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A novel entropy-based decision support framework for uncertainty resolution in the initial subjective evaluations of experts: The NATO enlargement problem

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ABSTRACT

We introduce a novel decision support framework that allows decision makers (DMs) to assess the informativeness of a ranking of alternatives provided by different experts and to extrapolate additional evaluations based on the distributional bias and entropy inherent to those received from the experts. In the proposed framework, expert analysts rank several alternatives within opportunities versus threats space of potential evaluations. The resulting rankings of alternatives vary among experts due to differences in the set of evaluation criteria chosen and to the subjective weights assigned by the experts to the criteria considered. As a result, DMs observe biases in the evaluations provided by the experts and variations in the degree of informativeness among alternatives. Thus, DMs must use the information available to them to assess the reliability of the ranking obtained from the experts' evaluations. In this regard, the distributional bias generated by the evaluations received will be used to define the dynamic structure of an algorithm that allows DMs to extrapolate additional expected evaluations and modify the initial ranking proposed by the experts accordingly. At the same time, the entropy generated by the evaluations will be used to validate the reliability of the resulting rankings and to determine the stopping rule for the data generating algorithm. A numerical example based on the North Atlantic Treaty Organization (NATO) membership enlargement problem is presented, where several teams of experts provide different evaluations on a set of applicant countries. A battery of Monte Carlo simulations has been performed, and alternative biased approaches have been followed. The rankings obtained have been compared with those resulting from our framework.

1. Introduction

In this study, we introduce a novel decision support framework that allows decision makers (DMs) to assess the ranking of alternatives provided by different experts and gain additional information by taking into account the distributional bias and entropy inherent to the decision making process. The high degree of subjectivity inherent to the judgments of experts when providing decision support has been consistently analyzed in the systems literature [1], particularly when dealing with the outcomes of multi-criteria decision problems [2]. Thus, it is widely acknowledged that the rankings of alternatives will vary among experts due to both the different evaluation criteria adopted and the subjective weights assigned to them by the experts. As a result, DMs will observe biases in the evaluations provided by the experts and variations in the degree of informativeness of such evaluations.

The decision support system (DSS) literature generally assumes that once the main opinions of the experts are presented to the DM, he will either utilize them [3] or ignore them [4]. Moreover, the DSS literature attempts to infer the preferences of the experts and maximize consensus when studying the rankings that are based upon subjective preferences [5,6]. In this setting, experts will provide judgments based on both their personal preferences and the information subsets (subjectively) chosen to assign their evaluations.

We move beyond the standard DSS setting and consider the formation of expectations by DMs and their capacity to extrapolate additional observations when presented with the evaluations of several alternatives by various experts. We must, therefore, consider two main sources

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of uncertainty: first, the uncertainty arising from the selection criteria and experts' subjective initial evaluations; and, second, the uncertainty due to the DM making additional observations.

Extensive literature studies DMs forecasting abilities when faced with a limited amount of *objective* information (see [7] for a comprehensive review). In this line of research, past observations by the DMs are essential for developing the forecasting process [8]. We will focus on the informativeness and the relative entropy of the evaluations. Both of these properties may reflect biases due to opinion similarities among the experts [9, 10], subjective weights assigned to the evaluations [11,12], or frictions in the aggregation process of the evaluations [13,14].

Our model exploits the biases of the experts observed in the dispersion and symmetry of the evaluations as well as the entropy associated with them. This latter measure of informativeness will be used to validate both the extrapolation of additional evaluations (allowing us to determine the stopping rule of the evaluation-generating process) and the subjective dispersion of the set of potential realizations considered by the DMs.

Following the DSS literature [15], we assume the DMs to expect ranking disagreements among the experts, despite the fact that the experts' subjective evaluations should all be equally reliable. As a result, DMs should try to infer a particular pattern in the evaluations to generate a ranking potentially more reliable than the one provided by the experts.

The main contribution of the ranking evaluation and validation method proposed in this paper is that it provides a quantifiable alternative to existing approaches relying on an objective distribution of the experts' evaluations.

Indeed, several methods already exist, such as PERT [16], designed to generate limit distributions, even when dealing with large degrees of data subjectivity. However, these Monte Carlo-based methods require a given probability distribution to define the data-generating process [17]. Thus, even if the information is based on a given density function, the subjectivity of the subset of data chosen and their weights would distort the process of obtaining a reliable limit distribution.

To support our analysis, we will perform a battery of Monte Carlo simulations and provide several alternative extrapolation approaches. The different rankings obtained will be compared with those resulting from the newly proposed framework.

Several attempts to deal with the subjectivity of the evaluation process and objectivize indexes generated through a common weighting criterion are also present in the empirical social science literature [18]. Nevertheless, economists expect a certain degree of subjectivity in the evaluations, which is considered to be optimal if each expert uses an empirically valid model [19]. As a result, economists concentrate on identifying and strategically selecting the expert whose preferences are closer to those of the DM [20–22]. The results obtained in this paper are, therefore, applicable to these branches, along with the general DSS literature.

The main contributions of the paper can be summarized as follows:

- We present a novel approach that allows DMs to assess the informativeness and reliability of a ranking based on the distributional bias of the evaluations received from the experts and their entropy.
- We illustrate how the distributional bias generated by the evaluations can be used to define a dynamic algorithm that allows DMs to extrapolate additional expected evaluations and modify the initial ranking proposed by the experts.
- We show that the entropy generated by the evaluations can be used to validate the reliability of the rankings extrapolated by the DMs and determine the stopping rule for the evaluation-generating algorithm.

We use a numerical example to demonstrate the applicability of the proposed framework. This example extends the analysis performed by [23] on the North Atlantic Treaty Organization (NATO) enlargement process.

The paper proceeds as follows. Section 2 introduces the basic geometric evaluation environment. Section 3 defines the first loop within the extrapolation process of the DMs. Section 4 describes the dynamic structure that allows the DMs to extrapolate further evaluations. Section 5 provides a numerical example with a ranking validation criterion based on the entropy generated by the evaluations. Section 6 concludes.

2. Geometric environment: defining expected evaluations

Consider a standard Euclidean plane. Let a value on the horizontal axis represent the opportunity score achieved by a given alternative and a value on the vertical axis represent the value of the threats (or threat level) posed by the alternative (see [24,25]). This setting is represented in Fig. 1a and b. Throughout the paper, both the evaluation of the opportunity score and that of the threat level of an alternative will be normalized. Thus, we will actually work in the Euclidean square $[0,1]^2 = [0,1] \times [0,1]$. In particular, at the point (1,0), opportunities are maximized and threats minimized; thus (1,0) can be referred to as the optimal reference point.

Let G be a group of experts assigning evaluations to a given set of alternatives. Let A_i stand for the i -th alternative being evaluated. Moreover, for every i, let

- x_i and y_i be the generic value that can be assigned to the alternative A_i as opportunity score (the x-variable) and threat level (the y-variable), respectively;
- x_i^m and x_i^M be the lowest and highest evaluations provided by G for the opportunity score of A_i;
- y_i^m and y_i^M be the lowest and highest evaluations provided by G for the threat level of A_i;
- $\bar{x}_i$ and $\bar{y}_i$ be the averaged values of the opportunity scores and threat levels assigned by G to A_i;
- $(\bar{x}_i,\ \bar{y}_i)$ be the initial position assigned by the DM to A_i after receiving the evaluations from G;
- $[x_i^m,\ x_i^M]$ and $[y_i^m, y_i^M]$ be the initial reference intervals considered by the DM for A_i;
- $[m, M]^2 = [x_i^m, x_i^M] \times [y_i^m, y_i^M]$ be the initial-evaluation-constrained domain;
- C_i be the circumference centered at (1,0) and passing through the point $(\bar{x}_i, \bar{y}_i)$;
- r_i be the radius of the circumference C_i, that is, $r_i = \sqrt{(\bar{x}_i-1)^2 + (\bar{y}_i-0)^2}$.

Fig. 1a and b illustrate the initial position $(\bar{x}_i, \bar{y}_i)$ assigned by the DM to the alternative A_i, the set of all points that represent potential improvements of $(\bar{x}_i, \bar{y}_i)$; that is, $\{(x, y) \in [0, 1]^2 : (x - 1)^2 + (y - 0)^2 \le r_i^2\}$, and the restriction of this set to the initial reference intervals; that is, $\{(x, y) \in [m, M]^2 : (x - 1)^2 + (y - 0)^2 \le r_i^2\}$. Also note that $(\bar{x}_i, \bar{y}_i)$ determines a division of $[0,1]^2$ and $[m, M]^2$ in four quadrants.

Throughout the evaluation process, we assume the DM to assign subjective probabilities to the set of potential realizations of both opportunity scores and threat levels for each alternative under analysis. Probabilities are updated as new information is received. In particular, we consider two subsequent phases.

- *Initial subjective probabilities*: We assume that the DM assigns initially, i.e., before receiving any evaluation, the same uniform density to each opportunity and threat scores defined in $[0,1]^2$. That is, the value of the density function at each potential opportunity and threat realization is initially equal to one. In other words, the DM faces complete uncertainty regarding the potential realizations of the evaluations before any information is provided.
- *Updated subjective probabilities*: We assume that after receiving the experts' evaluations, the DM subjectively assesses potential opportunity and threat realizations for each alternative on the base of the dispersion of the evaluations received.

To allow for a formalization of the second phase, which will consist of two loops, we need to introduce further notations and concepts.

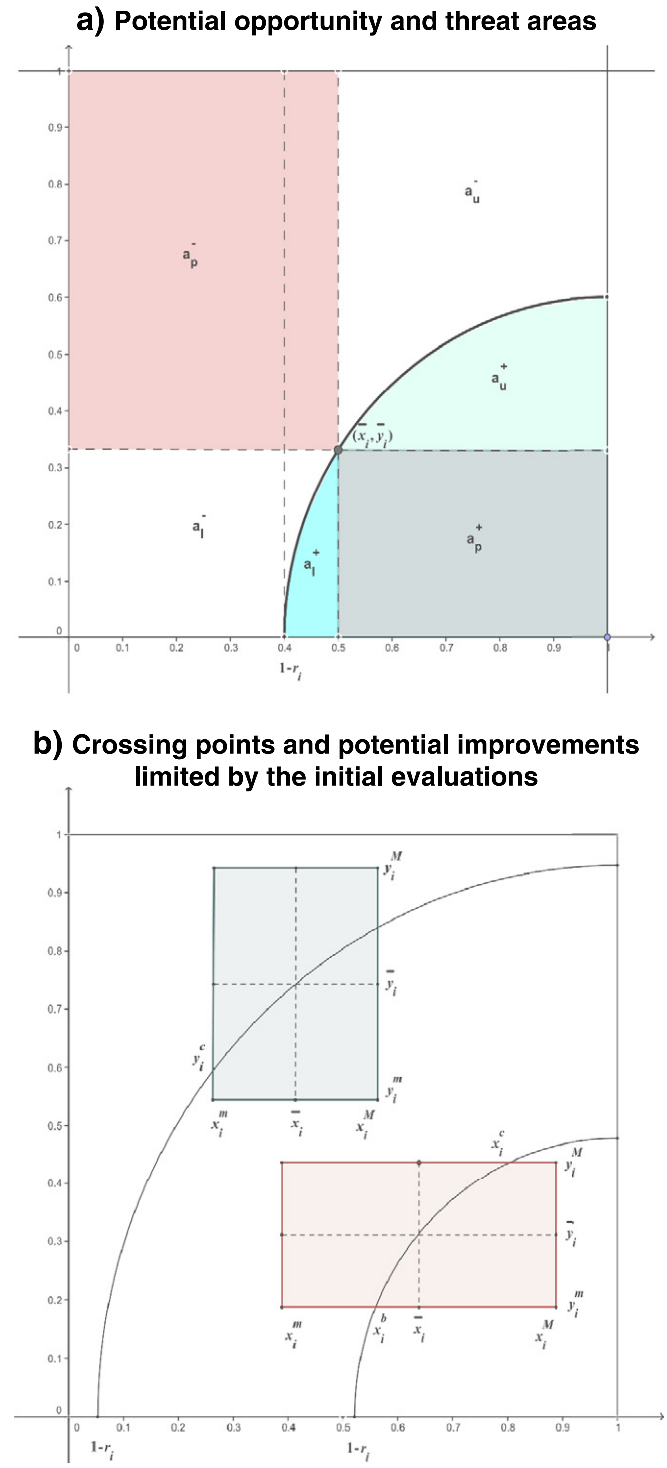

Fig. 1. Basic geometric evaluation environment. 1a. Potential opportunity and threat areas. 1b. Crossing points and potential improvements limited by the initial evaluations.

For every alternative A_i, we divide the Euclidean square $[0,1]^2$ in different areas determined by the potential realizations of the opportunity score and threat levels with respect to its initial position $(\bar{x}_i, \ \bar{y}_i)$. These areas are represented in Fig. 1a. The plus and minus super-indexes stand for the improvements and deteriorations with respect to the initial reference point $(\bar{x}_i, \ \bar{y}_i)$, respectively.

- $\bar{x}_i^+$ and $\bar{x}_i^-$ will denote potential opportunity score realizations to the right and the left of $\bar{x}_i$, respectively. That is, $\bar{x}_i^+ \in (\bar{x}_i, \ x_i^M]$ and $\bar{x}_i^- \in [x_i^m, \ \bar{x}_i)$.
- $\mu_x(\bar{x}_i^+)$ and $\mu_x(\bar{x}_i^-)$ will denote the *subjective* probability densities assigned by the DM to $\bar{x}_i^+$ and $\bar{x}_i^-$, respectively.
- $\bar{y}_i^+$ and $\bar{y}_i^-$ will denote potential threat level realizations to the right and left of $\bar{y}_i$, respectively. That is, $\bar{y}_i^+ \in (\bar{y}_i, \ y_i^M]$ and $\bar{y}_i^- \in [y_i^m, \ \bar{y}_i)$. Note that the threat levels improve as we move below $\bar{y}_i$ and worsen as we move above $\bar{y}_i$.
- $\mu_x(\bar{y}_i^+)$ and $\mu_x(\bar{y}_i^-)$ will denote the *subjective* probability densities assigned by the DM to $\bar{y}_i^+$ and $\bar{y}_i^-$, respectively.
- $1 - r_i$ is the value of the x-coordinate of the intersection point, if it exists, between C_i and the x-axis.
- $1-\sqrt{r_i{}^2-1}$ is the value of the x-coordinate of the intersection point, if it exists, between C_i and the horizontal line $y = 1$.

Using the same index notation as for the areas introduced in Fig. 1, the expected improvements computed by the DM can be defined as follows.

(i) If $(\bar{x}_i, \bar{y}_i) \in \left\{(x, \ y) \in [0, \ 1]^2 : \sqrt{(x-1)^2+(y-0)^2} \le 1\right\}$, then

$$E_p^+ = \int_{\bar{x}_i}^{1}\int_{0}^{\bar{y}_i} \mu_x\left(\bar{x}_i{}^+\right)\ \mu_y\left(\bar{y}_i{}^+\right)\ \sqrt{(x_i-1)^2+(y_i-0)^2}\ dy_i dx_i$$

$$E_u^+ = \int_{\bar{x}_i}^{1}\int_{\bar{y}_i}^{y_i=\sqrt{r_i^2-(x_i-1)^2}} \mu_x\left(\bar{x}_i{}^+\right)\ \mu_y(\bar{y}_i{}^-)\ \sqrt{(x_i-1)^2+(y_i-0)^2}\ dy_i dx_i$$

$$E_l^+ = \int_{1-r_i}^{\bar{x}_i}\int_{0}^{y_i=\sqrt{r_i^2-(x_i-1)^2}} \mu_x(\bar{x}_i{}^-)\ \mu_y\left(\bar{y}_i{}^+\right)\ \sqrt{(x_i-1)^2+(y_i-0)^2}\ dy_i dx_i. \tag{1}$$

(ii) If $(\bar{x}_i, \bar{y}_i) \in \left\{(x, \ y) \in [0, \ 1]^2 : \sqrt{(x-1)^2+(y-0)^2} > 1\right\}$, then

$$E_p^+ = \int_{\bar{x}_i}^{1}\int_{0}^{\bar{y}_i} \mu_x\left(\bar{x}_i{}^+\right)\ \mu_y\left(\bar{y}_i{}^+\right)\ \sqrt{(x_i-1)^2+(y_i-0)^2}\ dy_i dx_i$$

$$E_u^+ = \int_{\bar{x}_i}^{1-\sqrt{r_i^2-1}}\int_{\bar{y}_i}^{y_i=\sqrt{r_i^2-(x_i-1)^2}} \mu_x\left(\bar{x}_i{}^+\right)\ \mu_y(\bar{y}_i{}^-)\ \sqrt{(x_i-1)^2+(y_i-0)^2}\ dy_i dx_i + \int_{1-\sqrt{r_i^2-1}}^{1}\int_{\bar{y}_i}^{1} \mu_x\left(\bar{x}_i{}^+\right)\ \mu_y(\bar{y}_i{}^-)\ \sqrt{(x_i-1)^2+(y_i-0)^2}\ dy_i dx_i \tag{2}$$

$$E_l^+ = \int_{0}^{\bar{x}_i}\int_{0}^{y_i=\sqrt{r_i^2-(x_i-1)^2}} \mu_x(\bar{x}_i{}^-)\ \mu_y\left(\bar{y}_i{}^+\right)\ \sqrt{(x_i-1)^2+(y_i-0)^2}\ dy_i dx_i.$$

These expressions can be used to determine the total improvements expected to be achieved by each alternative. More precisely, the total improvement of A_i can be obtained by averaging the resulting improvement scores over the number of quadrants considered. At the same time, the DM should recalculate the initial ranking based on the effect that the expected realizations have on the variables $\bar{x}_i$ and $\bar{y}_i$ for each alternative A_i. Note that the basic environment just described may be regarded as a static one. That is, the DM accounts for just one set of expected evaluations and does not consider any further potential realizations that would result from new average evaluations.

2.1. Distributional biases and entropy inherent to the evaluations of experts

The DM receives a series of evaluations for each alternative A_i from several experts. He does not have any additional information regarding A_i. However, new experts could provide additional information, even though there is no guarantee that the evaluations provided by any of these new experts will fall within the initial reference intervals $[x_i^m, x_i^M]$ and $[y_i^m, y_i^M]$. This is due to the subjectivity involved in the processes of criteria selection and weighting applied by the different experts. Thus, the DM must take advantage of the available information when computing the set of potential evaluations that may be obtained from additional experts. Fig. 2 illustrates the extrapolation and validation processes that compose the decision making framework described through the paper. The number of the section corresponding to each step of the process has been specified in the figure so the process can be understood.

We will base our extrapolation process on three different measures of information dispersion and their effect on the subjective probability functions defined by the DM. We will consider observational biases determining the distributional properties of these probability functions together with the informativeness of the observations measured in terms of the relative entropy of the distributions. Finally, we will center our analysis on the opportunities. Clearly, a similar analysis can be performed to account for the dispersion exhibited by the threats. The measures of information dispersion that we will consider are the distributional bias of the evaluations, their entropy, and the spread of their domain considered by the DMs.

2.1.1. Distributional bias

We start by introducing the density function that the DM is assumed to define on the set $[x_i^m, x_i^M]$ of potential realizations for the opportunity scores of the alternative A_i. Trivially, the density associated with each of the two subintervals $[x_i^m, \bar{x}_i]$ and $[\bar{x}_i, x_i^M]$ must be different, unless the evaluations are equidistant from each other. Thus, the probability mass accumulated on $[x_i^m, \bar{x}_i]$ and $[\bar{x}_i, x_i^M]$ must be given by $\frac{\bar{x}_i - x_i^m}{x_i^M - x_i^m}$ and $\frac{x_i^M - \bar{x}_i}{x_i^M - x_i^m}$, respectively. This creates a distributional bias in the evaluations received by the experts, a bias that the DMs must account for when defining the dynamic structure of the algorithm that will allow them to extrapolate additional expected evaluations.

Henceforth, to simplify the reasoning and the presentation of the model, we will assume that the set of initial evaluations provided for each alternative A_i consists of n opportunity score values, $\{x_i^j : j = 1, \ldots, n\}$, and n threat level values, $\{y_i^j : j = 1, \ldots, n\}$. Thus, $\bar{x}_i = \frac{\sum_{j=1}^n x_i^j}{n}$ and $\bar{y}_i = \frac{\sum_{j=1}^n y_i^j}{n}$. Note that, to simplify notations, in the equations we will use $\bar{x}_i = \frac{\sum_{j=1}^n x_j}{n}$ and $\bar{y}_i = \frac{\sum_{j=1}^n y_j}{n}$.

Given an alternative A_i, we assume the DM to define the following density function on $[x_i^m, x_i^M]$.

$$\mu_x(x_i) = \begin{cases} \dfrac{1}{x_i^M - \bar{x}_i} = \dfrac{1}{x_i^M - \dfrac{\sum_{j=1}^n x_j}{n}} & \text{if } x_i = \bar{x}_i^+ \\[2ex] \dfrac{1}{\bar{x}_i - x_i^m} = \dfrac{1}{\dfrac{\sum_{j=1}^n x_j}{n} - x_i^m} & \text{if } x_i = \bar{x}_i^- \end{cases} \quad (3)$$

Hence, the value of the density function at each potential evaluation is determined by the distance between $\bar{x}_i$ and the extremes of

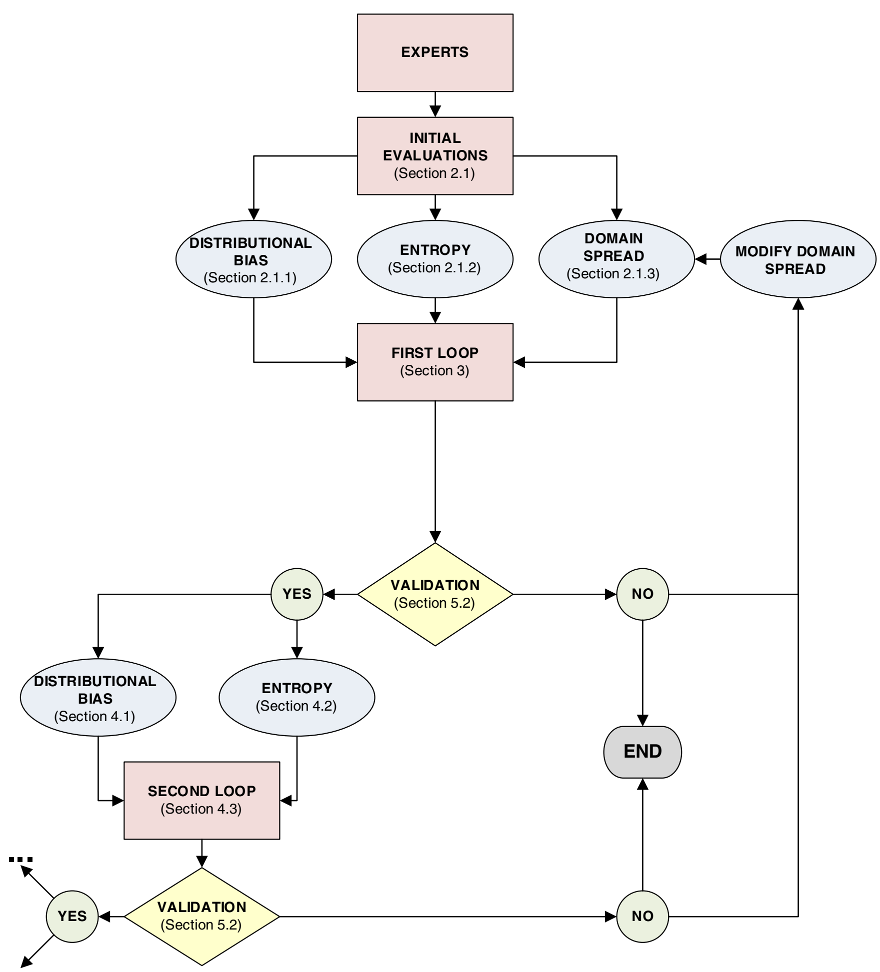

Fig. 2. Proposed decision making framework: extrapolation and validation processes.

the initial reference interval. The density function μ_y is defined in a similar way.

Finally, note that we have defined the density function on $[x_i^m, x_i^M]$, but we will also allow for this domain to expand to $[0,1]^2$. In this case, the densities will be modified accordingly to account for the differences in the spread of the evaluation domains assumed by the DM. This widening in the spread of potential evaluations will give rise to different extrapolation scenarios together with their corresponding rankings. As a result, we will introduce an *objective* entropy-based criterion that must be satisfied by all *subjective* rankings extrapolated by the DMs.

2.1.2. Entropy

We consider the entropy generated by the initial evaluations received for each alternative as a measure of their relative informativeness. The four initial evaluations considered per alternative in Eq. (5) shown below correspond to those provided by the research teams in [23]. A more general definition of information entropy within the current environment is presented in Subsection 4.2. Given the four evaluations received for either the opportunity score or the threat level of alternative A_i, the entropy of A_i will be defined in terms of the distances between the evaluations and their average value. That is:

$$e(x_i) = -K\sum_{j=1}^{4} \frac{\left|x_i^j - \overline{x}_i\right|}{\left(x_i^M - x_i^m\right) + \left(x_i^+ - x_i^-\right)} \ \ln \frac{\left|x_i^j - \overline{x}_i\right|}{\left(x_i^M - x_i^m\right) + \left(x_i^+ - x_i^-\right)} \tag{4}$$

where $x_i^m = x_i^1$, $x_i^- = x_i^2$, $x_i^+ = x_i^3$, and $x_i^M = x_i^4$ represent the initial evaluations received from the different experts regarding alternative A_i ordered from the lowest to the highest. The smaller the value of $e(x_i)$, the larger the amount of information being transmitted by the experts analyzing the alternative A_i. The maximum value for the entropy is given by $e_{\max} = 1\text{n}\ n$. We apply the normalization process described by [25,26] and set $K = \frac{1}{e_{\max}}$, leading to $0 \le e(x_i) \le 1$. After normalizing the entropy values derived from the evaluations received for each alternative, we can consider the total entropy of the set of all the alternatives, that is, $E = \sum_i e(x_i)$.

Tavana [25,26] uses the normalized and total entropy values to create a normalized index for each alternative A_i, which is a decreasing function of the entropy of A_i:

$$f(x_i) = \frac{1 - e(x_i)}{1 - E}. \tag{5}$$

The entropy inherent to the information received is therefore used to produce a credibility ranking for the alternatives. This ranking is based on the relative informativeness derived from the evaluations received by the experts. In the current setting, lower entropy values (i.e., higher information content) should reduce the distance between the alternative and the ideal point (1,0). Thus, we redefine the normalized index as follows:

$$1 - f(x_i) = 1 - \frac{1 - e(x_i)}{1 - E} = \frac{e(x_i) - E}{1 - E}. \tag{6}$$

A similar normalized index can be defined for the threat level evaluations of the alternative A_i.

2.1.3. Domain spread

In this section we provide the notations and the intuition for the case when one new evaluation is added to the set of initial evaluations, $\{x_i^j : j = 1, \ldots, n\}$, for a given alternative A_i. This new evaluation, x_i^{n+1}, can be either due to a new expert joining the group G or be extrapolated by the DM.

The averaged values of the opportunity scores corresponding to the $n+1$ evaluations in the set $\{x_i^j : j = 1, \ldots, n+1\}$ will be denoted by $\overline{x}_i^{n+1}$. Similar notations will be used for the threat levels.

In both cases, we assume the DM to shift from $[m, M]^2$ to the entire square $[0,1]^2$ when considering potential new evaluations for the alternative A_i.

Assuming a uniform density, the probability that a new evaluation will belong to the interval $\left[\overline{x}_i^{n+1}, \ 1\right]$ inherits the distributional bias exhibited by the previous evaluations belonging to $\left[\overline{x}_i^{n+1}, \ x_i^M\right]$. Similarly, the probability that a new potential evaluation belongs to $\left[0, \ \overline{x}_i^{n+1}\right]$ inherits the distributional bias exhibited by the evaluations located within $\left[x_i^m, \ \overline{x}_i^{n+1}\right]$.

Recall that the probability of getting an evaluation $x_i \in \left[x_i^m, \ x_i^M\right]$ is defined by:

$$\mu_{x+1}(x_i) = \begin{cases} \mu_{x+1}\left(x_i^M\right) = \dfrac{1}{x_i^M - \dfrac{\sum_{j=1}^{n} x_j + x_{n+1}}{n+1}} & \text{if } x_i \in \left[\overline{x}_i^{n+1}, \ x_i^M\right] \\ \mu_{x+1}\left(x_i^m\right) = \dfrac{1}{\dfrac{\sum_{j=1}^{n} x_j + x_{n+1}}{n+1} - x_i^m} & \text{if } x_i \in \left[x_i^m, \ \overline{x}_i^{n+1}\right]. \end{cases} \tag{7}$$

To gain more insight, assume that the average value is biased toward x_i^M, and, therefore, a higher density mass is accumulated on $\left[\overline{x}_i^{n+1}, \ x_i^M\right]$. Consequently, the following relationship between the density values $\mu_{x+1}(x_i^M)$ and $\mu_{x+1}(x_i^m)$ holds:

$$\left[\frac{1}{\dfrac{\sum_{j=1}^{n} x_j + x_{n+1}}{n+1} - x_i^m}\right]\alpha = \frac{1}{x_i^M - \dfrac{\sum_{j=1}^{n} x_j + x_{n+1}}{n+1}} \tag{8}$$

where the factor $a > 1$ is introduced to keep the density value assigned to any potential evaluation x_i within $\left[\overline{x}_i^{n+1}, \ x_i^M\right]$. Note that the entire probability mass distributed through the domain [0,1] must add up to 1. Thus, the following equation must hold for $a > 1$.

$$\int_{\overline{x}_i^{n+1}}^{1} \alpha p \ dx_i + \int_{0}^{\overline{x}_i^{n+1}} p \ dx_i = 1 \tag{9}$$

where p represents the density value assigned to an evaluation in $\left[0, \ \overline{x}_i^{n+1}\right]$ and αp the density value assigned to an evaluation in $\left[\overline{x}_i^{n+1}, \ 1\right]$. Solving the integrals and the resulting equation for p, we get:

$$p = \frac{1}{\alpha - [\alpha - 1]\overline{x}_i^{n+1}}. \tag{10}$$

Hence, by Eq. (8), we have:

$$\mu_{x+1}\left(x_i^m = 0\right) = \frac{1}{\left(\dfrac{\dfrac{\sum_{j=1}^{n} x_j + x_{n+1}}{n+1} - x_i^m}{x_i^M - \dfrac{\sum_{j=1}^{n} x_j + x_{n+1}}{n+1}}\right) - \left[\dfrac{\dfrac{\sum_{j=1}^{n} x_j + x_{n+1}}{n+1} - x_i^m}{x_i^M - \dfrac{\sum_{j=1}^{n} x_j + x_{n+1}}{n+1}}\right] \dfrac{\sum_{j=1}^{n} x_j + x_{n+1}}{n+1}} \tag{11}$$

and $\mu_{x+1}\left(x_i^M = 1\right) = \alpha\mu_{x+1}\left(x_i^m = 0\right)$.

3. First loop: initial expected evaluation

We make use of Fig. 1a and b to define the expected improvements resulting from the different areas over which the potential evaluations of each alternative may be realized. The sum of these expected improvements with respect to the initial position $(\bar{x}_i,\ \bar{y}_i)$ will be denoted by $EV(\bar{x}_i,\ \bar{y}_i)$. We will consider both the case where the potential evaluation is constrained to be in $[m, M]^2$ and the case where it can vary in the entire square $[0,1]^2$. In the first case, we must account for the crossing points between the boundary of $[m, M]^2$ and the circumference C_i. The x and y coordinates of these crossing points are illustrated in Fig. 1b and are the three main potential reference points that must be considered by the DM:

- x_i^b is the x-coordinate of the crossing point, if it exists, between C_i and the line $y = y_i^m$;
- x_i^c is the x-coordinate of the crossing point, if it exists, between C_i and the line $y = y_i^M$;
- y_i^c is the y-coordinate of the crossing point, if it exists, between C_i and the line $x = x_i^m$.

We are now ready to define the first loop of the evaluation algorithm that we propose in this paper (see Fig. 2). We start by defining the expected improvement $EV(\bar{x}_i,\ \bar{y}_i)$ in the case where a potential evaluation is constrained to be in $[m, M]^2$.

To evaluate $EV(\bar{x}_i,\ \bar{y}_i)$, the DM must account for the whole set of positions corresponding to potential improvements; that is, $\{(x, y) \in [m,\ M]^2 : (x-1)^2 + (y-0)^2 \le r_i^2\}$. This set is given by the union of the following three areas:

$$
\begin{aligned}
A_u^+ &= a_u^+ \cap \left[x_i^m,\ x_i^M\right] \times \left[y_i^m,\ y_i^M\right] \\
A_p^+ &= a_p^+ \cap \left[x_i^m,\ x_i^M\right] \times \left[y_i^m,\ y_i^M\right] \\
A_l^+ &= a_l^+ \cap \left[x_i^m,\ x_i^M\right] \times \left[y_i^m,\ y_i^M\right]
\end{aligned}
\tag{12}
$$

where the areas a_u^+, a_p^+, a_l^+ are those described in Fig. 1b. Note that each of these three areas must be weighted by 1/3, in order to provide an average of the evaluations obtained.

To fix the ideas, we evaluate explicitly the value of $EV(\bar{x}_i,\ \bar{y}_i)$ resulting from the potential position (x_i^{n+1}, y_i^{n+1}) varying in the following set:

$$
\left\{(x, y) \in [m,\ M]^2 : (x-1)^2 + (y-0)^2 \le r_i^2,\ x \le x_i^c,\ y \ge \bar{y}_i\right\}.
\tag{13}
$$

That is, the triangle-like shaped area bounded by the circumference C_i and the straight lines $y = \bar{y}_i$ and $x = x_i^c$. This is one of the two areas in which it is necessary to split A_u^+ when $x_i^c < x_i^M$. If $x_i^c \ge x_i^M$, this area coincides with A_u^+, and no splitting is necessary.

Note that for x_i^{n+1} to be the x-coordinate of a point in this area, it must vary in $\left[\bar{x}_i = \frac{\sum_{j=1}^n x_j}{n},\ x_i^c = 1 - \sqrt{r_i^2 - (y_i^M - 0)^2}\right]$, while the corresponding y-coordinate must belong to $\left[\bar{y}_i = \frac{\sum_{j=1}^n y_j}{n},\ y_i^{n+1} = \sqrt{r_i^2 - (x_i^{n+1} - 1)^2}\right]$. Thus, the $EV(\bar{x}_i,\ \bar{y}_i)$ value provided by the area of Eq. (13) is given by the following expression:

$$
\int_{\frac{\sum_{j=1}^n x_j}{n}}^{1-\sqrt{r_i^2-(y_i^M-0)^2}} \int_{\frac{\sum_{j=1}^n y_j}{n}}^{y_i^{n+1}=\sqrt{r_i^2-(x_i^{n+1}-1)^2}} [1-f(x_i)][1-f(y_i)] \left(\frac{1}{x_i^M - \frac{\sum_{j=1}^n x_j}{n}}\right) \left(\frac{1}{y_i^M - \frac{\sum_{j=1}^n y_j}{n}}\right) \left[\sqrt{(x_i^{n+1}-1)^2 + (y_i^{n+1}-0)^2}\right] dy_i^{n+1} dx_i^{n+1}.
\tag{14}
$$

It should be noted that Eq. (14) cannot be solved in closed form due to the square root in the integrand function. We will therefore simulate the Euclidean distances without accounting for the square root. This modification provides an equivalent ranking though defined for smaller distance values.

Consider now the case where a potential evaluation can belong to the entire square $[0,1]^2$. In this case, the expected improvement $EV(\bar{x}_i,\ \bar{y}_i)$ is computed considering the set $\{(x, y) \in [0,\ 1]^2 : (x-1)^2 + (y-0)^2 \le r_i^2\}$ as the area of potential improvement. This set is the union of a_u^+, a_p^+, and a_l^+ described in Fig. 1b. Thus, $EV(\bar{x}_i,\ \bar{y}_i)$ is given by the sum of the following three values, where the Euclidean distances have been already replaced with their squares.

$$
\begin{aligned}
E_p^+ &= \int_{\frac{\sum_{j=1}^n x_j}{n}}^{1} \int_{0}^{\frac{\sum_{j=1}^n y_j}{n}} \mu_x\left(x_i^M = 1\right)\ \mu_y\left(y_i^m = 0\right)[1-f(x_i)][1-f(y_i)]\left[\left(x_i^{n+1}-1\right)^2 + \left(y_i^{n+1}-0\right)^2\right] dy_i^{n+1} dx_i^{n+1}; \\
E_u^+ &= \int_{\frac{\sum_{j=1}^n x_j}{n}}^{1} \int_{\frac{\sum_{j=1}^n y_j}{n}}^{y_i^{n+1}=\sqrt{r_i^2-(x_i^{n+1}-1)^2}} \mu_x\left(x_i^M = 1\right)\ \mu_y\left(y_i^M = 1\right)[1-f(x_i)][1-f(y_i)]\left[\left(x_i^{n+1}-1\right)^2 + \left(y_i^{n+1}-0\right)^2\right] dy_i^{n+1} dx_i^{n+1}; \\
E_l^+ &= \int_{1-r_i}^{\frac{\sum_{j=1}^n x_j}{n}} \int_{0}^{y_i^{n+1}=\sqrt{r_i^2-(x_i^{n+1}-1)^2}} \mu_x\left(x_i^m = 0\right)\ \mu_y\left(y_i^m = 0\right)[1-f(x_i)][1-f(y_i)]\left[\left(x_i^{n+1}-1\right)^2 + \left(y_i^{n+1}-0\right)^2\right] dy_i^{n+1} dx_i^{n+1}.
\end{aligned}
\tag{15}
$$

4. Second loop: forward expected evaluation

4.1. Basic dynamic structure

After extrapolating his first evaluation, the DM must update the values of the corresponding averages and probability functions, together with the domain of the density functions. This procedure must be performed for each of the areas a_u^+, a_p^+, and a_l^+, and it can be applied recursively each time a new evaluation is extrapolated. Note that the limits defining the sets of potential realizations determine the behavior of the corresponding density functions. The domains of the densities considered through the second loop are given by $\left[x_i^m, \ \frac{\sum_{j=1}^n x_j + x_{n+1}}{n+1}\right]$ and $\left[\frac{\sum_{j=1}^n x_j + x_{n+1}}{n+1}, \ x_i^M\right]$, if the potential realization for the x-coordinate is constrained to be in $[x_i^m, \ x_i^M]$, and by $\left[0, \ \frac{\sum_{j=1}^n x_j + x_{n+1}}{n+1}\right]$ and $\left[\frac{\sum_{j=1}^n x_j + x_{n+1}}{n+1}, \ 1\right]$, if the potential realization can be in [0,1]. The same intuition applies to the densities that must be defined for the y-coordinate of the potential realization.

To simplify the presentation, we will use the following notation: $\mu_x^+ = \mu_x(\overline{x}_i^+)$ and $\mu_x^- = \mu_x(\overline{x}_i^-)$, and similarly for the y-variable. Recall that $\mu_{x+1}(x_i^M)$ and $\mu_{x+1}(x_i^m)$ represent the probability functions defined by the DM when the new realization $x_{n+1} \in [x_i^m, \ x_i^M]$, while $\mu_{x+1}(x_i^M = 1)$ and $\mu_{x+1}(x_i^m = 0)$ represent the probability functions when the new realization $x_{n+1} \in [0, \ 1]$.

For the sake of simplicity, we will not consider the entropy index through the current subsection. This implies, for instance, that the $EV(\overline{x}_i, \ \overline{y}_i)$ obtained from extrapolating a second evaluation when the first potential evaluations are realized within the set described in Eq. (13) is given by the expression:

$$\int_{\overline{x}_i}^{x_i^c} \int_{\overline{y}_i}^{y_i^{n+1} = \sqrt{r_i^2 - (x_i^{n+1} - 1)^2}} \mu_x^+ \mu_y^- \left[\int_{\overline{x}_i^{n+2}} \int_{\overline{y}_i^{n+2}} \left\{ E_p^+ + E_u^+ + E_l^+ | \mu_{x+1}\left(x_i^M\right), \mu_{y+1}\left(y_i^m\right) \right\} dy_i^{n+2} dx_i^{n+2} \right] dy_i^{n+1} dx_i^{n+1}. \tag{16}$$

The expression between brackets in Eq. (16) stands for the function derived from the set of potential improvements that may be obtained after extrapolating a second evaluation.

Note that for each and every possible realization of the first evaluation extrapolated, the DM must proceed as in the first loop and consider the values of x_i^b, x_i^c, and y_i^c in order to define the potential realization areas of the second evaluation. Thus, in order to fully illustrate the recursive behavior of Eq. (16), we must explicitly define the $EV(\overline{x}_i, \ \overline{y}_i)$ value that the DM computes in the second loop using the improvement areas generated by *each* first potential evaluation.

To simplify the presentation, we assume that $x_i^c > x_i^M$ (and also that $x_i^b > x_i^m$), so the DM does not have to compute two integrals to evaluate the value of $EV(\overline{x}_i, \ \overline{y}_i)$ corresponding to the area A_u^+.

(i) *First loop*: The first loop is determined by the expected evaluations located within the triangle-like shaped area represented by Eq. (13). This area coincides with A_u^+, if the expected evaluations x_{n+1} are constrained to be in $[x_i^m, \ x_i^M]$, and with a_u^+, if the expected evaluations x_{n+1} can be in $[0, \ 1]$. The corresponding values of $EV(\overline{x}_i, \ \overline{y}_i)$ are given by Eq. (14) and by E_u^+ in Eq. (15), respectively. Clearly, for a complete evaluation of $EV(\overline{x}_i, \ \overline{y}_i)$, the DM must also consider the other two areas of potential improvement; that is, A_p^+ and A_l^+ if $x_{n+1} \in [x_i^m, \ x_i^M]$, and, a_p^+ and a_l^+ if $x_{n+1} \in [0, \ 1]$.

(ii) *Second loop*: Given a potential evaluation x_i^{n+1} in A_u^+, the set of subsequent potential improvements must be computed using "again" the areas A_u^+, A_p^+, and A_l^+. Thus, when accounting for a second loop, the resulting value of $EV(\overline{x}_i, \ \overline{y}_i)$ is given by the following expression:

$$\begin{aligned}
&\frac{1}{3}\int_{\overline{x}_i}^{x_i^M} \int_{\overline{y}_i}^{y_i^{n+1} = \sqrt{r_i^2 - (x_i^{n+1} - 1)^2}} \left(\frac{1}{x_i^M - \frac{\sum_{j=1}^n x_j}{n}} \right) \left(\frac{1}{y_i^M - \frac{\sum_{j=1}^n y_j}{n}} \right) \Bigg\{ \Bigg[\frac{1}{3} \int_{\frac{\sum_{j=1}^n x_j + x_{n+1}}{n+1}}^{x_i^M} \int_{y_i^m}^{\frac{\sum_{j=1}^n y_j + y_{n+1}}{n+1}} \left(\frac{1}{x_i^M - \frac{\sum_{j=1}^n x_j + x_{n+1}}{n+1}} \right) \left(\frac{1}{\frac{\sum_{j=1}^n y_j + y_{n+1}}{n+1} - y_i^m} \right) \\
&\quad + \frac{1}{3} \int_{\frac{\sum_{j=1}^n x_{j+x_{n+1}}}{n+1}}^{x_i^M} \int_{\frac{\sum_{j=1}^n y_j + y_{n+1}}{n+1}}^{y_i^{n+2} = \sqrt{r_{i+1}^2 - (x_i^{n+2} - 1)^2}} \left(\frac{1}{x_i^M - \frac{\sum_{j=1}^n x_j + x_{n+1}}{n+1}} \right) \left(\frac{1}{y_i^M - \frac{\sum_{j=1}^n y_j + y_{n+1}}{n+1}} \right) \\
&\quad + \frac{1}{3} \int_{x_{i+1}^b}^{\frac{\sum_{j=1}^n x_j + x_{n+1}}{n+1}} \int_{y_i^m}^{y_i^{n+2} = \sqrt{r_{i+1}^2 - (x_i^{n+2} - 1)^2}} \left(\frac{1}{\frac{\sum_{j=1}^n x_j + x_{n+1}}{n+1} - x_i^m} \right) \left(\frac{1}{\frac{\sum_{j=1}^n y_j + y_{n+1}}{n+1} - y_i^m} \right) \Bigg] \left[\left(x_i^{n+2} - 1\right)^2 + \left(y_i^{n+2} - 0\right)^2 \right] dy_i^{n+2} dx_i^{n+2} \Bigg\} dy_i^{n+1} dx_i^{n+1}
\end{aligned} \tag{17}$$

where

$$r_{i+1} = \sqrt{\left(\frac{\sum_{j=1}^n x_j + x_{n+1}}{n+1} - 1 \right)^2 + \left(\frac{\sum_{j=1}^n y_j + y_{n+1}}{n+1} - 0 \right)^2}.$$

This expression, which extrapolates two additional reports by the experts, constitutes the basis of our analysis in the current paper, as well as of any potential extension. Note that Eq. (17) describes the second loop improvements obtained in the case where x_i^{n+1} is in A_u^+. The DM must also consider the case where x_i^{n+1} is in A_p^+ and that where x_i^{n+1} is in A_l^+. Thus, Eq. (17) accounts for only one-third of the total potential improvements that may be realized. A similar reasoning allows us to compute the value of $EV(\bar{x}_i, \bar{y}_i)$ when the potential evaluation x_i^{n+1} can belong to a_u^+, a_p^+ or a_l^+.

Finally, note that defining the complete forward improvements beyond the first loop, i.e., taking into account all improvement areas, will produce integrals whose integrand functions contain several square roots and, hence, are not numerically integrable. To remove these not integrable parts, we will concentrate on forward improvements obtained by considering either the area A_p^+ or a_p^+ in the numerical simulations section.

Note also that the above expressions are not integrable with respect to x_i^{n+1} and y_i^{n+1}, variables that will keep on getting more cumbersome as we add further loops. Thus, we are forced to operate with the expected value of x_i^{n+1}, i.e., $E\left[x_i^{n+1}\right]$, which is based on the densities generated by the different sets of evaluations. Consider, for example, the initial set of evaluations $\{x_i^j : j = 1, \ldots, n\}$. Computing $E\left[x_i^{n+1}\right]$ when constrained by the initial reference interval $[x_i^m, x_i^M]$ is straightforward. The corresponding extension to the entire plane [0,1] leads to the following expected value of the evaluation x_i^{n+1}:

$$E\left[x_i^{n+1}\right] = \frac{1}{2}\int_{\bar{x}_i}^{1} \alpha\ \mu(x_i^m = 0)\ x_i^{n+1} dx_i^{n+1} + \frac{1}{2}\int_{0}^{\bar{x}_i} \mu_x(x_i^m = 0)\ x_i^{n+1}\ dx_i^{n+1}. \tag{18}$$

Clearly, the value of $E\left[x_i^{n+1}\right]$ can be used to evaluate $E\left[x_i^{n+2}\right]$, $E\left[x_i^{n+2}\right]$ can be used to evaluate $E\left[x_i^{n+3}\right]$ and so on for as many loops as is needed.

4.2. On relative entropy

Similarly to Subsection 2.1.2, we consider the DM's ability to calculate the expected entropy defining the informativeness derived from the set of potential evaluations obtained after performing a second loop. The expected entropy of the potential evaluation x_i^{n+1} is given by:

$$e\left(x_i^{n+1}\right)_l = -\frac{1}{\ln(n+1)} \int_{x_i^m}^{\frac{\sum_{j=1}^n x_j}{n}} \left(\frac{1}{\frac{\sum_{j=1}^n x_j}{n} - x_i^m}\right) \left[\sum_{j=1}^{n+1} \frac{\left|x_i^j - \frac{\sum_{j=1}^n x_j + x_{n+1}}{n+1}\right|}{(x_i^M - x_i^m) + (x_i^+ - x_i^-) + \left|x_i^{n+1} - \frac{\sum_{j=1}^n x_j + x_{n+1}}{n+1}\right|} \right.$$
$$\left. \ln \frac{\left|x_i^j - \frac{\sum_{j=1}^n x_j + x_{n+1}}{n+1}\right|}{(x_i^M - x_i^m) + (x_i^+ - x_i^-) + \left|x_i^{n+1} - \frac{\sum_{j=1}^n x_j + x_{n+1}}{n+1}\right|}\right] dx_i^{n+1} \qquad \text{if } x_i^{n+1} \in \left[x_i^m, \bar{x}_i\right], \tag{19}$$

$$e\left(x_i^{n+1}\right)_h = -\frac{1}{\ln(n+1)} \int_{\frac{\sum_{j=1}^n x_j}{n}}^{x_i^M} \left(\frac{1}{x_i^M - \frac{\sum_{j=1}^n x_j}{n}}\right) \left[\sum_{j=1}^{n+1} \frac{\left|x_i^j - \frac{\sum_{j=1}^n x_j + x_{n+1}}{n+1}\right|}{(x_i^M - x_i^m) + (x_i^+ - x_i^-) + \left|x_i^{n+1} - \frac{\sum_{j=1}^n x_j + x_{n+1}}{n+1}\right|} \right.$$
$$\left. \ln \frac{\left|x_i^j - \frac{\sum_{j=1}^n x_j + x_{n+1}}{n+1}\right|}{(x_i^M - x_i^m) + (x_i^+ - x_i^-) + \left|x_i^{n+1} - \frac{\sum_{j=1}^n x_j + x_{n+1}}{n+1}\right|}\right] dx_i^{n+1} \qquad \text{if } x_i^{n+1} \in \left[\bar{x}_i, x_i^M\right]. \tag{20}$$

Therefore, the entropy of the potential evaluation x_i^{n+1} is obtained by averaging both values, $e\left(x_i^{n+1}\right)_l$ and $e\left(x_i^{n+1}\right)_h$. Integrability problems will also arise when considering further loops within this context. Thus, as we did in the previous subsection, we proceed by replacing x_i^{n+1} with $E\left[x_i^{n+1}\right]$ in the above expressions. This guarantees their integrability and the existence of a numerical solution.

Finally, as in Subsection 2.1.2, we redefine the index introduced by [25,26] to generate a normalized index accounting for the relative informativeness of the evaluations provided for each alternative through the second loop.

$$1 - f\left(x_i^{n+1}\right) = 1 - \frac{1 - e\left(x_i^{n+1}\right)}{1 - E^{n+1}} = \frac{e\left(x_i^{n+1}\right) - E^{n+1}}{1 - E^{n+1}} \tag{21}$$

where

$$E^{n+1} = \sum_i \left[\frac{1}{2} e\left(x_i^{n+1}\right)_l + \frac{1}{2} e\left(x_i^{n+1}\right)_h\right].$$

4.3. Full dynamic structure: potential ranking reorderings

We can now define the complete expression that must be computed by the DM when performing the second loop. It incorporates entropy into the basic dynamic structure introduced in Subsection 4.1. We assume that $x_i^c < x_i^M$ (and also that $x_i^b < x_i^m$). This will be the case for a subset of countries in the numerical section. Moreover, the case where $x_i^c > x_i^M$ (with $x_i^b > x_i^m$) is already described in Subsection 4.1. Also, we report the expression for the case where the potential realizations of the first loop must belong to the initial-evaluation-constrained domain $[m, M]^2$. A similar expression can be defined for the case where the potential realizations can vary in $[0,1]^2$.

In the numerical simulations, we will refer to this setting as the *Euclidean improvements setting*.

$$
\begin{aligned}
&\frac{1}{3}\int_{\frac{\sum_{j=1}^n x_j}{n}}^{x_i^c}\int_{\frac{\sum_{j=1}^n y_j}{n}}^{y_i^{n+1}=\sqrt{r_i^2-(x_i^{n+1}-1)^2}}[1-f(x_i)]\,[1-f(y_i)]\left(\frac{1}{x_i^M-\frac{\sum_{j=1}^n x_j}{n}}\right)\left(\frac{1}{y_i^M-\frac{\sum_{j=1}^n y_j}{n}}\right)\\
&\left[\int_{\frac{\sum_{j=1}^n x_j+x_{n+1}}{n+1}}^{x_i^M}\int_{y_i^m}^{\frac{\sum_{j=1}^n y_j+y_{n+1}}{n+1}}\left[1-f\left(x_i^{n+1}\right)\right]\left[1-f\left(y_i^{n+1}\right)\right]\left(\frac{1}{x_i^M-\frac{\sum_{j=1}^n x_j+x_{n+1}}{n+1}}\right)\left(\frac{1}{\frac{\sum_{j=1}^n y_j+y_{n+1}}{n+1}-y_i^m}\right)\right.\\
&\left.\left[\left(x_i^{n+2}-1\right)^2+\left(y_i^{n+2}-0\right)^2\right]dy_i^{n+2}dx_i^{n+2}\right]dy_i^{n+1}dx_i^{n+1}+\\
&\frac{1}{3}\int_{x_i^c}^{x_i^M}\int_{\frac{\sum_{j=1}^n y_j}{n}}^{y_i^M}[1-f(x_i)]\,[1-f(y_i)]\left(\frac{1}{x_i^M-\frac{\sum_{j=1}^n x_j}{n}}\right)\left(\frac{1}{y_i^M-\frac{\sum_{j=1}^n y_j}{n}}\right)\\
&\left[\int_{\frac{\sum_{j=1}^n x_j+x_{n+1}}{n+1}}^{x_i^M}\int_{y_i^m}^{\frac{\sum_{j=1}^n y_j+y_{n+1}}{n+1}}\left[1-f\left(x_i^{n+1}\right)\right]\left[1-f\left(y_i^{n+1}\right)\right]\left(\frac{1}{x_i^M-\frac{\sum_{j=1}^n x_j+x_{n+1}}{n+1}}\right)\left(\frac{1}{\frac{\sum_{j=1}^n y_j+y_{n+1}}{n+1}-y_i^m}\right)\right.\\
&\left.\left[\left(x_i^{n+2}-1\right)^2+\left(y_i^{n+2}-0\right)^2\right]dy_i^{n+2}dx_i^{n+2}\right]dy_i^{n+1}dx_i^{n+1}+\\
&\frac{1}{3}\int_{\frac{\sum_{j=1}^n x_j}{n}}^{x_i^M}\int_{y_i^m}^{\frac{\sum_{j=1}^n y_j}{n}}[1-f(x_i)]\,[1-f(y_i)]\left(\frac{1}{x_i^M-\frac{\sum_{j=1}^n x_j}{n}}\right)\left(\frac{1}{\frac{\sum_{j=1}^n y_j}{n}-y_i^m}\right)\\
&\left[\int_{\frac{\sum_{j=1}^n x_j+x_{n+1}}{n+1}}^{x_i^M}\int_{y_i^m}^{\frac{\sum_{j=1}^n y_j+y_{n+1}}{n+1}}\left[1-f\left(x_i^{n+1}\right)\right]\left[1-f\left(y_i^{n+1}\right)\right]\left(\frac{1}{x_i^M-\frac{\sum_{j=1}^n x_j+x_{n+1}}{n+1}}\right)\left(\frac{1}{\frac{\sum_{j=1}^n y_j+y_{n+1}}{n+1}-y_i^m}\right)\right.\\
&\left.\left[\left(x_i^{n+2}-1\right)^2+\left(y_i^{n+2}-0\right)^2\right]dy_i^{n+2}dx_i^{n+2}\right]dy_i^{n+1}dx_i^{n+1}+\\
&\frac{1}{3}\int_{x_i^m}^{\frac{\sum_{j=1}^n x_j}{n}}\int_{\sqrt{r_i^2-(x_i^m-1)^2}}^{y_i^{n+1}=\sqrt{r_i^2-(x_i^{n+1}-1)^2}}[1-f(x_i)]\,[1-f(y_i)]\left(\frac{1}{\frac{\sum_{j=1}^n x_j}{n}-x_i^m}\right)\left(\frac{1}{\frac{\sum_{j=1}^n y_j}{n}-y_i^m}\right)\\
&\left[\int_{\frac{\sum_{j=1}^n x_j+x_{n+1}}{n+1}}^{x_i^M}\int_{y_i^m}^{\frac{\sum_{j=1}^n y_{j+y_{n+1}}}{n+1}}\left[1-f\left(x_i^{n+1}\right)\right]\left[1-f\left(y_i^{n+1}\right)\right]\left(\frac{1}{x_i^M-\frac{\sum_{j=1}^n x_j+x_{n+1}}{n+1}}\right)\left(\frac{1}{\frac{\sum_{j=1}^n y_j+y_{n+1}}{n+1}-y_i^m}\right)\right.\\
&\left.\left[\left(x_i^{n+2}-1\right)^2+\left(y_i^{n+2}-0\right)^2\right]dy_i^{n+2}dx_i^{n+2}\right]dy_i^{n+1}dx_i^{n+1}
\end{aligned}
\tag{22}
$$

As already mentioned in Subsection 4.1, we will also compute a simplified version of the above expression, which applies under the assumption that, in the first loop, the DM considers only potential realizations located within the set $A_p^+ = [\bar{x}_i,\ x_i^M] \times [y_i^m,\ \bar{y}_i]$ and defines only the expected improvements E_p^+. To differentiate it from the Euclidean one, we will refer to this setting as the *quadrant improvements setting*. We have added this setting to illustrate the dependence of the results on the set of potential evaluations considered by the DMs.

In the following section, we compute one and two loops for each one of these settings and analyze the resulting reordered initial ranking:

(i) Quadrant improvements constrained by either $[m, M]^2$ or $[0,1]^2$;
(ii) Euclidean improvements constrained by either $[m, M]^2$ or $[0,1]^2$.

5. Numerical example: on the enlargement of NATO

Tavana and O'Connor [23] asked several groups of researchers to (subjectively) define different selection criteria for NATO membership application. The criteria selected by the researchers within each group were then divided binomially and categorized as opportunities or threats. Before arriving to the final ranking of candidates, each group of researchers applied Saaty's [27] analytic hierarchy process (AHP) to estimate the weights of the corresponding criteria based on the subjective importance assigned by the members of each group. The resulting MCDM model integrates the intuitive preferences of the researchers composing each group into a structured framework [26,28–30]. Finally, an aggregated (weighted-sum) opportunity and threat score was computed by each group of researchers for each of the applicant countries. The weighted-sum scores obtained by the different groups of researchers were averaged and used to rank the candidate countries in terms of their Euclidean distance from the optimal point (1,0) [24,25,31].

To provide additional intuition, the average scores assigned to each applicant country were presented and categorized in an opportunities versus threats Euclidean plane $[0,1] \times [0,1]$ (see Figs. 2 and 3 in [23]). The weighted-sum opportunity and threat scores obtained by the different groups of researchers for each applicant country are described in Table 1, which corresponds to Fig. 4 in [23].

We have highlighted the lowest opportunity and threat scores assigned to each country in blue and the highest ones in red. The results reported in Table 1 show the substantial variability in the evaluations provided by the different groups. In particular, the German group exhibits a tendency to consistently assign the lowest scores, particularly when evaluating opportunities. Thus, while having access to the same data and a similar academic formation, the subjective evaluations provided by each group of researches lead to substantially different results. For example, the French and German teams rank Sweden over Austria as the most preferred candidate, while the Swiss and American teams reverse this order. Thus, an obvious question one should ask is whether the opinion variability inherent to the evaluations may lead to rankings that differ from the one obtained by [23].

Through this section we extend numerically the NATO enlargement model developed by [23]. We report and validate the rankings that result from computing two additional loops. The evaluations used in the

Table 1
Initial rankings revisited.

Country	French teams		German teams		Swiss teams		American teams		All teams	
	Opportunity score	Threat score	Opportunity score	Threat score	Opportunity score	Threat score	Opportunity score	Threat score	Opportunity score	Threat score
Armenia	0.435	0.418	0.366	0.387	0.467	0.458	0.408	0.516	0.419	0.445
Austria	0.691	0.481	0.544	0.291	0.870	0.336	0.747	0.444	0.713	0.388
Azerbaijan	0.484	0.527	0.383	0.416	0.412	0.555	0.472	0.490	0.438	0.497
Belarus	0.480	0.333	0.305	0.356	0.452	0.395	0.455	0.452	0.423	0.384
Bosnia and Herzegovina	0.413	0.307	0.391	0.425	0.412	0.486	0.504	0.485	0.430	0.426
Cyprus	0.490	0.326	0.403	0.340	0.498	0.491	0.494	0.507	0.471	0.416
Finland	0.721	0.544	0.564	0.325	0.698	0.352	0.682	0.481	0.666	0.425
Georgia	0.370	0.442	0.340	0.483	0.349	0.577	0.426	0.551	0.371	0.513
Ireland	0.597	0.488	0.487	0.312	0.639	0.434	0.656	0.401	0.595	0.409
Kazakhstan	0.521	0.506	0.398	0.402	0.470	0.511	0.476	0.468	0.466	0.472
Kyrgyzstan	0.313	0.490	0.335	0.577	0.408	0.611	0.395	0.431	0.363	0.527
Macedonia	0.384	0.287	0.433	0.350	0.459	0.431	0.551	0.448	0.457	0.379
Malta	0.468	0.412	0.427	0.321	0.529	0.357	0.477	0.487	0.475	0.394
Moldova	0.380	0.391	0.247	0.407	0.328	0.388	0.353	0.456	0.327	0.411
Montenegro	0.358	0.483	0.320	0.446	0.361	0.472	0.427	0.632	0.367	0.508
Russia	0.578	0.526	0.646	0.570	0.600	0.656	0.685	0.618	0.627	0.592
Serbia	0.421	0.307	0.402	0.402	0.506	0.386	0.425	0.509	0.438	0.401
Sweden	0.895	0.426	0.621	0.2 95	0.814	0.319	0.705	0.420	0.759	0.365
Switzerland	0.758	0.477	0.531	0.253	0.782	0.493	0.661	0.474	0.683	0.424
Tajikistan	0.252	0.610	0.270	0.639	0.275	0.533	0.307	0.468	0.276	0.562
Turkmenistan	0.363	0.377	0.373	0.557	0.420	0.690	0.458	0.519	0.404	0.536
Ukraine	0.572	0.371	0.499	0.552	0.569	0.448	0.525	0.457	0.541	0.457
Uzbekistan	0.422	0.583	0.370	0.413	0.412	0.499	0.368	0.466	0.393	0.490

additional loops are extrapolated by the DMs and represent the evaluations that could be provided by two potential new groups of researchers (experts).

5.1. Numerical results: ranking variability

Table 2 presents the rankings obtained by the DMs after one and two loops have been performed within both the quadrant improvements setting and the Euclidean improvements setting. The ranking obtained by [23], referred to as "original ranking," is also reported in Table 2.

Consider first the *quadrant improvements* setting. We have highlighted in red the main modifications in the ordering relative to the original ranking, though it should be noted that these are not the only ones obtained. Among the main ranking modifications obtained, note the substantial improvement experienced by Bosnia and Herzegovina and Russia when allowing for potential new evaluations to vary in $[0, 1]^2$. This constitutes an important result, since both these countries experience only a slight improvement in the ranking when potential new evaluations are constrained to vary in the domain $[m, M]^2$. Thus, the domain on which potential evaluations may be defined has a substantial effect on the final ranking obtained.

Similar results are obtained when considering the *Euclidean improvements* setting. We have highlighted in blue the main modifications obtained in the previous [quadrant improvements] setting. The main modifications derived from the current one are highlighted in red. Note, for example, the improvement experienced by Finland after performing one loop within the domain constrained by the initial evaluations, $[m, M]^2$, and how this improvement vanishes when a second loop is performed. Therefore, deciding how many loops to perform or, in other words, how many additional evaluations the DMs want to extrapolate, determines the final ranking obtained.

In this scenario, Russia experiences a considerable improvement within both the initial-evaluation-constrained domain $[m, M]^2$ and the entire square $[0,1]^2$. The improvement is larger when the domain is extended and a second loop is performed. A considerable improvement is also shown by Turkmenistan within $[m, M]^2$, while Macedonia and Bosnia and Herzegovina experience moderate improvements only when considering the entire domain $[0,1]^2$. However, both of them perform worse than in the quadrant improvements setting with potential evaluations varying in $[m, M]^2$.

It therefore follows from these simulations that the domain fixed for the potential evaluations plays a substantial role in the resulting ranking. Table 2 clearly illustrates how the DMs' beliefs regarding the set of potential evaluations for a given alternative affect the resulting ranking. This is particularly important if we consider the substantial improvement experienced by Russia when DMs perform a second loop fixing $[0,1]^2$ as the domain for potential evaluations. Thus, we must define a validation criterion that allows the DMs to select the

Table 2
Forward-looking rankings: Quadrant and Euclidean improvements in $[m, M]^2$ and $[0,1]^2$.

Country	Original distance	Quadrant improvements $EV(\bar{x}_i, \bar{y}_i)$				Euclidean improvements $EV(\bar{x}_i, \bar{y}_i)$			
		$[m, M]^2$ First loop	$[m, M]^2$ Second loop	$[0, 1]^2$ First loop	$[0, 1]^2$ Second loop	$[m, M]^2$ First loop	$[m, M]^2$ Second loop	$[0, 1]^2$ First loop	$[0, 1]^2$ Second loop
Sweden	0.438	0.1414	0.1439	0.0052	0.0375	0.0989	0.0905	0.0045	0.0215
Austria	0.483	0.1636	0.1679	0.00898	0.0539	0.1107	0.1004	0.0072	0.0291
Switzerland	0.529	0.1915	0.1931	0.00899	0.0616	0.1491	0.1298	0.0082	0.0354
Finland	0.541	0.2376	0.2426	0.0221	0.1991	0.1416	0.1365	0.0141	0.0922
Ireland	0.575	0.2776	0.2862	0.0223	0.3000	0.1710	0.1659	0.0164	0.1508
Ukraine	0.647	0.3770	0.3898	0.0368	0.5713	0.1862	0.1865	0.0260	0.2798
Malta	0.656	0.3849	0.3980	0.0330	0.5802	0.2512	0.2486	0.0252	0.3012
Macedonia	0.662	0.3644	0.3733	0.0224	0.4096	0.2750	0.2625	0.0224	0.2582
Cyprus	0.673	0.4060	0.4118	0.0469	1.2819	0.2291	0 .2259	0.0333	0.6408
Serbia	0.690	0.4153	0.4272	0.0286	0.4802	0.2483	0.2427	0.0258	0.2775
Belarus	0.693	0.4350	0.4444	0.0543	1.1755	0.2541	0.2521	0.0375	0.5841
Russia	0.700	0.4348	0.4438	0.0322	0.2051	0.2297	0.2263	0.0306	0.1271
Bosnia and Herzegovina	0.711	0.4279	0.4343	0.0193	0.3899	0.2938	0.2787	0.0247	0.2925
Kazakhstan	0.712	0.4590	0.4728	0.0338	0.6075	0.3341	0.3292	0.0328	0.3685
Armenia	0.732	0.4902	0.5029	0.0542	1.0263	0.3040	0.3023	0.0401	0.5227
Azerbaijan	0.750	0.5048	0.5154	0.0476	0.8771	0.32 81	0.3219	0.0413	0.4927
Uzbekistan	0.780	0.5614	0.5747	0.0629	1.4816	0.2681	0.2685	0.0489	0.7740
Moldova	0.788	0.5903	0.6059	0.0925	2.2188	0.3622	0.3661	0.0582	1.0421
Turkmenistan	0.802	0.5433	0.5585	0.0607	0.9068	0.2535	0.2499	0.0522	0.5035
Georgia	0.812	0.5990	0.6111	0.0530	0.9248	0.3692	0.3633	0.0511	0.5554
Montenegro	0.812	0.6029	0.6179	0.0866	1.2202	0.3687	0.3657	0.0602	0.5914
Kyrgyzstan	0.827	0.6102	0.6207	0.0746	1.5851	0.3515	0.3451	0.0616	0.8640
Tajikistan	0.917	0.7819	0.8022	0.0979	3.6771	0. 3797	0.3810	0.0907	2.1714

correct ranking among those that can be extrapolated. This criterion must be based on the number of loops performed and the domain assumed for the densities, which also determines the spread of the potential evaluations obtained for each alternative.

5.2. Numerical results: ranking validation

The quality of the observations extrapolated from the evaluations provided by the experts must be verified, and any ranking redistribution considered to be less reliable than the one initially provided must be excluded. This quality evaluation problem requires an objective criterion that must allow for the validation of the ranking obtained after each additional loop. This criterion should be based on the entropy of the resulting observations and be sufficiently strict to prevent any less reliable ranking to be validated by the DMs. We will impose a twofold requirement for DMs to validate any ranking based on the extrapolated evaluations:

(i) The total entropy of the new ranking must be lower than that of the previous ranking. The total entropy of a ranking is defined as the product of the total entropies associated to both the opportunity scores and the threat levels of all the alternatives. Thus, in order for a ranking obtained after performing a second loop to be acceptable, the entropies derived from the evaluations must satisfy:

$$\left(\sum_i e\left(x_i^{n+1}\right)\right)\left(\sum_i e\left(y_i^{n+1}\right)\right) < \left(\sum_i e(x_i)\right)\left(\sum_i e(y_i)\right). \quad (23)$$

A stricter criterion requiring this inequality to be satisfied and both the total entropy of the opportunity scores and that of the threat levels to be lower than those of the previous ranking could also be defined.

(ii) The product of the entropies of each one of the alternatives whose relative position within the ranking is being modified must have a lower value. That is:

$$\forall i, \quad \text{if } p(i) \text{ holds} \quad \Rightarrow \quad e\left(x_i^{n+1}\right)e\left(y_i^{n+1}\right) < e(x_i)e(y_i) \quad (24)$$

where, $p(i)$ is the statement "the relative ranking position of A_i is modified."
Hence, ranking modifications cannot involve any alternative whose entropy has increased. A stricter criterion, similar to the one suggested in (i), could also be defined.

Table 3
Entropy evaluations for the Euclidean improvements setting with $[m, M]^2$ and $[0,1]^2$.

Rank	Country	Alternative ranking First loop $[m, M]^2$	Alternative ranking Second loop $[m, M]^2$	$e(x)$	$e(y_i)$	$e(x_i^{n+1})$ $[m, M]^2$	$e(y_i^{n+1})$ $[m, M]^2$	$e(x_i^{n+1})$ $[0,1]^2$	$e(y_i^{n+1})$ $[0,1]^2$
1	Sweden	Sweden	Sweden	0.9308	0.9917	0.8041	0.8753	0.8298	0.9514
2	Austria	Austria	Austria	0.7978	0.9721	0.7061	0.8465	0.9136	0.9389
3	Switzerland	Finland	Switzerland	0.8835	0.9105	0.8172	0.9360	0.9278	1.0850
4	Finland	Switzerland	Finland	0.8715	0.9716	0.8649	0.8626	0.9044	0.9267
5	Ireland	Ireland	Ireland	0.7737	0.7956	0.7786	0.7219	0.9507	0.9559
6	Ukraine	Ukraine	Ukraine	0.9623	0.6159	0.8723	0.5622	0.8334	0.7456
7	Malta	Cyprus	Cyprus	0.9148	0.9970	0.9373	0.8619	1.0114	0.9479
8	Macedonia	Russia	Russia	0.9360	0.9198	0.8336	0.8066	0.8487	0.7718
9	Cyprus	Serbia	Serbia	0.8680	0.6633	0.8718	0.6003	0.7687	0.8784
10	Serbia	Malta	Malta	0.6923	0.8899	0.6190	0.8050	0.7780	0.8272
11	Belarus	Turkmenistan	Turkmenistan	0.9411	0.7468	0.8555	0.6513	0.7250	0.9296
12	Russia	Belarus	Belarus	0.8954	0.8801	0.9045	0.8028	1.0794	0.7735
13	Bosnia and Herzegovina	Uzbekistan	Macedonia	0.7392	0.9451	0.6819	0.8557	0.8317	0.8998
14	Kazakhstan	Macedonia	Uzbekistan	0.9918	0.8050	0.8769	0.7278	0.7599	0.8406
15	Armenia	Bosnia and Herzegovina	Bosnia and Herzegovina	0.8820	0.7664	0.8844	0.7657	0.7303	0.9970
16	Azerbaijan	Armenia	Armenia	0.8683	0.8813	0.7706	0.7971	0.8889	0.8045
17	Uzbekistan	Azerbaijan	Azerbaijan	0.9722	0.8316	0.8636	0.7671	0.8280	0.9327
18	Moldova	Kazakhstan	Kazakhstan	0.7263	0.8238	0.6867	0.7710	0.9348	0.9487
19	Turkmenistan	Kyrgyzstan	Kyrgyzstan	0.9802	0.9516	0.8605	0.8454	0.8741	0.8561
20	Georgia	Moldova	Georgia	0.7718	0.9576	0.7784	0.8472	0.6789	0.8295
21	Montenegro	Montenegro	Montenegro	0.7628	0.8857	0.7233	0.8893	0.8515	0.6714
22	Kyrgyzstan	Georgia	Moldova	0.7502	0.8481	0.7395	0.8402	0.9599	0.8253
23	Tajikistan	Tajikistan	Tajikistan	0.7258	0.9375	0.6964	0.8433	0.8150	0.8693
Total entropy				19.6376	19.9877	18.4273	18.2822	19.7236	20.2069

Information entropy can therefore be used to define a ranking validation mechanism based on the informativeness derived from the set of evaluations assigned to each alternative. The physical meaning of the two requirements imposed can be observed in Table 3, where the entropy evaluations for the Euclidean improvements setting are reported.

Note that the entropy inherent to the ranking generated by the first loop corresponds to that of the initial evaluations received. The DMs extrapolate their first evaluation when performing the second loop. The total entropy of the extrapolated rankings is presented in the last row of Table 3. Clearly, the criterion imposed by Eq. (23) is satisfied within the initial-evaluation-constrained domain $[m, M]^2$.

At the same time, we observe that Bosnia shifts from an initial entropy of 0.6759 ($e(x_i)e(y_i) = 0.8820 \cdot 0.7664$) to an entropy of 0.6771 ($e(x_i^{n+1})e(y_i^{n+1}) = 0.8844 \cdot 0.7657$) when a second loop is performed within the domain $[m, M]^2$. Given the validation requirements described in Eq. (24), the DMs cannot validate any ranking that modifies the initial relative position of Bosnia. This means that the initial ranking position of Bosnia cannot be modified, and all the other countries whose ranking positions are modified must exhibit lower entropy values from their extrapolated evaluations. Table 3 illustrates that this criterion is also satisfied: the entropy values of the individual countries satisfying Eq. (24) are reported in the seventh and eighth columns of this table. Thus, the DMs will validate the ranking obtained after performing a second loop in the $[m, M]^2$ setting.

The last row of Table 3 also shows that the total entropy generated in the $[0,1]^2$ setting through its second loop is larger than the initial one. Based on the criterion defined by Eq. (23), this implies that the evaluations extrapolated using the $[0,1]^2$ domain distort the initial ranking and, therefore, must not be considered by the DMs.

Finally, the entropy criteria (i) and (ii) can also be used to validate the spread of the domain considered by the DMs. In this regard, the choice of the domain could be based on both the evaluations received and those extrapolated, following the approach suggested by [32]. Moreover, a range of potential domains based on the initial evaluations of the experts could be considered by the DMs, extending our paper into the field of fuzzy decision support systems [33]. In this case, our model could be modified to account for information regarding the preferences of the experts and their effect on the final rankings obtained by the DMs.

5.3. Alternative numerical evaluations

In this section, we perform a battery of Monte Carlo simulations to generate comparable rankings based on the potential trends that may be statistically inferred from the data. These alternative numerical evaluations are described in Table 4. The main modifications relative to the ranking based on the initial evaluations have been highlighted in red.

We generate 100 trial runs of two evaluations for each country so as to maintain a dispersed set of realizations that does not fully converge to the density determining the Monte Carlo process. The results remain unchanged when running 100 trials of 10 evaluations. The Monte Carlo uniform scenario assumes a uniform density on the intervals $[x_i^m, x_i^M]$ and $[y_i^m, y_i^M]$. It then reports the Euclidean distances based on the mean of the simulated runs. The Monte Carlo normal scenario assumes a normal distribution on the same intervals, while the PERT one assumes a Beta distribution to which a low confidence level $\lambda = 4$ has been assigned. This latter distribution, based on the three-point $([x_i^m, \bar{x}_i, x_i^M])$ estimation technique, is generally used in situations relying on a limited number of observations [16].

We must emphasize the dependence of the Monte Carlo simulations on a given probability distribution. The numerical results illustrate a relative improvement in the ranking positions of Russia and Bosnia as the dispersion of the density defining the simulations increases through the domains. Given the significant degree of subjectivity inherent to the judgment of the experts, the PERT evaluation method may seem the most reliable one, but we lack a criterion to determine which ranking should be chosen by the DMs.

Table 4
Alternative ranking scenarios: Monte Carlo simulations and biased limit evaluations.

Country	Monte Carlo uniform	Monte Carlo normal	Monte Carlo PERT	Limit algorithm	Limit algorithm entropy	Limit distance algorithm	Limit distance entropy
Sweden	0.4317	0.4414	0.4363	0.4375	0.4393	0.5714	0.5738
Austria	0.4845	0.4814	0.4833	0.4826	0.4886	0.5687	0.5758
Switzerland	0.4946	0.5297	0.5213	0.5296	0.5354	0.5564	0.5626
Finland	0.5617	0.5353	0.5479	0.5407	0.5453	0.5519	0.5565
Ireland	0.5832	0.5718	0.5790	0.5756	0.5889	0.6057	0.6197
Ukraine	0.6498	0.6525	0.6500	0.6476	0.6620	0.6937	0.7091
Malta	0.6587	0.6567	0.6576	0.6564	0.6711	0.7642	0.7813
Macedonia	0.6466	0.6598	0.6571	0.6623	0.6736	0.7767	0.7898
Cyprus	0.6859	0.6750	0.6785	0. 6728	0.6760	0.6636	0.6668
Serbia	0.6810	0.6856	0.6874	0.6901	0.7073	0.8371	0.8580
Belarus	0.7211	0.6947	0.7031	0.6931	0.7014	0.6912	0.6995
Russia	0.6953	0.6976	0.6986	0.6999	0.7053	0.7364	0.7421
Bosnia and Herzegovina	0.6834	0.7104	0.7011	0.7115	0.7248	0.8509	0.8668
Kazakhstan	0.7084	0.7147	0.7110	0.7124	0.7296	0.7573	0.7756
Armenia	0.7399	0.7318	0.7338	0.7317	0.7415	0.7747	0.7851
Azerbaijan	0.7460	0.7537	0.7489	0.7504	0.7582	0.7753	0.7834
Uzbekistan	0.7845	0.7872	0.7813	0.7802	0.7886	0.8261	0.8349
Moldova	0.8046	0.7922	0.7943	0.7884	0.8053	0.7825	0.7993
Turkmenistan	0.7931	0.7966	0.7994	0.8017	0.8150	0.8249	0.8385
Georgia	0.8022	0.8118	0.8079	0.8116	0.8234	0.8778	0.8905
Montenegro	0.8250	0.8176	0.8164	0.8121	0.8274	0.8259	0.8415
Kyrgyzstan	0.8238	0.8284	0.8261	0.8270	0.8300	0.8182	0.8212
Tajikistan	0.9062	0.9186	0.9139	0.9168	0.9333	0.9034	0.9197

Consider now the limit cases described in the last four columns of Table 4. The expected opportunity scores obtained in the *Limit Algorithm* follow from $\left(\frac{x_i^M-\bar{x}_i}{x_i^M-x_i^m}\right)x_i^m+\left(\frac{\bar{x}_i-x_i^m}{x_i^M-x_i^m}\right)x_i^M$, while in the *Limit Distance Algorithm* they are given by $\left(\frac{\bar{x}_i-x_i^m}{x_i^M-x_i^m}\right)x_i^M$. Similar definitions are used to determine the value of the expected threat levels. The entropy version of these two limit cases is obtained by multiplying the resulting distances by $[1-f(x_i)]\,[1-f(y_i)]$. Both cases report the results from performing one loop based on the limit evaluations assumed.

Both of these cases are based on extreme evaluations of the experts and have been included to highlight the potential distributional frictions that are not captured by the Monte Carlo simulations. It should be emphasized that the differences between the resulting limit rankings do not rely only on the bias generated by the limit distance structure. The differences in the rankings derived in the quadrant and Euclidean improvements settings underscore this point. Quadrant improvements, which are based on a biased approach similar to the limit distance case, fail to account for all the potential improvements of the alternatives. For example, this bias leads Macedonia to improve its ranking position (see Table 2), while in the limit distance case it suffers a considerable fall in the ranking. Thus, absent entropy considerations, accounting for all potential improvements is essential to generating reliable rankings.

6. Conclusion

We have introduced a novel decision support framework that allows DMs to assess the informativeness of a ranking of alternatives provided by different experts and to extrapolate additional evaluations based on the distributional bias and entropy inherent to those received from the experts. Alternatives have been evaluated in terms of their Euclidean distance from an ideal state, which endows the current setting with the visual and intuitive power of TOPSIS type environments [34].

We have illustrated the diversity of potential rankings available to the DMs by extending the NATO enlargement analysis performed by [23], where several groups of researchers provide different evaluations of candidate countries within an opportunities versus threats framework. In this regard, the entropy generated by the evaluations has been used as a criterion to validate the reliability of the rankings obtained and to determine the stopping rule for the data generating algorithm.

We have performed a battery of Monte Carlo simulations and provided several alternative extrapolation approaches in order to illustrate the distributional effects that can be captured and validated using the current approach without imposing an exogenous structure on the evaluation probabilities considered by the DMs.

It should be emphasized that the current analysis adds a strategic component to the display of evaluations among alternatives. That is, evaluations may be displayed strategically such that one of the alternatives tops the resulting ranking after an endogenously selected number of experts have delivered their evaluations. This extension relates the current paper to the idea of manipulation in social sciences, which is usually relegated to social choice theory and the design of voting mechanisms [35–37]. In this regard, the economic literature has consistently illustrated the susceptibility of DMs to information from biased third parties, even if this information is assumed to be verifiable [38–40].

The fact that evaluations are determined by experts whose priorities and preferences may differ from those of the DMs and, hence, lead them to accept a potentially suboptimal alternative, implies that control instruments accounting for opinion volatility should be incorporated in the design of DSS. The current paper has examined the trend that may be extrapolated from the evaluations of several experts in order to generate rankings that are more consistent and reliable for the DMs. Future research in this area should aim to incorporate insights from the economics (and game-theoretic) literature in the design of strategy-proof DSS.

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