Optimistic and pessimistic performance and congestion analysis in fuzzy data envelopment analysis

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Abstract: The notion of input congestion in data envelopment analysis (DEA) is analogous to the 'law of diminishing returns' in the classical economic theory of production which states that if a single input is increased while other inputs are held constant, the marginal product of the variable input diminishes. Congestion has been an under-researched topic in economic theory especially when there is a need for augmenting inputs to serve important objectives besides output maximisation. We propose a fuzzy DEA model and represent the imprecise and ambiguous input and output data with fuzzy numbers. We solve the model with an α-cut approach and obtain the value of input congestion for the optimistic and pessimistic cases. The fundamental idea in this paper is to transform the fuzzy DEA model into a crisp linear programming model using the α-cut approach. Two auxiliary crisp models are solved to obtain optimistic and pessimistic values of congestion for evaluating the decision-making units (DMUs). We use a numerical example from the literature to demonstrate the applicability of the proposed method and exhibit the efficacy of the procedures and algorithms.

Keywords: data envelopment analysis; DEA; fuzzy inputs and outputs; α-cut approach; optimistic and pessimistic congestion; grey system theory.

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1 Introduction

Data envelopment analysis (DEA) was originated by Charnes et al. (1978) and later developed by Banker et al. (1984) to evaluate the relative efficiency of a set of decision-making units (DMUs) involved in a production process. The DEA models provide efficiency scores that assess the performance of different DMUs in terms of either the use of several inputs or the production of certain outputs (Khodabakhshi et al., 2010). Cooper et al. (2004) argues that congestion has been an under-researched topic in the economic theory of production especially when there is a need for augmenting inputs to serve important objectives besides output maximisation. Congestion literally means overcrowding or concentration of some material objects in a small space. Färe and Svensson (1980) originally introduced the notion of input congestion by referring to the 'law of diminishing returns'. Brue (1993) and Färe (1980) presents a historical review and formal treatment of this classic economic concept. The law of diminishing returns states that if a single input is increased while other inputs are held constant, the marginal product of the variable input diminishes (Cherchye et al., 2001; Färe, 1980).

In the early 1980's, Färe and Grosskopf (1983) and Färe et al. (1985) suggested a practical and functional approach for analysing congestion in DEA and used several models and methods to evaluate production efficiency. Later, Cooper et al. (1996) proposed an alternative DEA approach for analysing and studying congestion. They drew a comparison between the two approaches and used numerical examples to depict the advantages of their method. They then introduced a 'one model approach' to unify the two models provided in Cooper et al.'s (1996) approach. Brocket et al. (1998) applied Cooper et al.'s (1996) approach to analyse employee productivity. Brocket et al. (2004) used DEA to identify congestion, estimate its amounts and distinguish it from other forms of inefficiency. They also used DEA to manage congestion by estimating input decreases and output increases.

Cooper et al. (2001) examined congestion management in several Chinese industries. They illustrated how to remove managerial inefficiency in the textile and automobile industry by boosting output without trimming the workforce. This congestion problem in the Chinese textile and automobile industry was re-investigated by Jahanshahloo and Khodabakhshi (2004) who showed that flexibility in using inputs can significantly increase output and can occur multiple times because of new input combinations. Färe and Svensson (1980) defined and developed the topic of congestion using a model based on variable proportions. Färe and Grosskopf (1983, 2001) suggested a procedure for identifying those input factors responsible for congestion. Brocket et al. (1998) and Cooper et al. (2000) proposed new DEA frameworks for capturing input congestion.

Wei and Yan (2004) and Tone and Sahoo (2004) developed another method by following Färe and Grosskopf (1983) and Cooper et al.'s (2000) approaches. Both studies were conducted from the output perspective and considered excessive input to explain the effects of congestion. Sueyoshi and Sekitani (2009) suggested another approach for measuring the value of congestion under the occurrence of multiple solutions. Fleg and Alen (2007) explored these three approaches and studied the rapid growth of the number of the students in British universities which had led to congestion.

Fuzzy set theory, introduced by Zadeh (1965) has been used to measure and evaluate efficiency in problems with imprecise and vague data. Sengupta (1992) originated the fuzzy mathematical programming approach with non-crisp constraints and non-crisp objective functions. Sengupta (1992) considered both objectives and constraints as fuzzy and analysed the resulting fuzzy DEA model using Zimmermann's (1976) method. Kao and Liu (2000) developed a method for finding the membership function of the fuzzy efficiency scores when some observations are fuzzy numbers. Congestion can be considered a severe form of inefficiency in that a reduction in one or more input results in an increase in one or more outputs – without deteriorating other inputs or outputs, as opposed to technical inefficiency which simply represents an excess of some inputs or a shortfall in some outputs (Cooper et al., 2004).

The conventional DEA methods require precise measurement for both the inputs and outputs. However, the observed values of the input and output data in real-world problems are sometimes imprecise or vague. Imprecise evaluations may be the result of unquantifiable, incomplete and/or non-obtainable information. Researchers have proposed various strategies such as stochastic, interval, grey and fuzzy data (Khodabakhshi and Asgharian, 2009; Khodabakhshi et al., 2010) for dealing with the impreciseness and ambiguity in DEA.

Stochastic output and input variation has been commonly used in DEA (Cooper et al., 2004; Huang and Li, 1996; Asgharian et al., 2010; Khodabakhshi, 2010). Cooper et al. (2004) showed how to identify congestion with deterministic models rather than their stochastic counterparts under suitable assumptions. Huang and Li (1996) discussed the relationship between the conventional DEA models and the general stochastic DEA models and proposed stochastic models in DEA by considering random variations in the input and output data. Asgharian et al. (2010) explained that input relaxation DEA models and their stochastic version were more flexible in dealing with input combination changes for maximising outputs. They studied congestion issues in this setting and obtained a deterministic equivalent to the stochastic congestion model.

Interval data are also commonly used to identify imprecise input and output data (Entani et al., 2002; Jahanshahloo et al., 2004, 2011; Jahanshahloo and Khodabakhshi, 2004; Smirlis et al., 2006; Entani and Tanaka, 2006). Entani et al. (2002) formulated a DEA model with interval efficiency which consisted of efficiencies obtained from the optimistic and pessimistic viewpoints. Jahanshahloo et al. (2004) studied sensitivity and stability analysis in DEA with interval data and suggested a modified CCR model for analysing the sensitivity of the DMUs with interval data. Jahanshahloo and Khodabakhshi (2004) applied ranking methods based on the comparison of the α-cut. Smirlis et al. (2006) introduced an approach based on interval DEA that allowed the evaluation of the DMUs with missing values. The missing values were replaced by intervals and the constant bounds of the intervals were estimated by using statistical or experiential techniques. Entani and Tanaka (2006) improved the efficiency interval of a DMU by adjusting its given inputs and outputs. They formulated an interval DEA model and obtained an efficiency interval consisting of evaluations from both the optimistic and pessimistic viewpoints. Jahanshahloo et al. (2011) formulated an interval DEA model to obtain an efficiency interval consisting of evaluations from both the optimistic and the pessimistic viewpoints. The points obtained by this method are called ideal points which are used to rank the DMUs.

Grey system theory, initiated by Julong (1982), is another alternative method for solving problems under uncertain conditions using fragmented data and incomplete information. Model parameters in multiple criteria decision-making can involve uncertainty with regards to both performance of alternatives on the attributes and the attribute weights. DEA is commonly used as an objective method for deriving these attribute weights. Wu and Olson (2010) proposed a grey-based fuzzy set method and incorporated DEA to objectively rank the alternatives. They focused on identifying the most efficient alternatives with respect to the decision maker's preferences. They demonstrated their method on a multi-attribute problem and simulation was used to validate the efficiency of the model. Chen and Chen (2011) used grey theory and applied the DEA and Malmquist productivity index to explore the operation performances in wafer fabrication companies. The input variables were total assets, operation costs and selling and administrative expenditures and the output variable was net sales. Wu (2011) proposed a solution that involved applying a variety of objective weighting methods including DEA, grey system theory and artificial neural networks to produce practical rankings, as well as using the Borda count methodology to combine these rankings.

The remainder of this paper is organised as follows. In Section 2, we review the chance constrained programming approach to congestion proposed by Cooper et al. (2004). In Section 3, we propose a procedure based on Cooper et al.'s (2001) approach for determining optimistic and pessimistic congestion in problems with fuzzy inputs and outputs. A numerical example taken from the literature is presented in Section 4 to demonstrate the applicability of the proposed method and exhibit the efficacy of the procedures and algorithms. In Section 5, we present our conclusions and future research directions.

2 Background

Suppose we have n DMUs with m inputs and s outputs and that the vectors $x_{ij} = (x_{1j}, x_{2j}, \ldots x_{mj})^T$ and $y_{rj} = (y_{1j}, y_{2j} \ldots y_{sj})^T$ denote the input and output values of DMU$_j$ (where $j = 1,2,\ldots n$), respectively. Cooper et al. (2004) used the output oriented BCC model of Banker et al. (1984) with variable returns to scale to calculate the efficiency of the DMUs:

$$
\begin{aligned}
&\text{Max } \varphi_o + \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right)\\
&\text{s.t}\\
&\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = \varphi_o y_{ro},\\
&\sum_{j=1}^{n} \lambda_j x_{rj} + s_i^- = x_{io},\\
&\sum_{j=1}^{n} \lambda_j = 1,\\
&\lambda_j \geq 0, \quad s_r^+ \geq 0, \quad s_i^- \geq 0\\
&i = 1,2,\ldots,m, \quad j = 1,2,\ldots,n, \quad r = 1,2,\ldots,s.
\end{aligned}
\tag{1}
$$

where $(\varphi^*, s_r^{*+}, s_i^{*-}, \lambda_j^*)$ is optimal solution for model (1).

Inefficiency is a necessary condition for the presence of congestion; therefore, we first use (1) to identify whether DMU is inefficient. If it is found to be inefficient, then we will use (2) to obtain $\hat{x}_{io}, \hat{y}_{ro}$ using s_i^{*-}. If we do not have any inefficiency $(s_i^{*-} \neq 0)$, we will not be able to continue because congestion is related to the inefficiency in the DMUs.

$$\begin{aligned} &\sum_{j=1}^{n} y_{rj}\lambda_j^* = \varphi^* y_{ro} + s_r^{+*}, \quad r = 1, 2, \dots, s \\ &\sum_{j=1}^{n} x_{ij}\lambda_j^* = x_{io} - s_i^{-*}, \qquad i = 1, 2, \dots, m \end{aligned} \tag{2}$$

Hence, we can apply the value of left-hand side in (2) to define the new output and inputs.

$$\begin{aligned} &\hat{y}_{ro} = \varphi_o^* y_{ro} + s_r^{+*} \geq y_{ro}, \quad r = 1, 2, \dots, s \\ &\hat{x}_{io} = x_{io} - s_i^{-*} \leq x_{io}, \qquad i = 1, 2, \dots, m \end{aligned} \tag{3}$$

After that, we solve (4) to identify $\delta_i'^-$ (without congestion):

$$\begin{aligned} &\text{Max } \delta_i^- \\ &\text{s.t.} \\ &\hat{y}_{ro} = \sum_{j=1}^{n} y_{rj}\lambda_j', \\ &\hat{x}_{io} = \sum_{j=1}^{n} x_{ij}\lambda_j' - \delta_i^-, \\ &\sum_{j=1}^{n} \lambda_j' = 1, \\ &\lambda_j \geq 0, \quad \delta_i^- \geq 0 \\ &i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n, \quad r = 1, 2, \dots, s. \end{aligned} \tag{4}$$

The value of $s_i^{-c} \geq 0$ represents the amount of congestion in input i for DMUo, while $\delta_i^- \geq 0$ represents the amount of technical inefficiency in this same input. We have, therefore, decomposed the total slack s_i^{*-} obtained for each DMUo from (1) into two components represented as follows:

$$s_i^{-c} = s_i^{-*} - \delta_i^{-*}, \quad i = 1, 2, \dots, m \tag{5}$$

3 Proposed method

In this section, we propose a fuzzy model for evaluating the amount of congestion with fuzzy inputs and outputs. Consider n DMUs, each of which uses m different fuzzy inputs to secure s different fuzzy outputs. The fuzzy model that we propose is based on the following model developed by Cooper et al. (2001):

$$
\begin{aligned}
&\text{Max } \varphi_p + \sum_{i=1}^{m}\left(s_i^-, s_i^-, s_i^-, s_i^-\right) + \sum_{r=1}^{s}\left(s_r^+, s_r^+, s_r^+, s_r^+\right) \\
&\text{s.t.} \\
&\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} + s_i^- = \tilde{x}_{ip}, \\
&\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} - s_r^+ = \varphi_p \tilde{y}_{rp}, \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\lambda_j \geq 0, \quad s_i^- \geq 0, \quad s_r^+ \geq 0, \\
&i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n; \quad \text{and} \quad r = 1, 2, \ldots, s.
\end{aligned}
\tag{6}
$$

where $\tilde{x}_{ij} = (x_{ij}^{m1}, x_{ij}^{m2} x_{ij}^{l}, x_{ij}^{u})$ and $\tilde{y}_{rj} = (y_{rj}^{m1}, y_{rj}^{m2} y_{rj}^{l}, y_{rj}^{u})$, the fuzzy input and fuzzy output values of the j^{th} DMU, respectively; are expressed as trapezoidal fuzzy numbers. Now, we show the crisp slacks s_r^- and s_i^+ in the above model as $s_r^+ = (s_r^+, s_r^+, s_r^+, s_r^+)$ and $s_i^+ = (s_i^-, s_i^-, s_i^-, s_i^-)$, respectively; therefore, model (6) can be written as follows:

$$
\begin{aligned}
&\text{Max } \varphi_p + \left(\sum_{i=1}^{m}\left(s_i^-, s_i^-, s_i^-, s_i^-\right) + \sum_{r=1}^{s}\left(s_r^+, s_r^+, s_r^+, s_r^+\right)\right) \\
&\text{s.t.} \\
&\sum_{j=1}^{n} \lambda_j \left(x_{ij}^{m1}, x_{ij}^{m2} x_{ij}^{l}, x_{ij}^{u}\right) + \left(s_i^-, s_i^-, s_i^-, s_i^-\right) = \left(x_{ip}^{m1}, x_{ip}^{m2}, x_{ip}^{l}, x_{ip}^{u}\right), \\
&\sum_{j=1}^{n} \lambda_j \left(y_{rj}^{m1}, y_{rj}^{m2} y_{rj}^{l}, y_{rj}^{u}\right) - \left(s_r^+, s_r^+, s_r^+, s_r^+\right) = \varphi_p \left(y_{rp}^{m1}, y_{rp}^{m2}, y_{rp}^{l}, y_{rp}^{u}\right), \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\lambda_j \geq 0, \quad \tilde{s}_i^- \geq 0, \quad \tilde{s}_r^+ \geq \tilde{0}, \\
&i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n; \quad \text{and} \quad r = 1, 2, \ldots, s.
\end{aligned}
\tag{7}
$$

Model (7) can be solved in two stages as follows.

In the first stage:

$$
\begin{aligned}
& \text{Max } \varphi_p \\
& \text{s.t.} \\
& \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \le \tilde{x}_{ip}, \\
& \sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} \ge \varphi_p \tilde{y}_{rp}, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \ge 0, \\
& i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n, \quad r = 1, 2, \ldots, s.
\end{aligned} \tag{8}
$$

In the second stage:

$$
\begin{aligned}
& \text{Max } \varphi_p \\
& \text{s.t.} \\
& \sum_{j=1}^{n} \lambda_j \left(x_{ij}^{m1}, x_{ij}^{m2} x_{ij}^{l}, x_{ij}^{u} \right) \le \left(x_{ip}^{m1}, x_{ip}^{m2}, x_{ip}^{l}, x_{ip}^{u} \right), \\
& \sum_{j=1}^{n} \lambda_j \left(y_{rj}^{m1}, y_{rj}^{m2} y_{rj}^{l}, y_{rj}^{u} \right) \ge \varphi_p \left(y_{rp}^{m1}, y_{rp}^{m2}, y_{rp}^{l}, y_{rp}^{u} \right), \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \ge 0, \\
& i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n, \quad r = 1, 2, \ldots, s.
\end{aligned} \tag{9}
$$

We can solve model (9) by applying the following four approaches proposed in the fuzzy DEA literature: the fuzzy ranking approach, the possibility approach, the tolerance approach and the α-level-based approach. Since the other three approaches will culminate in losing some information when they produce crisp results, we apply the α-level approach to take into consideration all the fuzzy information in the performance assessment (Hatami-Marbini et al., 2012).

$$
\begin{aligned}
&\text{Max } \varphi_p \\
&\text{s.t.} \\
&\sum_{j=1}^{n} \lambda_j \left(\alpha x_{ij}^{m1} + (1-\alpha) x_{ij}^{l}, \alpha x_{ij}^{m2} + (1-\alpha) x_{ij}^{u} \right) \\
&\qquad \le \left(\alpha x_{ip}^{m1} + (1-\alpha) x_{ip}^{l}, \alpha x_{ip}^{m2} + (1-\alpha) x_{ip}^{u} \right), \\
&\sum_{j=1}^{n} \lambda_j \left(\alpha y_{rj}^{m1} + (1-\alpha) y_{rj}^{l}, \alpha y_{rj}^{m2} + (1-\alpha) y_{rj}^{u} \right) \\
&\qquad \ge \varphi_p \left(\alpha y_{rp}^{m1} + (1-\alpha) y_{rp}^{l}, \alpha y_{rp}^{m2} + (1-\alpha) y_{rp}^{u} \right), \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\lambda_j \ge 0, \\
&i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n, \quad r = 1, 2, \ldots, s.
\end{aligned}
\tag{10}
$$

Model (10) is an interval linear programming model that cannot be solved by standard optimisation methods. Hence, we convert the objective function into an interval bounded by the lower and upper bound efficiencies as follows:

$$
\begin{aligned}
&\text{Max } \left[\varphi_p^L, \varphi_p^U \right] \\
&\text{s.t.} \\
&\sum_{j=1}^{n} \lambda_j \left[x_{ij}^L, x_{ij}^U \right] \le \left[x_{ip}^L, x_{ip}^U \right], \\
&\sum_{j=1}^{n} \lambda_j \left[y_{rj}^L, y_{rj}^U \right] \ge \varphi_p \left[y_{rj}^L, y_{rj}^U \right], \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\lambda_j \ge 0, \\
&i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n, \quad r = 1, 2, \ldots, s.
\end{aligned}
\tag{11}
$$

In the first stage (12), we obtain φ_p^{L*}, and in the second stage (13) we change the objective function to find s_r^{*+} and s_i^{*-} :

$$
\begin{aligned}
&\varphi_p^{L*} = \max \varphi_p^l \\
&\text{s.t.} \\
&\sum_{j=1}^{n} \lambda_j x_{ij}^L \le x_{ip}^U, \\
&\sum_{j=1}^{n} \lambda_j y_{rj}^U \ge \varphi_p^L y_{rp}^L, \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n, \quad r = 1, 2, \ldots, s.
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
&\text{Max } \left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \right) \\
&\text{s.t.} \\
&\sum_{j=1}^{n} \lambda_j x_{ij}^L + s_i^- = x_{ip}^U, \\
&\sum_{j=1}^{n} \lambda_j y_{rj}^U - s_r^+ = \varphi_p^{L*} y_{rp}^L, \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\lambda_j \ge 0, \quad s_i^- \ge 0, \quad s_r^+ \ge 0, \\
&i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n, \quad r = 1, 2, \ldots, s.
\end{aligned} \tag{13}
$$

Note that $s_i^{-*} \neq 0$ is a necessary condition for obtaining x'_{ip} and y'_{rp} in (14). Next, we obtain $\delta_i^{L^{-*}}$ and $\delta_i^{U^{-*}}$ which represents the amount of technical inefficiency. Therefore, we can apply the left-hand side value in (14) to define new outputs and inputs:

$$
\begin{aligned}
&\sum_{j=1}^{n} \lambda_j^* x_{ij}^L = x_{ip}^U - s_i^{*-} = x_{ip}'^U, \quad i = 1, 2, \ldots, m \\
&\sum_{j=1}^{n} \lambda_j^* y_{rj}^U = \varphi_p^* y_{rp}^L + s_r^{*+} = y_{rp}'^L, \quad j = 1, 2, \ldots, n
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
&\text{Max } \sum_{i=1}^{m} \delta_{i}^{-L} \\
&\text{s.t.} \\
&\sum_{j=1}^{n} \lambda_j x_{ij}^{L} - \delta_i^{L-} = x_{ip}'^{U}, \\
&\sum_{j=1}^{n} \lambda_j y_{rj}^{U} = y_{rp}'^{L}, \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\lambda_j \geq 0, \\
&\delta_i^{-L} \leq s_i^{*-}, \\
&i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n, \quad r = 1, 2, \dots, s.
\end{aligned} \tag{15}
$$

Finally we obtain s_i^{-pc} which represents the pessimistic congestion for input *i*:

$$
s_i^{-pc} = s_i^{*-} - \delta_i^{*-L}, \quad i = 1, 2, \dots, m. \tag{16}
$$

Similarly, we obtain s_i^{-oc}, which represents the optimistic congestion.

$$
\begin{aligned}
&\varphi_p^{U*} = \max \varphi_p^{U} \\
&\text{s.t.} \\
&\sum_{j=1}^{n} \lambda_j x_{ij}^{U} \leq x_{ip}^{L}, \\
&\sum_{j=1}^{n} \lambda_j y_{rj}^{L} \geq \varphi_p^{U} y_{rp}^{U}, \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\lambda_j \geq 0, \quad s_i^{-} \geq 0, \quad s_r^{+} \geq 0, \\
&i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n, \quad r = 1, 2, \dots, s.
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
&\text{Max}\left(\sum_{i=1}^{n} s_i^- + \sum_{r=1}^{s} s_r^+\right)\\
&\text{s.t.}\\
&\sum_{j=1}^{n} \lambda_j x_{ij}^U + s_i^- = x_{ip}^L,\\
&\sum_{j=1}^{n} \lambda_j y_{rj}^L - s_r^+ = \varphi_p^{U*} y_{rp}^U,\\
&\sum_{j=1}^{n} \lambda_j = 1,\\
&\lambda_j \geq 0, \quad s_i^- \geq 0, \quad s_r^+ \geq 0,\\
&i = 1,2,\ldots,m, \quad j = 1,2,\ldots,n, \quad r = 1,2,\ldots,s.
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
&\sum_{j=1}^{n} \lambda_j^* x_{ij}^U = x_{ip}^L - s_i^{*-} = x_{ip}'^L, \qquad i = 1,2,\ldots,m\\
&\sum_{j=1}^{n} \lambda_j^* y_{rj}^L = \varphi_p^* y_{rp}^U + s_r^{*+} = y_{rp}'^U, \quad r = 1,2,\ldots,s
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&\text{Max}\sum_{i=1}^{m} \delta_i^{-U}\\
&\text{s.t.}\\
&\sum_{j=1}^{n} \lambda_j x_{ij}^U - \delta_i^{-U} = x_{ip}'^L,\\
&\sum_{j=1}^{n} \lambda_j y_{rj}^L = y_{rp}'^U,\\
&\sum_{j=1}^{n} \lambda_j = 1,\\
&\lambda_j \geq 0,\\
&\delta_i^{-U} \leq s_i^{*-},\\
&i = 1,2,\ldots,m, \quad j = 1,2,\ldots,n, \quad r = 1,2,\ldots,s.
\end{aligned}
\tag{20}
$$

Finally, the optimistic congestion is measured as:

$$
s_i^{-oc} = s_i^{*-} - \delta_i^{U*}, \quad i = 1,2,\ldots,m \tag{21}
$$

Remark: we should note that the formulas (5) and (20) by which the amount of technical inefficiency in the i^{th} input is measured are always feasible and always have a feasible solution. As indicated earlier, we know that if (17) and (18) have feasible solutions and if the optimal solution for this problem is (λ, s^{*+}, s^{*-}, φ^*), then, it will be a feasible problem. We can define the following vector ($\lambda = \lambda^*$, $\delta^{*-} = 0$) for the above model. In this case (λ^*, δ^{*-}) is one of the optimal solutions for this problem; therefore, this problem is always feasible.

4 Numerical example

In this section, we study a problem with two inputs, two outputs and five DMUs introduced by Guo and Tanka (2001) and used by Saati et al. (2002). As shown in Table 1, the inputs and outputs data are all triangular fuzzy numbers. Note that triangular fuzzy numbers are a special case of trapezoidal fuzzy numbers. In fact, when the first two numbers (x_{ij}^{m1} and x_{ij}^{m2}) in $\tilde{x}_{ij} = (x_{ij}^{m1}, x_{ij}^{m2}, x_{ij}^{l}, x_{ij}^{u})$ are equal, then, $\tilde{x}_{ij}$ has a triangular form. Table 2 shows the amount of congestion under crisp conditions obtained by using model (5) for $\alpha = 1$.

Table 1 Fuzzy inputs and outputs

DMU	*Inputs*		*Outputs*	
	I1	*I2*	*O1*	*O2*
1	(4.0, 3.5, 4.5)	(2.1, 1.9, 2.3)	(2.6, 2.4, 2.8)	(4.1, 3.8, 4.4)
2	(2.9, 2.9, 2.9)	(1.5, 1.4, 1.6)	(2.2, 2.2, 2.2)	(3.5, 3.3, 3.7)
3	(4.9, 4.4, 5.4)	(2.6, 2.2, 3.0)	(3.2, 2.7, 3.7)	(5.1, 4.3, 5.9)
4	(4.1, 3.4, 4.8)	(2.3, 2.2, 2.4)	(2.5, 2.9, 2.3)	(5.7, 5.5, 5.9)
5	(6.5, 5.9, 7.1)	(4.1, 3.6, 4.6)	(5.1, 4.4, 5.8)	(7.4, 6.5, 8.3)

Table 2 Results of model (5)

DMU	φ^*	s_1^{-*}	s_2^{-*}	s_1^{+*}	s_2^{+*}	s_1^{-c}	s_2^{-c}
1	1.08	0.25	0	0	0.73	0.25	0
2	1	0	0	0	0	0	0
3	1.04	0.46	0	0	1.30	0.10	0
4	1	0	0	0	0	0	0
5	1	0.30	0.25	0.35	0.45	0	0.13

Tables 3–7 show the amounts of optimistic and pessimistic congestion for DMUs 1–5 obtained by applying formulas (16) and (21) at four different α-levels (i.e., α = 0, α = 0.2, α = 0.5 and α = 0.7). We should note that using α = 1.0 results in crisp values.

Table 3 Results of model (16) and (21) for DMU1

	DMU1													
α	*Pessimistic*							*Optimistic*						
	φ^*	s_1^{-*}	s_2^{-*}	s_1^{+*}	s_2^{+*}	s_1^{pc}	s_2^{pc}	φ^*	s_1^{-*}	s_2^{-*}	s_1^{+*}	s_2^{+*}	s_1^{oc}	s_2^{oc}
0	1.54	0.115	0	0	0.48	0.10	0	0.86	0.51	0.30	0	0	0.05	0
0.2	1.45	0	0	0	0.68	0	0	0.88	0.18	0.26	0	0	0	0
0.5	1.27	0.24	0	0	0	0.02	0	0.95	0.18	0	0	0.33	0	0
0.7	1.01	0.33	0	0.20	0.13	0.10	0	1.02	0.53	0	0	0	0.04	0

Table 4 Results of (16) and (21) for DMU2

	DMU2													
α	*Pessimistic*							*Optimistic*						
	φ^*	s_1^{-*}	s_2^{-*}	s_1^{+*}	s_2^{+*}	s_1^{pc}	s_2^{pc}	φ^*	s_1^{-*}	s_2^{-*}	s_1^{+*}	s_2^{+*}	s_1^{oc}	s_2^{oc}
0	1	0	0.20	0	0.40	0	0.20	0.89	0	0	0	0	0	0
0.2	1	0	0.16	0	0.32	0	0	0.91	0	0.84	0.19	0.94	0	0
0.5	1	0	0.10	0	0.20	0	0.10	1	0	0	0	0	0	0
0.7	1	0	0.60	0	0.12	0	0	0.96	0	0	0	0	0	0

Table 5 Results of (16) and (21) for DMU3

	DMU3													
α	*Pessimistic*							*Optimistic*						
	φ^*	s_1^{-*}	s_2^{-*}	s_1^{+*}	s_2^{+*}	s_1^{pc}	s_2^{pc}	φ^*	s_1^{-*}	s_2^{-*}	s_1^{+*}	s_2^{+*}	s_1^{oc}	s_2^{oc}
0	1	0	0.20	0	0.40	0	0.20	0.71	0.46	0.03	0	0	0	0
0.2	1.54	0.39	0	0	0.13	0	0	0.74	0.60	0.44	0	0	0	0
0.5	1.32	0.36	0	0	0	0.24	0	0.84	0.50	0	0	0.24	0	0
0.7	1.15	0.44	0	0	0.12	0.44	0	0.85	0.15	0	0	0.49	0.15	0

Table 6 Results of (16) and (21) for DMU4

	DMU4													
α	*Pessimistic*							*Optimistic*						
	φ^*	s_1^{-*}	s_2^{-*}	s_1^{+*}	s_2^{+*}	s_1^{pc}	s_2^{pc}	φ^*	s_1^{-*}	s_2^{-*}	s_1^{+*}	s_2^{+*}	s_1^{oc}	s_2^{oc}
0	1.69	0.56	0	0	0.56	0	0	0.65	0.37	0.40	0.33	0	0.03	0
0.2	1.10	0.85	0	0.55	0	0	0	0.70	0.30	0.36	0.27	0	0	0
0.5	1.05	0.54	0	0.35	0.50	0	0	0.79	0	0.26	0.17	0.24	0	0
0.7	1.01	0.33	0	0.20	0	0	0	0.86	0	0.17	0.11	0.31	0	0

Table 7 Results of (16) and (21) for DMU5

	DMU5													
α	*Pessimistic*							*Optimistic*						
	φ^*	s_1^{-*}	s_2^{-*}	s_1^{+*}	s_2^{+*}	s_1^{pc}	s_2^{pc}	φ^*	s_1^{-*}	s_2^{-*}	s_1^{+*}	s_2^{+*}	s_1^{oc}	s_2^{oc}
0	1	1.20	0.50	0	0	1.20	0.50	0.65	0.27	0.21	0	0.18	0.08	0
0.2	1.21	0.96	0.80	0.16	0.37	0	0	0.68	0.24	0.62	0	0.17	0	0
0.5	0.78	0.60	0	0	0.29	0	0.13	1.20	0.60	0.50	0.70	0.90	0	0
0.7	1.07	0.36	0.30	0.77	0.40	0.10	0.03	0.86	0.42	0	0	0.18	0	0

In summary, as shown in Tables 3–7, the values of pessimistic congestions are worse than the values of optimistic congestions. For example, for DMU 1 in Table 3, when $\alpha =$ 0, the values of pessimistic and optimistic congestions for the first input are 0.10 and 0.05, respectively. On the other hand, the values of pessimistic and optimistic congestions for the second input are zeros. As can be seen in Tables 3–7, the amount of pessimistic congestion for each input is worse than (or equal to) the amount of optimistic congestions for all cases.

5 Conclusions and further research directions

The observed values of the input and output data in real-world DEA problems are sometimes imprecise or vague. Imprecise evaluations may be the result of unquantifiable, incomplete and/or non-obtainable information. Researchers have proposed various strategies such as stochastic, interval, grey and fuzzy data for dealing with the impreciseness and ambiguity in DEA. Congestion has been an under-researched topic in the economic theory of production especially when there is a need for augmenting inputs to serve important objectives besides output maximisation. In this paper, we studied congestion in DEA and proposed a fuzzy DEA model that represented the imprecise and ambiguous input and output data with fuzzy numbers. We solved the model with an α-cut approach and obtained the value of input congestion for the optimistic and pessimistic cases by transforming the fuzzy DEA model into a crisp linear programming model. We solved two auxiliary crisp models and obtained optimistic and pessimistic values of congestion for evaluating the DMUs. We also used a numerical example from the literature to demonstrate For further research, we plan to investigate congestion from an optimistic and pessimistic viewpoint in the stochastic and grey DEA models.

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