}$$

The objective value φ in model (9) is the maximum possible efficiency $\left(\sum_{r=1}^{s} u_r \tilde{y}_{rp} \right)$ of the DMU under evaluation relative to other DMUs at the possibility level δ or higher, subject to the possibility levels of the first and the second group of constraints being at least δ.

5.3 Chance-constrained CE with fuzzy input-output and fuzzy input price variables

In this section, by using the concepts of chance constrained programming and the possibility of fuzzy events, the weight restriction (multiplier) of the generalised theory of CE for situations where the input and output data as well as the input prices appear in the form of fuzzy variables becomes the following weight restriction possibility CE model:

$$\begin{aligned}
&\max \varphi \\
&\text{subject to:} \\
&Pos\left(\varphi \le \sum_{r=1}^{s} u_r \tilde{y}_{rk} \right) \ge \delta, \\
&Pos\left(\sum_{i=1}^{m} v_i \tilde{x}_{ik} = 1 \right) \ge \delta, \\
&Pos\left(\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \le 0 \right) \ge \delta, \quad j = 1, \ldots, n, \\
&Pos\left(v_{i^a} - \frac{\tilde{P}_{i^a k}}{\tilde{P}_{i^b k}} v_{i^b} = 0 \right) \ge \delta, \quad i^a < i^b, i^a, i^b = 1, \ldots, m, \\
&u_r, v_i \ge \varepsilon, \quad r = 1, \ldots, s; i = 1, \ldots, m.
\end{aligned} \tag{10}$$

We convert the constraints in this model into their respective crisp equivalents to solve the possibility constrained programming models (8), (9), and (10). Thereby, the Lemma proposed by Sakawa (1993) plays a pivotal role in solving the proposed models.

Lemma: Let $\bar{\lambda}_1$ and $\bar{\lambda}_2$ be two independent fuzzy numbers with continuous membership functions. For a given confidence level $\alpha \in [0, 1]$,

$$Pos\{\bar{\lambda}_1 \ge \bar{\lambda}_2\} \ge \alpha \text{ if and only if } \lambda_{1,\alpha}^R \ge \lambda_{2,\alpha}^L,$$

where $\lambda_{1,\alpha}^L, \lambda_{1,\alpha}^R$ and $\lambda_{2,\alpha}^L, \lambda_{2,\alpha}^R$ are the left and the right side extreme points of the α-level sets $\bar{\lambda}_1$ and $\bar{\lambda}_2$, respectively, and $Pos\{\bar{\lambda}_1 \ge \bar{\lambda}_2\}$ represents the degree of possibility.

We apply the Lemma to the fuzzy CE model (9) and model (10) to obtain the deterministic equivalent models in the presence of fuzzy inputs, outputs and the input prices. In addition, the following corresponding intervals of $\sum_{r=1}^{s} v_i \tilde{x}_{ij}$ and $\sum_{r=1}^{s} u_r \tilde{y}_{rp}$ result from the α-cut (see Definition 7):

$$\left[\left(\sum_{i=1}^{m} v_i \tilde{x}_{ij}\right)^L, \left(\sum_{i=1}^{m} v_i \tilde{x}_{ij}\right)^R\right]$$
$$=\left[\sum_{i=1}^{m} v_i x_{ij}^{m_1} - L^{-1}(\delta)\sum_{i=1}^{m} v_i x_{ij}^{\alpha}, \sum_{i=1}^{m} v_i x_{ij}^{m_2} + R^{-1}(\delta)\sum_{i=1}^{m} v_i x_{ij}^{\beta}\right]$$

$$\left[\left(\sum_{r=1}^{s} u_r \tilde{y}_{rj}\right)^L, \left(\sum_{r=1}^{s} u_r \tilde{y}_{rj}\right)^R\right]$$
$$=\left[\sum_{r=1}^{s} u_r y_{rp}^{m_1} - L^{-1}(\delta)\sum_{r=1}^{s} u_r y_{rj}^{\alpha}, \sum_{r=1}^{s} u_r y_{rp}^{m_2} + R^{-1}(\delta)\sum_{r=1}^{s} u_r y_{rj}^{\beta}\right]$$

If $\tilde{p}_{i^a k}$ and $\tilde{p}_{i^b k}$ are fuzzy variables, then $\tilde{p}_{i^a k}(\div)\tilde{p}_{i^b k}$ must be a fuzzy variable according to the concept of fuzzy arithmetic defined by using the approximation of Dubois and Prade (1980). That is, if the i^afuzzyinput price and the i^b fuzzy input for DMU_k are denoted as $\left(p_{i^a k}^{\alpha}, p_{i^a k}^{m_1}, p_{i^a k}^{m_2}, p_{i^a k}^{\beta}\right)_{LR}$ and $\left(p_{i^b k}^{\alpha}, p_{i^b k}^{m_1}, p_{i^b k}^{m_2}, p_{i^b k}^{\beta}\right)_{LR}$, respectively, then the fuzzy arithmetic yields the following: (font size)

$$\tilde{p}_{i^a k}(\div)\tilde{p}_{i^b k} = \left(p_{i^a k}^{\alpha}, p_{i^a k}^{m_1}, p_{i^a k}^{m_2}, p_{i^a k}^{\beta}\right)_{LR}(\div)\left(p_{i^b k}^{\alpha}, p_{i^b k}^{m_1}, p_{i^b k}^{m_2}, p_{i^b k}^{\beta}\right)_{LR}$$
$$\simeq\left(\frac{p_{i^a k}^{m_1} p_{i^b k}^{\beta} + p_{i^b k}^{m_2} p_{i^a k}^{\alpha}}{p_{i^b k}^{m_2}\left(p_{i^b k}^{m_2} + p_{i^b k}^{\beta}\right)}, \frac{p_{i^a k}^{m_1}}{p_{i^b k}^{m_2}}, \frac{p_{i^a k}^{m_2}}{p_{i^b k}^{m_1}}, \frac{p_{i^a k}^{m_2} p_{i^b k}^{\alpha} + p_{i^b k}^{m_1} p_{i^a k}^{\beta}}{p_{i^b k}^{m_1}\left(p_{i^b k}^{m_1} - p_{i^b k}^{\alpha}\right)}\right)$$

Let

$$\overline{p}_{i^a k}^{\alpha} = \frac{p_{i^a k}^{m_1} p_{i^b k}^{\beta} + p_{i^b k}^{m_2} p_{i^a k}^{\alpha}}{p_{i^b k}^{m_2}\left(p_{i^b k}^{m_2} + p_{i^b k}^{\beta}\right)}, \ \overline{p}_{i^a k}^{m_1} = \frac{p_{i^a k}^{m_1}}{p_{i^b k}^{m_2}}, \ \overline{p}_{i^a k}^{m_2} = \frac{p_{i^a k}^{m_2}}{p_{i^b k}^{m_1}} \text{ and } \overline{p}_{i^a k}^{\beta} = \frac{p_{i^a k}^{m_2} p_{i^b k}^{\alpha} + p_{i^b k}^{m_1} p_{i^a k}^{\beta}}{p_{i^b k}^{m_1}\left(p_{i^b k}^{m_1} - p_{i^b k}^{\alpha}\right)}.$$

Based on the Lemma, the first constraint and last constraint in model (10),

$$Pos\left(\sum_{i=1}^{m} v_i \tilde{x}_{ik} = 1\right) \geq \delta \text{ and } Pos\left(v_{i^a} - \frac{\tilde{P}_{i^a k}}{\tilde{P}_{i^b k}} v_{i^b} = 0\right) \geq \delta$$

can also be transformed into the following constraints:

$$Pos\left(\sum_{i=1}^{m} v_i \tilde{x}_{ik} \geq 1\right) \geq \delta, \ Pos\left(\sum_{i=1}^{m} v_i \tilde{x}_{ik} \leq 1\right) \geq \delta,$$
$$Pos\left(v_{i^a} - \frac{\tilde{P}_{i^a k}}{\tilde{P}_{i^b k}} v_{i^b} \geq 0\right) \geq \delta \text{ and } Pos\left(v_{i^a} - \frac{\tilde{P}_{i^a k}}{\tilde{P}_{i^b k}} v_{i^b} \leq 0\right) \geq \delta,$$

respectively. These constraints can be rewritten as the following constraints based upon the Lemma:

$$\left(\sum_{i=1}^{m} v_i \tilde{x}_{ik}\right)^{R} \geq 1 \Leftrightarrow \sum_{i=1}^{m} v_i \left(x_{ik}^{m_2} + R^{-1}(\delta) x_{ik}^{\beta}\right) \geq 1$$

and

$$\left(\sum_{i=1}^{m} v_i \tilde{x}_{ik}\right)^{L} \leq 1 \Leftrightarrow \sum_{i=1}^{m} v_i \left(x_{ik}^{m_2} - L^{-1}(\delta) x_{ik}^{\beta}\right) \leq 1$$

$$v_{i^a} \geq \left(\frac{\tilde{P}_{i^a k}}{\tilde{P}_{i^b k}} v_{i^b}\right)^{L} \Leftrightarrow v_{i^a} \geq \frac{p_{i^a k}^{m_1}}{p_{i^b k}^{m_2}} v_{i^b} - L^{-1}(\delta)\left(\frac{p_{i^a k}^{m_1} p_{i^b k}^{\beta} + p_{i^b k}^{m_2} p_{i^a k}^{\alpha}}{p_{i^b k}^{m_2}\left(p_{i^b k}^{m_2} + p_{i^b k}^{\beta}\right)}\right) v_{i^b},$$

and

$$\left(\frac{\tilde{P}_{i^a k}}{\tilde{P}_{i^b k}} v_{i^b}\right)^{R} \geq v_{i^a} \Leftrightarrow \frac{p_{i^a k}^{m_2}}{p_{i^b k}^{m_1}} v_{i^b} + R^{-1}(\delta)\left(\frac{p_{i^a k}^{m_2} p_{i^b k}^{\alpha} + p_{i^b k}^{m_1} p_{i^a k}^{\beta}}{p_{i^b k}^{m_1}\left(p_{i^b k}^{m_1} - p_{i^b k}^{\alpha}\right)}\right) v_{i^b} \geq v_{i^a}.$$

Based on the Lemma, model (10) is equivalent to the following equations:

$$\begin{aligned}
&\max \varphi \\
&\text{subject to:} \\
&\varphi \leq \sum_{r=1}^{s} u_r y_{rk}^{m} + R^{-1}(\delta) \sum_{r=1}^{s} u_r y_{rj}^{\beta}, \\
&\sum_{i=1}^{m} v_i \left(x_{ik}^{m} + R^{-1}(\delta) x_{ik}^{\beta}\right) \geq 1, \\
&\sum_{i=1}^{m} v_i \left(x_{ik}^{m} - L^{-1}(\delta) x_{ik}^{\alpha}\right) \leq 1, \\
&\sum_{r=1}^{s} u_r \left(y_{rj}^{m} - L^{-1}(\delta) y_{rj}^{\alpha}\right) - \sum_{i=1}^{m} v_i \left(x_{ij}^{m} + R^{-1}(\delta) x_{ij}^{\beta}\right) \leq 0, \quad j = 1, \ldots, n, \\
&v_{i^a} \geq \frac{p_{i^a k}^{m_1}}{p_{i^b k}^{m_2}} v_{i^b} - L^{-1}(\delta)\left(\frac{p_{i^a k}^{m_1} p_{i^b k}^{\beta} + p_{i^b k}^{m_2} p_{i^a k}^{\alpha}}{p_{i^b k}^{m_2}\left(p_{i^b k}^{m_2} + p_{i^b k}^{\beta}\right)}\right) v_{i^b}, \quad i^a < i^b, i^a, i^b = 1, \ldots, m, \\
&\frac{p_{i^a k}^{m_2}}{p_{i^b k}^{m_1}} v_{i^b} + R^{-1}(\delta)\left(\frac{p_{i^a k}^{m_2} p_{i^b k}^{\alpha} + p_{i^b k}^{m_1} p_{i^a k}^{\beta}}{p_{i^b k}^{m_1}\left(p_{i^b k}^{m_1} - p_{i^b k}^{\alpha}\right)}\right) v_{i^b} \geq v_{i^a}, \quad i^a < i^b, i^a, i^b = 1, \ldots, m, \\
&u_r, v_i \geq \varepsilon, \quad r = 1, \ldots, s; i = 1, \ldots, m.
\end{aligned} \tag{11}$$

The fuzzy CE model (6) with deterministic input-output and fuzzy input prices and the CE model (7) with fuzzy input-output and exact input prices can be similarly transformed into the following two deterministic models, respectively:

$$
\begin{aligned}
&\max \sum_{r=1}^{s} u_r y_{rk} \\
&\text{subject to:} \\
&\sum_{i=1}^{m} v_i x_{ik} = 1, \\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j = 1, \dots, n, \\
&v_{i^a} \ge \frac{p_{i^a k}^{m_1}}{p_{i^b k}^{m_2}} v_{i^b} - L^{-1}(\delta) \left(\frac{p_{i^a k}^{m_1} p_{i^b k}^{\beta} + p_{i^b k}^{m_2} p_{i^a k}^{\alpha}}{p_{i^b k}^{m_2} \left(p_{i^b k}^{m_2} + p_{i^b k}^{\beta} \right)} \right) v_{i^b}, \quad i^a < i^b, i^a, i^b = 1, \dots, m, \\
&\frac{p_{i^a k}^{m_2}}{p_{i^b k}^{m_1}} v_{i^b} + R^{-1}(\delta) \left(\frac{p_{i^a k}^{m_2} p_{i^b k}^{\alpha} + p_{i^b k}^{m_1} p_{i^a k}^{\beta}}{p_{i^b k}^{m_1} \left(p_{i^b k}^{m_1} - p_{i^b k}^{\alpha} \right)} \right) v_{i^b} \ge v_{i^a}, \quad i^a < i^b, i^a, i^b = 1, \dots, m, \\
&u_r, v_i \ge \varepsilon, \quad r = 1, \dots, s; i = 1, \dots, m.
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
&\max \varphi \\
&\text{subject to:} \\
&\varphi \le \sum_{r=1}^{s} u_r y_{rk}^{m} + R^{-1}(\delta) \sum_{r=1}^{s} u_r y_{rj}^{\beta}, \\
&\sum_{i=1}^{m} v_i \left(x_{ik}^{m} + R^{-1}(\delta) x_{ik}^{\beta} \right) \ge 1, \\
&\sum_{i=1}^{m} v_i \left(x_{ik}^{m} - L^{-1}(\delta) x_{ik}^{\alpha} \right) \le 1, \\
&\sum_{r=1}^{s} u_r \left(y_{rj}^{m} - L^{-1}(\delta) y_{rj}^{\alpha} \right) - \sum_{i=1}^{m} v_i \left(x_{ij}^{m} + R^{-1}(\delta) x_{ij}^{\beta} \right) \le 0, \quad j = 1, \dots, n, \\
&v_{i^a} - \frac{P_{i^a k}}{P_{i^b k}} v_{i^b} = 0, \quad i^a < i^b, i^a, i^b = 1, \dots, m, \\
&u_r, v_i \ge \varepsilon, \quad r = 1, \dots, s; i = 1, \dots, m.
\end{aligned} \tag{13}
$$

6 Fuzzy FDH-CE

In the previous section we discussed the CE model with fuzzy variables. In this section, we develop an imprecise FDH-based formulation for dealing with the fuzzy parameters on a possibility space (Θ, *P*(Θ), *Pos*). Let us extend the FDH-CE formulation with fuzzy variables.

The FDH approach, an alternative method to DEA, was first introduced by Deprins et al. (1984). Tulkens (1993) formulated the FDH problem as a mixed integer linear program and proposed a numeration method for solving it with a simple trade-off between the DMUs without any need for solving the corresponding integer programme. Agrell and Tind (2001) formulated a FDH VRS model using a linear programme and Leleu (2006) extended their method and introduced several linear programming models to include other types of returns to scale in FDH. Briec et al. (2004) developed a series of nonparametric, non-convex models and cost functions with crisp data, and derived some analytical formulas for the non-convex cost functions without solving any mathematical programming models. Leleu (2006) obtained the FDH cost functions by linear programming.

6.1 The weight-restricted FDH-CE model

Leleu (2006) provided the following linear programming model to obtain the FDH-cost function of DMU$_k$:

$$\begin{aligned}
&C_{FDH\text{-}RTS} = \text{Min} \sum_{i=1}^{m}\sum_{j=1}^{n} p_{ik}\bar{x}_{ij} \\
&\text{subject to:} \\
&\left(\lambda_j + \omega_j\right) y_{rj} \geq \lambda_j y_{rk}, \quad r = 1,\ldots,s; \quad j = 1,\ldots,n, \\
&\left(\lambda_j + \omega_j\right) x_{ij} \leq \bar{x}_{ij}, \quad i = 1,\ldots,m; \quad j = 1,\ldots,n, \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\lambda_j \geq 0, \bar{x}_{ij} \geq 0, \quad i = 1,\ldots,m; \ j = 1,\ldots,n, \\
&\omega_j \in \Gamma_j, \Gamma_j \in \{NIRS, NDRS, CRS, VRS\}, \quad \forall j, \\
&NIRS = \left\{\omega_j : \omega_j \leq 0\right\}, \quad NDRS = \left\{\omega_j : \omega_j \geq 0\right\}, \\
&CRS = \left\{\omega_j : \omega_j \ unconstrained\right\}, \quad VRS = \left\{\omega_j : \omega_j = 0\right\}.
\end{aligned} \tag{14}$$

Assume that $(\lambda_j^*, \omega_j^*, \bar{x}_{ij}^*)$ is the optimal solution of model (1) for DMU$_k$. The FDH-CE of DMU$_k$ can be obtained as follows: ('CE', see below)

$$\text{FDH-CE of DMU}_k = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} p_{ik}\bar{x}_{ij}^*}{\sum_{i=1}^{m} p_{ik} x_{ik}} \tag{15}$$

Recently, Paryab et al. (2012) showed that the FDH-CE can be obtained by solving the following linear programming problem:

$$\begin{aligned}
&\min \sum_{j=1}^{n} \bar{C}_j \left(\lambda_j + \omega_j\right) \\
&\text{subject to:} \\
&\left(\lambda_j + \omega_j\right) y_{rj} \geq \lambda_j y_{rk}, \quad r = 1,\ldots,s; \ j = 1,\ldots,n,
\end{aligned} \tag{16}$$

$$\sum_{j=1}^{n} \lambda_j = 1,$$
$$\lambda_j \geq 0; \quad \omega_j \in \Gamma_j;\ j = 1,\ldots,n.$$

In model (16), $\bar{C}_j = \frac{\sum_{i=1}^{m} p_{ik} x_{ij}}{\sum_{i=1}^{m} p_{ik} x_{ik}}$, j = 1,…,n. p_{ik} are the input prices observed for DMU$_k$.

The objective function of model (16) can be used to assess the FDH-CE as follows:

$$\text{Min} \sum_{j=1}^{n} \bar{C}_j \left(\lambda_j + \omega_j\right)$$
subject to:
$$\left(\lambda_j + \omega_j\right) y_{rj} \geq \lambda_j y_{rk}, \quad r = 1,\ldots,s;\ j = 1,\ldots,n, \tag{17}$$
$$\sum_{j=1}^{n} \lambda_j = 1,$$
$$\lambda_j \geq 0; \quad \omega_j \in \Gamma_j;\ j = 1,\ldots,n.$$

By writing the dual of above model we have as follows:

$$FDH\text{-}CE_k^{Multiplier} = \max\ \rho$$
subject to:
$$\sum_{r=1}^{s} u_{rj} \left(y_{rj} - y_{rk}\right) + \rho \leq \frac{\sum_{i=1}^{m} p_{ik} x_{ij}}{\sum_{i=1}^{m} p_{ik} x_{ik}}, \quad j = 1,\ldots,n, \tag{18}$$
$$u_{rj}, v_{ij} \geq 0, \quad r = 1,\ldots,s;\ i = 1,\ldots,m;\ j = 1,\ldots,n.$$

and

$$\text{NIRS}: \quad \sum_{r=1}^{s} u_{rj} y_{rj} - \frac{\sum_{i=1}^{m} p_{ik} x_{ij}}{\sum_{i=1}^{m} p_{ik} x_{ik}} \geq 0, \quad j = 1,\ldots,n,$$

$$\text{NDRS}: \quad \sum_{r=1}^{s} u_{rj} y_{rj} - \frac{\sum_{i=1}^{m} p_{ik} x_{ij}}{\sum_{i=1}^{m} p_{ik} x_{ik}} \leq 0, \quad j = 1,\ldots,n,$$

$$\text{CRS}: \quad \sum_{r=1}^{s} u_{rj} y_{rj} - \frac{\sum_{i=1}^{m} p_{ik} x_{ij}}{\sum_{i=1}^{m} p_{ik} x_{ik}} = 0, \quad j = 1, \ldots, n,$$

$$\text{VRS}: \quad \sum_{r=1}^{s} u_{rj} y_{rj} - \frac{\sum_{i=1}^{m} p_{ik} x_{ij}}{\sum_{i=1}^{m} p_{ik} x_{ik}}, \quad j = 1, \ldots, n.$$

Letting $v_i = \frac{p_{ik}}{\sum_{i=1}^{m} p_{ik} x_{ik}}$, we can convert Model (18) into the following equivalent linear programming model:

$$\begin{aligned}
& FDH\text{-}CE_k^{Multiplier} = \max \ \rho \\
& \text{subject to:} \\
& \sum_{i=1}^{m} v_i x_{ik} = 1, \quad j = 1, \ldots, n, \\
& \sum_{r=1}^{s} u_{rj} \left(y_{rj} - y_{rk} \right) - \sum_{i=1}^{m} v_i x_{ij} + \rho \leq 0, \quad j = 1, \ldots, n, \\
& v_{i^a} - \frac{p_{i^a k}}{p_{i^b k}} v_{i^b} = 0, \quad j = 1, \ldots, n, \, i^a < i^b, \, i^a, i^b = 1, 2, \ldots, m, \\
& u_{rj}, v_i \geq 0, \quad r = 1, \ldots, s, \, i = 1, \ldots, m, \, j = 1, \ldots, n.
\end{aligned} \tag{19}$$

and

$$\text{NIRS}: \quad \sum_{r=1}^{s} u_{rj} y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \geq 0, \quad j = 1, \ldots, n,$$

$$\text{NDRS}: \quad \sum_{r=1}^{s} u_{rj} y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, \ldots, n,$$

$$\text{CRS}: \quad \sum_{r=1}^{s} u_{rj} y_{rj} - \sum_{i=1}^{m} v_i x_{ij} = 0, \quad j = 1, \ldots, n,$$

$$\text{VRS}: \quad \sum_{r=1}^{s} u_{rj} y_{rj} - \sum_{i=1}^{m} v_i x_{ij}, \quad j = 1, \ldots, n.$$

□

Above discussion yields the following theorem:

Theorem: The optimal objective function value of (19) is equal to that obtained by (15).

We use (19) to develop a sequence of FDH-CE measures with fuzzy variables in this section. We have three types of models including: FDH-CE with fuzzy input prices similar to the fuzzy CE models in the convex approach: fuzzy FDH-CE models with fuzzy input-output data, and fuzzy input-output as well as fuzzy input prices. These three models are as follows:

6.1.1 FDH-CE model with fuzzy input prices:

$$
\begin{aligned}
&\max \rho \\
&\text{subject to:} \\
&\sum_{i=1}^{m} v_{ij} x_{ik} = 1, \quad j = 1, \ldots, n, \\
&\sum_{r=1}^{s} u_{rj} \left(y_{rj} - y_{rk} \right) - \sum_{i=1}^{m} v_{ij} x_{ij} + \rho \le 0, \quad j = 1, \ldots, n, \\
&Pos\left(v_{i^a j} - \frac{\tilde{p}_{i^a k}}{\tilde{p}_{i^b k}} v_{i^b j} = 0 \right) \ge \delta, \quad j = 1, \ldots, n, \, i^a < i^b, \, i^a, i^b = 1, 2, \ldots, m, \\
&u_{rj}, v_{ij} \ge 0, \quad r = 1, \ldots, s; \, i = 1, \ldots, m; \, j = 1, \ldots, n.
\end{aligned} \tag{20}
$$

We have the following for the VRS assumptions:

$$
\text{NIRS}: \quad \sum_{r=1}^{s} u_{rj} y_{rj} - \sum_{i=1}^{m} v_{ij} x_{ij} \ge 0, \quad j = 1, \ldots, n,
$$

$$
\text{NDRS}: \quad \sum_{r=1}^{s} u_{rj} y_{rj} - \sum_{i=1}^{m} v_{ij} x_{ij} \le 0, \quad j = 1, \ldots, n,
$$

$$
\text{CRS}: \quad \sum_{r=1}^{s} u_{rj} y_{rj} - \sum_{i=1}^{m} v_{ij} x_{ij} = 0, \quad j = 1, \ldots, n,
$$

$$
\text{VRS}: \quad \sum_{r=1}^{s} u_{rj} y_{rj} - \sum_{i=1}^{m} v_{ij} x_{ij} = 0, \quad j = 1, \ldots, n.
$$

6.1.2 FDH-CE model with fuzzy input-output:

$$
\begin{aligned}
&\max \rho \\
&\text{subject to:} \\
&Pos\left(\sum_{i=1}^{m} v_{ij} \tilde{x}_{ik} = 1 \right) \ge \delta, \quad j = 1, \ldots, n, \\
&Pos\left(\sum_{r=1}^{s} u_{rj} \left(\tilde{y}_{rj} - \tilde{y}_{rk} \right) - \sum_{i=1}^{m} v_{ij} \tilde{x}_{ij} + \rho \le 0 \right) \ge \delta, \quad j = 1, \ldots, n, \\
&v_{i^a j} - \frac{p_{i^a k}}{p_{i^b k}} v_{i^b j} = 0, \quad j = 1, \ldots, n, \, i^a < i^b, \, i^a, i^b = 1, 2, \ldots, m, \\
&u_{rj}, v_{ij} \ge 0, \quad r = 1, \ldots, s; \, i = 1, \ldots, m; \, j = 1, \ldots, n.
\end{aligned} \tag{21}
$$

We have the following for the VRS assumptions:

$$\begin{aligned}
&\text{NIRS}: \quad \sum_{r=1}^{s} u_{rj}\tilde{y}_{rj} - \sum_{i=1}^{m} v_{ij}\tilde{x}_{ij} \geq 0, \quad j = 1,\ldots,n, \\
&\text{NDRS}: \quad \sum_{r=1}^{s} u_{rj}\tilde{y}_{rj} - \sum_{i=1}^{m} v_{ij}\tilde{x}_{ij} \leq 0, \quad j = 1,\ldots,n, \\
&\text{CRS}: \quad \sum_{r=1}^{s} u_{rj}\tilde{y}_{rj} - \sum_{i=1}^{m} v_{ij}\tilde{x}_{ij} = 0, \quad j = 1,\ldots,n, \\
&\text{VRS}: \quad \sum_{r=1}^{s} u_{rj}\tilde{y}_{rj} - \sum_{i=1}^{m} v_{ij}\tilde{x}_{ij} = 0, \quad j = 1,\ldots,n.
\end{aligned} \tag{22}$$

6.1.3 FDH-CE model with fuzzy input-output and fuzzy input price

$$\begin{aligned}
&\max \rho \\
&\text{subject to:} \\
&Pos\left(\sum_{i=1}^{m} v_{ij}\tilde{x}_{ik} = 1\right) \geq \delta, \quad j = 1,\ldots,n, \\
&Pos\left(\sum_{r=1}^{s} u_{rj}\left(\tilde{y}_{rj} - \tilde{y}_{rk}\right) - \sum_{i=1}^{m} v_{ij}\tilde{x}_{ij} + \rho \leq 0\right) \geq \delta, \quad j = 1,\ldots,n, \\
&Pos\left(v_{i^a j} - \frac{\tilde{p}_{i^a k}}{\tilde{p}_{i^b k}} v_{i^b j} = 0\right) \geq \delta, \quad j = 1,\ldots,n, i^a < i^b, i^a, i^b = 1, 2,\ldots,m, \\
&u_{rj}, v_{ij} \geq 0, \quad r = 1,\ldots,s; i = 1,\ldots,m; j = 1,\ldots,n.
\end{aligned} \tag{23}$$

We have the following for the VRS assumptions:

$$\begin{aligned}
&\text{NIRS}: \quad Pos\left(\sum_{r=1}^{s} u_{rj}\tilde{y}_{rj} - \sum_{i=1}^{m} v_{ij}\tilde{x}_{ij} \geq 0\right) \geq \delta, \quad j = 1,\ldots,n, \\
&\text{NDRS}: \quad Pos\left(\sum_{r=1}^{s} u_{rj}\tilde{y}_{rj} - \sum_{i=1}^{m} v_{ij}\tilde{x}_{ij} \leq 0\right) \geq \delta, \quad j = 1,\ldots,n, \\
&\text{CRS}: \quad Pos\left(\sum_{r=1}^{s} u_{rj}\tilde{y}_{rj} - \sum_{i=1}^{m} v_{ij}\tilde{x}_{ij} = 0\right) \geq \delta, \quad j = 1,\ldots,n, \\
&\text{VRS}: \quad Pos\left(\sum_{r=1}^{s} u_{rj}\tilde{y}_{rj} - \sum_{i=1}^{m} v_{ij}\tilde{x}_{ij} = 0\right) \geq \delta, \quad j = 1,\ldots,n.
\end{aligned} \tag{24}$$

Similar to the solution method in model (10), we use the Lemma for the constraints of fuzzy FDH-CE models (20), (21) and (23) to obtain the following deterministic models:

$$
\begin{aligned}
& FDH\text{-}CE_k^{Multiplier} = \max\ \rho \\
& \text{subject to:} \\
& \sum_{i=1}^{m} v_{ij} x_{ik} = 1, \quad j = 1,\ldots,n, \\
& \sum_{r=1}^{s} u_{rj}\left(y_{rj} - y_{rk}\right) - \sum_{i=1}^{m} v_{ij} x_{ij} + \rho \le 0, \quad j = 1,\ldots,n, \\
& v_{i^a j} \ge \frac{p_{i^a k}^{m_1}}{p_{i^b k}^{m_2}} v_{i^b j} - L^{-1}(\delta)\left(\frac{p_{i^a k}^{m_1} p_{i^b k}^{\beta} + p_{i^b k}^{m_2} p_{i^a k}^{\alpha}}{p_{i^b k}^{m_2}\left(p_{i^b k}^{m_2} + p_{i^b k}^{\beta}\right)}\right) v_{i^b j}, \quad i^a < i^b,\ i^a, i^b = 1,\ldots,m, \\
& \frac{p_{i^a k}^{m_2}}{p_{i^b k}^{m_1}} v_{i^b j} + R^{-1}(\delta)\left(\frac{p_{i^a k}^{m_2} p_{i^b k}^{\alpha} + p_{i^b k}^{m_1} p_{i^a k}^{\beta}}{p_{i^b k}^{m_1}\left(p_{i^b k}^{m_1} - p_{i^b k}^{\alpha}\right)}\right) v_{i^b j} \ge v_{i^a j}, \quad i^a < i^b,\ i^a, i^b = 1,\ldots,m, \\
& u_{rj}, v_{ij} \ge 0, \quad r = 1,\ldots,s;\ i = 1,\ldots,m;\ j = 1,\ldots,n.
\end{aligned}
\tag{25}
$$

and

$$
\begin{aligned}
& \text{NDRS}: \quad \sum_{r=1}^{s} u_{rj} y_{rj} - \sum_{i=1}^{m} v_{ij} x_{ij} \le 0, \quad j = 1,\ldots,n, \\
& \text{NIRS}: \quad \sum_{r=1}^{s} u_{rj} y_{rj} - \sum_{i=1}^{m} v_{ij} x_{ij} \ge 0, \quad j = 1,\ldots,n, \\
& \text{CRS}: \quad \sum_{r=1}^{s} u_{rj} y_{rj} - \sum_{i=1}^{m} v_{ij} x_{ij} = 0; \quad j = 1,\ldots,n,
\end{aligned}
$$

$$
\begin{aligned}
& \max\ \rho \\
& \text{subject to:} \\
& \sum_{i=1}^{m} v_{ij}\left(x_{ik}^{m_2} + R^{-1}(\delta) x_{ik}^{\beta}\right) \ge 1, \quad j = 1,\ldots,n, \\
& \sum_{i=1}^{m} v_{ij}\left(x_{ik}^{m_1} - L^{-1}(\delta) x_{ik}^{\alpha}\right) \le 1, \quad j = 1,\ldots,n, \\
& \sum_{r=1}^{s} u_{rj}\left(\left(y_{rj}^{m_1} - y_{rk}^{m_2}\right) - L^{-1}(\delta)\left(y_{rj}^{\alpha} + y_{rk}^{\beta}\right)\right) \\
& - \sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_2} + R^{-1}(\delta) x_{ij}^{\beta}\right) + \rho \le 0, \quad j = 1,\ldots,n, \\
& v_{i^a j} - \frac{p_{i^a k}}{p_{i^b k}} v_{i^b j} = 0, \quad j = 1,\ldots,n,\ i^a < i^b,\ i^a, i^b = 1, 2,\ldots,m, \\
& u_{rj}, v_{ij} \ge 0, \quad r = 1,\ldots,s;\ i = 1,\ldots,m;\ j = 1,\ldots,n.
\end{aligned}
\tag{26}
$$

and

$$\text{NDRS}: \quad \sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_1} - L^{-1}(\delta) y_{rj}^{\alpha}\right) - \sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_2} + R^{-1}(\delta) x_{ij}^{\beta}\right) \le 0, \quad j = 1,\ldots,n,$$

$$\text{NIRS}: \quad \sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_2} + R^{-1}(\delta) y_{rj}^{\beta}\right) - \sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_1} - L^{-1}(\delta) x_{ij}^{\alpha}\right) \ge 0, \quad j = 1,\ldots,n,$$

$$\text{CRS}: \quad \begin{cases} \sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_1} - L^{-1}(\delta) y_{rj}^{\alpha}\right) - \sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_2} + R^{-1}(\delta) x_{ij}^{\beta}\right) \le 0; \quad j = 1,\ldots,n, \\ \sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_2} + R^{-1}(\delta) y_{rj}^{\beta}\right) - \sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_1} - L^{-1}(\delta) x_{ij}^{\alpha}\right) \ge 0; \quad j = 1,\ldots,n. \end{cases}$$

$$\begin{aligned}
&\max\ \rho \\
&\text{subject to:} \\
&\sum_{i=1}^{m} v_{ij}\left(x_{ik}^{m_2} + R^{-1}(\delta) x_{ik}^{\beta}\right) \ge 1, \quad j = 1,\ldots,n, \\
&\sum_{i=1}^{m} v_{ij}\left(x_{ik}^{m_1} - L^{-1}(\delta) x_{ik}^{\alpha}\right) \le 1, \quad j = 1,\ldots,n, \\
&\sum_{r=1}^{s} u_{rj}\left(\left(y_{rj}^{m_1} - y_{rk}^{m_2}\right) - L^{-1}(\delta)\left(y_{rj}^{\alpha} + y_{rk}^{\beta}\right)\right) \\
&-\sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_2} + R^{-1}(\delta) x_{ij}^{\beta}\right) + \rho \le 0, \ j = 1,\ldots,n, \\
&v_{i^a j} \ge \frac{p_{i^a k}^{m_1}}{p_{i^b k}^{m_2}} v_{i^b j} - L^{-1}(\delta)\left(\frac{p_{i^a k}^{m_1} p_{i^b k}^{\beta} + p_{i^b k}^{m_2} p_{i^a k}^{\alpha}}{p_{i^b k}^{m_2}\left(p_{i^b k}^{m_2} + p_{i^b k}^{\beta}\right)}\right) v_{i^b j}, \quad i^a < i^b, i^a, i^b = 1,\ldots,m, \\
&\frac{p_{i^a k}^{m_2}}{p_{i^b k}^{m_1}} v_{i^b j} + R^{-1}(\delta)\left(\frac{p_{i^a k}^{m_2} p_{i^b k}^{\alpha} + p_{i^b k}^{m_1} p_{i^a k}^{\beta}}{p_{i^b k}^{m_1}\left(p_{i^b k}^{m_1} - p_{i^b k}^{\alpha}\right)}\right) v_{i^b j} \ge v_{i^a j}, i^a < i^b, i^a, i^b = 1,\ldots,m, \\
&u_{rj}, v_{ij} \ge 0, \quad r = 1,\ldots,s; i = 1,\ldots,m; j = 1,\ldots,n.
\end{aligned} \tag{27}$$

and

$$\text{NDRS}: \quad \sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_1} - L^{-1}(\delta) y_{rj}^{\alpha}\right) - \sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_2} + R^{-1}(\delta) x_{ij}^{\beta}\right) \le 0, \quad j = 1,\ldots,n,$$

$$\text{NIRS}: \quad \sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_2} + R^{-1}(\delta) y_{rj}^{\beta}\right) - \sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_1} - L^{-1}(\delta) x_{ij}^{\alpha}\right) \ge 0, \quad j = 1,\ldots,n,$$

$$\text{CRS}: \quad \begin{cases} \sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_1} - L^{-1}(\delta) y_{rj}^{\alpha}\right) - \sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_2} + R^{-1}(\delta) x_{ij}^{\beta}\right) \le 0; \quad j = 1,\ldots,n, \\ \sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_2} + R^{-1}(\delta) y_{rj}^{\beta}\right) - \sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_1} - L^{-1}(\delta) x_{ij}^{\alpha}\right) \ge 0; \quad j = 1,\ldots,n. \end{cases}$$

7 Numerical illustrations

In this section, we use a hypothetical example to examine the applicability of the proposed models. Consider ten DMUs with two fuzzy symmetrical triangular inputs-outputs and two fuzzy symmetrical triangular input prices as reported in Table 1.

Table 1 The fuzzy input-output and input price of DMUs

DMU	*Fuzzy input 1*	*Fuzzy input 2*	*Fuzzy input price 1*	*Fuzzy input price 2*	*Fuzzy output 1*	*Fuzzy output 2*
1	(53, 3.6)	(45.0, 4.5)	(5.0, 0.3)	(5.0, 0.5)	(77.5, 7.5)	(35.4, 6.6)
2	(25, 5.0)	(46.5, 4.9)	(5.0, 1.5)	(6.0, 1.5)	(49.6, 8.1)	(34.7, 5.8)
3	(18, 6.5)	(15.7, 3.8)	(8.0, 0.8)	(7.0, 0.7)	(26.5, 4.9)	(37.6, 8.2)
4	(18, 5.9)	(25.5, 5.4)	(9.0, 1.8)	(5.5, 0.5)	(37.5, 3.3)	(47.5, 6.6)
5	(32, 2.9)	(25.0, 4.7)	(3.0, 0.2)	(2.0, 0.25)	(64.0, 4.8)	(76.4, 3.5)
6	(56, 5.2)	(45.1, 3.5)	(6.0, 0.6)	(4.0, 0.75)	(35.3, 2.7)	(42.5, 5.5)
7	(24, 6.4)	(17.5, 3.9)	(2.0, 0.4)	(2.0, 0.5)	(82.9, 7.3)	(38.5, 5.3)
8	(78, 6.8)	(23.9, 5.4)	(3.0, 1.2)	(3.9, 0.9)	(66.0, 6.0)	(47.4, 9.2)
9	(52, 6.5)	(19.8, 6.1)	(5.0, 0.2)	(9.8, 1.8)	(56.5, 5.4)	(56.0, 4.7)
10	(49, 4.4)	(20.6, 4.3)	(2.9, 0.9)	(2.6, 0.6)	(46.5, 8.5)	(38.0, 5.9)

This data is denoted by (m, α) where m is the centre value and α is the spread value. models (11) and (27) have been solved using GAMS software to obtain the values of efficiency.

Four different possibility (threshold) levels of $\delta = 0$, $\delta = 0.25$, $\delta = 0.75$, and $\delta = 1$ are considered to compare the results. The result for model (11) is reported in Table 2.

Table 2 The CE model results

Δ	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 0.75$	$\delta = 1.0$
DMU 1	0.7091	0.6259	0.5498	0.4799	0.4191
DMU 2	1.2639	1.0738	0.9118	0.773	0.6439
DMU 3	1.8936	1.5354	1.2607	1.0446	0.8111
DMU 4	2.0817	1.7417	1.4719	1.2536	1.0000
DMU 5	1.4009	1.2836	1.1787	1.0846	1.0000
DMU 6	0.4538	0.4153	0.3799	0.3473	0.3064
DMU 7	2.0946	1.7371	1.4443	1.2021	1.0000
DMU 8	0.5709	0.4943	0.4310	0.3767	0.3290
DMU 9	0.7499	0.6733	0.6055	0.5452	0.4749
DMU 10	0.6849	0.5941	0.5179	0.4526	0.3962

As shown in Table 2, all the possibility levels DMU 4, DMU 5, DMU 7 are cost efficient, while DMUs 1, 6, 8 and 10 are always cost inefficient. Interestingly, the CE measure strictly decreases when δ is increased. For instance, efficiency results for DMU 1 under possibility levels decreases from 0.7091 to 0.4191. It is obvious that if the DMU under possibility level $\delta = 0$ is cost inefficient, then the DMU under all possibility levels

(δ = 0.25, 0.5, 0.75, and 1) is cost inefficient. And also if the DMU under possibility level δ = 1 is cost efficient, then the DMU under all possibility levels (δ = 0.25, 0.5, 0.75, and 1) is cost efficient.

Similar to the fuzzy CE model, now we compute the efficiency measures of the fuzzy FDH-cost model. The result for the fuzzy FDH-CE based on the CRS approach is reported in Table 3.

Table 3 The fuzzy FDH-CE for the CRS approach results

δ	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 0.75$	$\delta = 1.0$
DMU 1	0.7382	0.6475	0.5633	0.4852	0.4191
DMU 2	1.6201	1.4100	1.1772	0.9759	0.7959
DMU 3	2.0792	1.6700	1.3479	1.0742	0.8111
DMU 4	2.2014	1.8321	1.5324	1.2528	1.0000
DMU 5	1.4248	1.3052	1.1953	1.0939	1.0000
DMU 6	0.4959	0.4482	0.4022	0.3588	0.3170
DMU 7	2.1366	1.7707	1.4680	1.2145	1.0000
DMU 8	0.6955	0.6190	0.5564	0.4761	0.4045
DMU 9	0.8328	0.7539	0.6841	0.6187	0.5552
DMU 10	0.8599	0.7577	0.6672	0.5847	0.5081

The result in Table 3 indicates that the efficiency status differs between the two models. The difference occurs because the fuzzy FDH-CE approach is constructed relative to the non-convex models, but the fuzzy CE is constructed based on the convex models. Also the efficiency values of the fuzzy FDH-cost models are equal to or greater than the fuzzy CE models.

8 Conclusions and future research directions

Crisp input and output data are fundamentally indispensable in traditional DEA evaluation processes. However, the observed values of the input and output data in real-world problems are sometimes imprecise or vague. Imprecise evaluations may be the result of unquantifiable, incomplete and non-obtainable information. In this paper, we proposed two CE approaches based on the convex DEA and non-convex FDH models with fuzzy variables. We developed a CE analysis for nonparametric convex models based on fuzzy set theory. We then extended the convex fuzzy CE approach into a non-convex FDH approach, which could then deal with the VRS assumptions. Similar to the DEA case, we provided a crisp model for solving the fuzzy FDH-cost models. Finally, we presented a numerical example and demonstrated the applicability of the proposed models and exhibited the efficacy of the proposed models. For further research, we plan to extend the proposed approach to other types of imprecise data such as random or fuzzy random data and investigate the practical applications of the model with a real-world problem.

Acknowledgements

The authors would like to thank the anonymous reviewers and the editor for their insightful comments and suggestions.

References

Agrell, P.J. and Tind, J. (2001) 'A dual approach to nonconvex frontier models', *Journal of Productivity Analysis*, Vol. 16, No. 2, pp.129–147.

Bagherzadeh Valami, H. (2009) 'Cost efficiency with triangular fuzzy number input prices: an application of DEA', *Chaos, Solitons and Fractals*, Vol. 42, No. 3, pp.1631–1637.

Banker, R.D., Charnes, A. and Cooper, W.W. (1984) 'Some method for estimating technical and scale inefficiencies in data envelopment analysis, *Management Science*, Vol. 30, No. 9, pp.1078–1092.

Briec, W., Kerstens, K. and Vanden Eechaut, P. (2004) 'Non-convex technologies and cost functions: definitions, duality and nonparametric tests of convexity', *Journal of Economics*, Vol. 81, No. 2, pp.155–192.

Camanho, A.S. and Dyson, R.G. (2005) 'Cost efficiency measurement with price uncertainty: a DEA application to bank branch assessments', *European Journal of Operational Research*, Vol. 161, No. 2, pp.432–446.

Charnes, A. and Cooper, W.W. (1959) 'Deterministic equivalents for optimizing and satisfying under chance constraints'. *Operations Research*, Vol. 11, No. 1, pp.18–39.

Charnes, A., Cooper, W.W. and Rhodes, E.L. (1978) 'Measuring the efficiency of decision making units', *European Journal of Operational Research*, Vol. 2, No. 6, pp.429–444.

Deprins, D., Simar, L. and Tulkens, H. (1984) 'Measuring labor efficiency in post offices', in March, M., Pestieu, P. and Tulkens, H. (Eds.): *The Performance of Public Enterprises: Concepts and Measurements*, pp.247–263, North Holland, Amsterdam.

Dubois, D. and Prade, H. (1980) *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Academic Press, New York.

Fang, L. and Li, H. (2013) 'Duality and efficiency computations in the cost efficiency model with price uncertainty', *Computers & Operations Research*, Vol. 40, No. 2, pp.594–602.

Färe, R., Grosskopf, S. and Lovell, C.A.K. (1985) *The Measurement of Efficiency of Production*, Klumer-Nijhoff Publishing, Boston.

Farrell, M.J. (1957) 'The measurement of productive efficiency', *Journal of Royal Statistical Society*, Series A, Vol. 120, No. 3, pp.253–281.

Guo, P. and Tanaka, H. (2001) 'Fuzzy DEA: a perceptual evaluation method', *Fuzzy Sets and Systems*, Vol. 119, No. 1, pp.149–160.

Jahanshahloo, G.R., Soleimani-damaneh, M. and Mostafaee, A. (2007a) 'On the computational complexity of cost efficiency analysis models', *Applied Mathematics and Computation*, Vol. 188, No. 1, pp.638–640.

Jahanshahloo, G.R., Soleimani-damaneh, M. and Mostafaee, A. (2007b) 'Cost efficiency analysis with ordinal data: a theoretical and computational view', *International Journal of Computer Mathematics*, Vol. 84, No. 4, pp.553–562.

Jahanshahloo, G.R., Soleimani-damaneh, M. and Mostafaee, A. (2008) 'A simplified version of the DEA cost efficiency model', *European Journal of Operational Research*, Vol. 184, No. 2, pp.814–815.

Kao, C. and Liu, S.T. (2003) 'A mathematical programming approach to fuzzy efficiency ranking', *International Journal of Production Economics*, Vol. 86, No. 2, pp.45–154.

Kao, C. and Liu, S.T. (2005) 'Data envelopment analysis with imprecise data: an application of Taiwan machinery firms', *International Journal of Uncertainty, Fuzziness and Knowledge Based Systems*, Vol. 13, No. 2, pp.225–240.

Kuosmanen, T. (2003) 'Duality theory of non-convex technologies', *Journal of Productivity Analysis*, Vol. 20, No. 3, pp.273–304.

Kuosmanen, T. and Post, T. (2001) 'Measuring economic efficiency with incomplete price information: with an application to European commercial banks', *European Journal of Operational Research*, Vol. 134, No. 1, pp.43–58.

Leleu, H. (2006) 'A linear programming framework for free disposal hull technologies and cost functions: primal and dull models', *European Journal of Operational Research*, Vol. 168, No. 2, pp.340–344.

León, T., Liern, V., Ruiz, J.L. and Sirvent, I. (2003) 'A fuzzy mathematical programming approach to the assessment of efficiency with DEA models', *Fuzzy Sets and Systems*, Vol. 139, No. 2, pp.407–419.

Lertworasirikul, S., Shu-Cherng, F., Joines, J.A. and Nuttle, H.L.W. (2003a) 'Fuzzy data envelopment analysis (DEA): a possibility approach', *Fuzzy Sets and Systems*, Vol. 139, No. 2, pp.379–394.

Lertworasirikul, S., Shu-Cherng, Nuttle, H.L.W.F. and Joines, J.A. (2003b) 'Fuzzy BCC model for data envelopment analysis', *Fuzzy Optimization and Decision Making*, Vol. 2, No. 4, pp.337–358.

Liu, B. (2002) *Theory and Practice of Uncertain Programming*, Physica-Verlag, New York.

Liu, B. and Liu, Y.K. (2002) 'Expected value of fuzzy variable and fuzzy expected value models', *IEEE Transactions on Fuzzy Systems*, Vol. 10, No. 4, pp.445–450.

Mostafaee, A. and Saljooghi, F.H. (2010) 'Cost efficiency measures in data envelopment analysis with data uncertainty', *European Journal of Operational Research*, Vol. 202, No. 2, pp.595–603.

Paryab, K.H., Khanjani Shiraz, R. and Jalalzadeh, L. (2012) 'An improvement model for assessing FDH-cost efficiency', *Asia-Pacific Journal of Operational Research*, Vol. 29, No. 4, 5pp, DOI: 10.1142/S0217595912500224.

Sakawa, M. (1993) *Fuzzy Sets and Interactive Multiobjective Optimization*, Plenum Press, New York.

Schaffnit, C., Rosen, D. and Paradi, J.C. (1997) 'Best practice analysis of bank branches: an application of DEA in a large Canadian bank', *European Journal of Operational Research*, Vol. 98, No. 2, pp.269–289.

Sengupta, J.K. (1992) 'A fuzzy systems approach in data envelopment analysis', *Computers and Mathematics with Applications*, Vol. 24, No. 8, pp.259–266.

Tavana, M., Khanjani Shiraz, R., Hatami-Marbini, A., Agrell, P.J. and Paryab, K. (2012) 'Fuzzy stochastic data envelopment analysis with application to base realignment and closure (BRAC)', *Expert Systems and Applications*, Vol. 39, No. 15, pp.12247–12259.

Thompson, R.G., Dharmapala, P.S., Humphrey, D.B., Taylor, W.M., and Thrall, R.M. (1996) 'Computing DEA/AR efficiency and profit ratio measures with an illustrative bank application', *Annals of Operations Research,* Vol. 68, No. 3, pp.303–327.

Triantis, K. and Girod, O. (1998) 'A mathematical programming approach for measuring technical efficiency in a fuzzy environment', *Journal of Productivity Analysis*, Vol. 10, No. 1, pp.85–102.

Tulkens, H. (1993) 'On FDH efficiency: some methodological issues and application to retail banking, courts, and urban transit', *Journal of Productivity Analysis*, Vol. 4, No. 1, pp.183–210.

Wang, Y.M. and Chin, K.S. (2011) 'Fuzzy data envelopment analysis: a fuzzy expected value approach', *Expert Systems with Applications*, Vol. 38, No. 9, pp.11678–11685.

Zadeh, L.A. (1965) 'Fuzzy sets', *Information and Control*, Vol. 8, No. 3, pp.338–353.

Zadeh, L.A. (1978) 'Fuzzy sets as a basis for a theory of possibility', *Fuzzy Sets and Systems*, Vol. 1, No. 1, pp.3–28.

Zimmermann, H.J. (1996) *Fuzzy Set Theory and its Applications*, 2nd ed., Kluwer Academic Publishers, Dordrecht, The Netherlands.