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An integrated data envelopment analysis and mixed integer non-linear programming model for linearizing the common set of weights

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Abstract The problem of ranking efficient decision making units (DMUs) is of interest from both theoretical and practical points of view. In this paper, we propose an integrated data envelopment analysis and mixed integer non-linear programming (MINLP) model to find the most efficient DMU using a common set of weights. We linearize the MINLP model to an equivalent mixed integer linear programming (MILP) model by eliminating the non-linear constraints in which the products of variables are incorporated. The formulated MILP model is simpler and computationally more efficient. In addition, we introduce a model for finding the value of epsilon, since the improper choice of the non-Archimedean epsilon may result in infeasible conditions. We use a real-life facility layout problem to demonstrate the applicability and exhibit the efficacy of the proposed model.

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1 Introduction

Data Envelopment Analysis (DEA) is a mathematical programming technique for measuring the relative efficiency score of homogeneous decision making units (DMUs) with multiple inputs and outputs. Standard DEA models evaluate the relative efficiency of each DMU using a scalar measure ranging between zero and one. DEA assigns an efficiency score of one to efficient DMUs and an efficiency score lower than one to inefficient units. The efficiency scores of the inefficient DMUs are used to rank them accordingly. However, no ranking can be derived for the efficient DMUs with an efficiency score equal to one. For this reason, several methods have been developed to discriminate among the efficient DMUs.

Motivated by this multiplicity problem, Adler et al. (2002) conducted a review of the different ranking methods in DEA and divided them into six groups. The authors concluded that while these techniques were useful for specific applications, none of them could be considered as a comprehensive ranking methodology. As a result, several DMUs are generally found to be efficient, which weakens the capacity of DEA to identify the most efficient unit (Doyle and Green 1994; Foroughi 2013). Indeed, different efficient solutions can be derived from different DEA models, requiring subjective simulations to classify the alternatives into an overall group ranking (Ebrahimnejad et al. 2016). This multiplicity drawback becomes particularly important when considering, for example, the application of DEA to measure the environmental performance of DMUs on which to base policies designed to promote sustainable development (Ramilan et al. 2011).

Many DEA models assign optimal variable weights separately to each DMU under consideration, which is like the use of a variable set of optimal weights in some ranking methods. However, models based on a common set of weights have been recently proposed to evaluate all the DMUs with a set of weights under a sharing situation. In some cases, identifying only one DMU as the most efficient one is desirable. For this reason, an integrated model is implemented instead of solving at least one problem for each DMU (Lam 2015). Li and Reeves (1999), Amin et al. (2006), Ertay et al. (2006), Amin and Toloo (2007) , Amin (2009), Toloo and Nalchigar (2009), and Toloo (2012) have developed various methods for identifying the most efficient DMU. We must highlight the increasing importance gained by the DEA models based on a common set of weights in the literature (Sun et al. 2013; Wu et al. 2016). Indeed, these models have been recently extended to account for imprecise data and fuzzy environments (Azar et al. 2015).

The model defined in the current paper is based on that of Ertay et al. (2006), who introduced a minimax model to discriminate among the efficient units selected by the standard DEA model. The hybrid Analytic Hierarchy Process (AHP)–DEA model designed by Ertay et al. (2006) has been successfully applied in several research areas, including the measurement of airport efficiency (Lai et al. 2015), the evaluation of different lean tools (Anvari et al. 2014), and the selection of emerging technologies (Yu and Lee 2013), to cite just a few. Two different types of frictions can arise from the integration of the AHP within a DEA framework. On one hand, differences in the importance assigned to the priority weights of inputs and outputs determine the evaluation and ranking position of the DMUs (Pakkar 2015). On the other hand, multiple efficient alternatives can be obtained when solving the DEA model, requiring the subjective choice of a parameter k via trial and error to discriminate among them. We focus on this second type of friction.

In particular, the specific improvements of the DEA model with a common set of weights introduced in the current paper are sequentially defined as follows:

- Ertay et al. (2006): complete ranking with multiple efficient DMUs $\rightarrow$ trial and error parameter k and one linear programming problem per DMU $\Rightarrow$
- $\Rightarrow$ Amin and Toloo (2007): eliminate the k parameter and reduce the computational burden $\rightarrow$ provide a unique efficient DMU but not a complete ranking. Also, potential multiplicity of efficient DMUs $\Rightarrow$
- $\Rightarrow$ Amin (2009): complete ranking with a unique efficient DMU $\rightarrow$ nonlinear model $\Rightarrow$
- $\Rightarrow$ Current paper: generates a complete ranking with a unique efficient DMU and is linear.

That is, the linear DEA model introduced in the current paper provides a complete ranking of the DMUs and selects a unique efficient alternative without solving a linear programming problem per DMU or relying on a parameter (subjectively) defined by trial and error.

The remainder of this paper is organized as follows. Section 2 describes the motivation behind the model being introduced as well as its main contribution. In Sect. 3, we formulate an MILP model that is equivalent to the non-linear one of Amin (2009) under the constant returns-to-scale (CRS) assumption. Section 4 extends the proposed CRS approach into a variable returns-to-scale (VRS) one. In Sect. 5, we use a real-life problem with 19 facility layout alternatives to demonstrate the applicability and exhibit the efficacy of the proposed model. Section 6 concludes and suggests future research directions.

2 Motivation and contribution

The model constituting the base of the current research is that of Ertay et al. (2006), who introduced deviation from efficiency variables in a conventional DEA environment and proposed a minimax model that minimized the maximum of these variables. Given the multiple efficient solutions obtained, Ertay et al. (2006) modified the objective function by adding a constant parameter obtained by trial and error to reach a single relatively efficient DMU. To overcome the difficulty of (subjectively) determining the value of the parameter in Ertay's model, Amin and Toloo (2007) proposed the following integrated DEA model to identify the most efficient DMU:

$$\begin{aligned}
&\min d_{\max}\\
&s.t.\\
&d_{\max} - d_j \geq 0, \quad j = 1, \ldots, n\\
&\sum_{i=1}^{m} x_{ij} w_i \leq 1 \quad j = 1, \ldots, n\\
&\sum_{r=1}^{s} y_{rj} u_r - \sum_{i=1}^{m} x_{ij} w_i + d_j - \beta_j = 0, \quad j = 1, \ldots, n\\
&\sum_{j=1}^{n} d_j = n - 1\\
&0 \leq \beta_j \leq 1, d_j \in \{0, 1\}, \quad j = 1, \ldots, n\\
&w_i \geq \varepsilon^*, \quad i = 1, \ldots, m\\
&u_r \geq \varepsilon^*, \quad r = 1, \ldots, s
\end{aligned} \tag{1}$$

where $x_{ij}(i = 1, \ldots, m)$ and $y_{rj}(r = 1, \ldots, s)$ are respectively the input and output vectors of DMU_j, $(j = 1, \ldots, n)$, and ε^* is the maximum non-Archimedean epsilon.

Amin and Toloo (2007) claimed that the binary variable d_j represents the deviation from the efficiency of DMU_j and concluded that DMU_j is efficient if and only if $d_j^* = 0$. However, their evaluation of the DMUs was defined in terms of a binary variable, which prevents the definition of a ranking of alternatives in terms of their deviation from efficiency. Moreover, Amin (2009) showed that the application of Model (1) may lead to more than one efficient DMU and modified the model of Amin and Toloo (2007) as follows:

$$\begin{aligned}
&\min d_{\max}\\
&s.t.\\
&d_{\max} - d_j \geq 0, \quad j = 1, \ldots, n\\
&\sum_{i=1}^{m} x_{ij} w_i \leq 1, \quad j = 1, \ldots, n\\
&\sum_{r=1}^{s} y_{rj} u_r - \sum_{i=1}^{m} x_{ij} w_i + d_j = 0, \quad j = 1, \ldots, n\\
&\sum_{j=1}^{n} \theta_j = n - 1\\
&\theta_j - d_j \beta_j = 0, \quad j = 1, \ldots, n\\
&d_j \geq 0, \beta_j \geq 1, \theta_j \in \{0, 1\}, \quad j = 1, \ldots, n\\
&w_i \geq \varepsilon^*, \quad i = 1, \ldots, m\\
&u_r \geq \varepsilon^*, \quad r = 1, 2 \ldots, s
\end{aligned} \tag{2}$$

Table 1 Improvements upon the models considered

Model	Parameter k	Multiplicity	No ranking	Non linearity
Ertay et al. (2006)	•	•	✓	✓
Amin and Toloo (2007)	✓	•	•	✓
Amin (2009)	✓	✓	✓	•
Current paper	✓	✓	✓	✓

where the variable d_j represents the deviation from efficiency of DMU_j, implying that a unit is efficient if and only if $d_j^* = 0$. The variable β_j has been included in the model so as to generate a complete ranking of alternatives. It should be noted that Toloo (2014) analyzed the infeasibility of model (2) and formulated a model to provide a suitable value for the non-Archimedean epsilon ε^*.

Taking the constraint $\theta_j - d_j\beta_j = 0$ into account, if $\theta_j = 0$, then d_j has to take a zero value; otherwise, we should have $\beta_j = 0$, which violates the constraint $\beta_j \geq 1$. On the other hand, if $\theta_j = 1$, then, $d_j = \frac{\theta_j}{\beta_j} > 0$ (Amin 2009). Hence, the following property is imposed on the model:

$$\begin{cases} d_j = 0, & \text{if } \theta_j = 0 \\ d_j > 0, & \text{if } \theta_j = 1 \end{cases} \tag{3}$$

The property delivers a single efficient unit since there is a unique zero binary variable (θ) at each optimal solution. However, the product of variables defined in the fifth set of constraints $(\theta_j - d_j\beta_j = 0,\ j = 1, \ldots, n)$ implies that Model (2) is non-linear. In this paper, we linearize this model using mixed integer linear programming (MILP).

More precisely, Table 1 describes the main drawbacks exhibited by the models being considered and the improvements defined by the current one. Note that the current model can be interpreted as the result of a sequential improvement process based upon the model of Ertay et al. (2006). In this regard, it provides a complete ranking of the DMUs—with a unique efficient alternative—that does not rely on any subjective parametric judgment defined by trial and error. Moreover, it prevents the potential insolvability of the model that may arise due to the inclusion of nonlinear constraints.

3 Proposed approach

This section introduces a method that incorporates the products of variables in order to linearize the non-linear constraints in Model (2). In addition, a MILP model is proposed to calculate the infinitesimal non-Archimedean ε that appears in the linearized model. We begin with the following elementary property of Model (2).

Lemma 1 *In Model* (2), $\forall j, d_j \leq 1$.

Proof From the fifth set of constraints in Model (2) we have $\theta_j = d_j\beta_j$, $j = 1, \ldots n$, or equivalently, $d_j = \frac{\theta_j}{\beta_j}$. The fact that $\theta_j \in \{0, 1\}$ and $\beta_j \geq 1$ complete the proof. □

This lemma will be used in the proof of the following theorem, which deals with the linearization of Model (2).

Theorem 1 *Model (2) is equivalent to the following MILP model:*

$$\begin{aligned}
&\min d_{\max} \\
&s.t. \\
&d_{\max} - d_j \geq 0, \quad j = 1, \ldots, n \\
&\sum_{i=1}^{m} x_{ij} w_i \leq 1, \quad j = 1, \ldots, n \\
&\sum_{r=1}^{s} y_{rj} u_r - \sum_{i=1}^{m} x_{ij} w_i + d_j = 0, \quad j = 1, \ldots, n \\
&\sum_{j=1}^{n} \theta_j = n - 1 \\
&d_j \leq \theta_j \leq M d_j, \quad j = 1, \ldots, n \\
&d_j \geq 0, \theta_j \in \{0, 1\}, \quad j = 1, \ldots, n \\
&w_i \geq \varepsilon^*, \quad i = 1, \ldots, m \\
&u_r \geq \varepsilon^*, \quad r = 1, \ldots, s.
\end{aligned} \tag{4}$$

where M is a large enough positive number.

Proof From the fifth set of constraints, which cause Model (2) to be non-linear, we have $\theta_j = d_j\beta_j (j = 1, \ldots, n)$. Let M be a large number such that $\beta_j \leq$ M. More importantly, β_j appears in only one constraint, plus $0 \leq d_j$, $1 \leq \beta_j$. Then, multiplying $1 \leq \beta_j \leq$ M by d_j leads to $d_j \leq \theta_j \leq \mathrm{M}d_j$. As a result, adding the constraint $d_j \leq \theta_j \leq \mathrm{M}d_j$ is sufficient to eliminate the non-linear term $\theta_j - d_j\beta_j = 0$.

It now suffices to show that the optimal solution of Model (4) can be obtained directly from the optimal solution of Model (2) and vice versa. To do this, let us assume that $(d^*_{\max}, \mathbf{d}^*, \mathbf{u}^*, \mathbf{w}^*, \boldsymbol{\theta}^*)$ is an optimal solution of Model (4) where $\mathbf{d}^* = (d^*_1, \ldots, d^*_n)$, $\mathbf{u}^* = (u^*_1, \ldots, u^*_s)$, $\mathbf{w}^* = (w^*_1, \ldots, w^*_m)$, and $\boldsymbol{\theta}^* = (\theta^*_1, \ldots, \theta^*_n)$. Let:

$$\beta^*_j = \begin{cases} \frac{1}{d^*_j} & \text{if } d^*_j > 0 \\ 1 & \text{otherwise} \end{cases}.$$

Notice that Lemma 1 implies that $\beta^*_j \geq 1$. It is easy to verify that $(d^*_{\max}, \mathbf{d}^*, \mathbf{u}^*, \mathbf{w}^*, \boldsymbol{\theta}^*, \boldsymbol{\beta}^*)$ is an optimal solution of Model (2), where $\boldsymbol{\beta}^* = (\beta^*_1, \ldots, \beta^*_n)$. It is clear on inspection that the reverse is also true. □

We should note that Model (4) is an MILP model that is computationally more efficient than the MINLP Model (2). However, it has just been proved that Model (2) and Model (4) are equivalent. Model (4) can be independently analyzed as follows:

In the proposed MILP model, if $\theta_j^* = 0$, then the constraint $d_j \leq \theta_j$ implies that d_j^* is equal to zero. On the other hand, if $\theta_j^* = 1$, then the constraint $\theta_j \leq \mathrm{M}d_j$ forces d_j^* to take a positive value and the constraint $d_j \leq \theta_j$ becomes redundant. Hence, in Model (4) we have:

$$\begin{cases} d_j = 0, & \text{if } \theta_j = 0 \\ d_j > 0, & \text{if } \theta_j = 1 \end{cases}$$

Model (4) selects DMU_k as the most efficient unit if and only if $d_k^* = 0$. The constraint $\sum_{j=1}^{n} \theta_j = n - 1$ implies that the model can select a unique efficient DMU as the most efficient one.

Theorem 2 *Solving Model* (4) *provides a unique efficient DMU.*

Proof Assume that $(d_{\max}^*, \mathbf{d}^*, \mathbf{u}^*, \mathbf{w}^*, \boldsymbol{\theta}^*)$ is an optimal solution of Model (4). Let $\theta_k^* = 0$ and $\theta_j^* = 1, \forall j \neq k$, implying that $d_k^* = 0$, $d_j^* > 0, \forall j \neq k$ and $\mathbf{u}^* y_k - \mathbf{w}^* x_k = 0$. It is clear that taking the common set of optimal weights $(\mathbf{u}^*, \mathbf{w}^*)$ into consideration, DMU_k is an efficient unit and, in conclusion, there exists at least one efficient DMU. On the contrary, suppose that both DMU_k and DMU_l are efficient DMUs based on the underlying model. The third set of constraints implies that $d_k^* = d_l^* = 0$, and, therefore, $\theta_k^* = \theta_l^* = 0$, which is in contradiction with the fourth set of constraints**.** Consequently, DMU_k and DMU_l cannot be simultaneously efficient DMUs by the common set of optimal weights $(\mathbf{u}^*, \mathbf{w}^*)$, which completes the proof. □

We should emphasize that the value of ε^* plays a critical role in DEA models such as Model (4). In other words, an improper choice of ε^* can cause the model to be infeasible (see Amin and Toloo 2004). Foroughi (2011) showed that the value of ε that is obtained in the model of Amin and Toloo (2007) may cause infeasibility of Model (2). It therefore follows that using such an ε may lead to infeasible conditions in Model (4), since this model is equivalent to Model (2). As a result, we propose the following MILP model to compute the non-Archimedean ε in Model (4):

$$\begin{aligned}
&\max \varepsilon \\
&s.t. \\
&\sum_{i=1}^{m} x_{ij} w_i \leq 1, \quad j = 1, \ldots, n \\
&\sum_{r=1}^{s} y_{rj} u_r - \sum_{i=1}^{m} x_{ij} w_i + d_j = 0, \quad j = 1, \ldots, n \\
&\sum_{j=1}^{n} \theta_j = n - 1 \\
&d_j \leq \theta_j \leq M d_j, \quad j = 1, \ldots, n
\end{aligned}$$

$$
\begin{aligned}
& d_j \geq 0, \theta_j \in \{0, 1\}, \quad j = 1, \ldots, n \\
& w_i \geq \varepsilon, \quad i = 1, \ldots, m \\
& u_r \geq \varepsilon, \quad r = 1, \ldots, s.
\end{aligned} \tag{5}
$$

Theorem 3 *Model (5) is feasible.*

Proof Since in DEA models the efficiency score of each DMU is calculated relative to other DMUs, there clearly exists at least one efficient DMU. Moreover, we can find at least one extreme efficient DMU among the efficient DMUs (Cooper et al. 2007). Now, assume that DMU_k is an extreme efficient unit and $(\mathbf{u}^*, \mathbf{w}^*)$ is the vector of the corresponding multipliers, which is obtained by the conventional CCR model. If $\sum_{i=1}^{m} x_{ij} w_i^* \leq 1$, then the first set of constraints in Model (5) is satisfied. Otherwise, we let $t = \max_{j=1,\ldots,n} \left\{\sum_{i=1}^{m} x_{ij} w_i^*\right\}$ and define $(\mathbf{u}^*, \mathbf{w}^*) = (\mathbf{u}^*/t, \mathbf{w}^*/t)$. Using this new definition, $(\mathbf{u}^*, \mathbf{w}^*)$ satisfies the first set of constraints in Model (5). It also follows from $(\mathbf{u}^*, \mathbf{w}^*)$ that $\sum_{r=1}^{s} y_{rk} u_r^* - \sum_{i=1}^{m} x_{ik} w_i^* = 0$ and $\sum_{r=1}^{s} y_{rj} u_r^* - \sum_{i=1}^{m} x_{ij} w_i^* \leq 0, \forall j \neq k$. Hence, the second set of constraints is also satisfied. Clearly, if we set $\theta_k = 0$ and $\theta_j = 1, \forall j \neq k$, the constraint $\sum_{j=1}^{n} \theta_j = n - 1$ is met. Finally, if DMU_k is an extreme efficient unit, then we must have that $\theta_k = d_k = 0$, while $\sum_{r=1}^{s} y_{rj} u_r^* + d_j = \sum_{i=1}^{m} x_{ij} w_i^* \leq 1$ for the other DMUs, which satisfies the fourth set of constraints. Thus, by defining $\varepsilon = \min_{r,i}\{u_r^*, w_i^*\}$, a feasible solution of Model (5) is obtained. This completes the proof. □

We have just proved that Model (5) is feasible. At the same time, if we compare the constraints of Models (4) and (5), we can easily discern that the constraints of these two models are identical except for the first set in Model (4). The following corollary can be obtained directly from Theorem 3:

Corollary 1 *Model (4) is feasible.*

4 Constant versus variable returns to scale

The models have been defined imposing a CRS assumption such that an increase in inputs results in a proportionate increase in the output levels. However, in some real-world problems this assumption may not hold, with an increase in inputs not necessarily leading to a proportional change in outputs. A VRS model takes this possibility into consideration. In order to introduce the VRS assumption into the multiplier form of the CRS framework, a free variable must be added to the model. As a result, we formulate the following MINLP model under the VRS assumption, which constitutes an extension of Amin's (2009) model:

$$
\begin{array}{ll}
\min d_{max} & \\
\text{s.t.} & \\
\sum_{i=1}^{m} v_i x_{ij} \le 1 & j = 1, \ldots, n \\
\sum_{r=1}^{s} u_r y_{rj} - u_0 - \sum_{i=1}^{m} v_i x_{ij} + d_j = 0 & j = 1, \ldots, n \\
d_{max} - d_j \ge 0 & j = 1, \ldots, n \\
\sum_{j=1}^{n} \theta_j = n - 1 & \\
\theta_j - d_j \beta_j = 0 & j = 1, \ldots, n \\
\theta_j \in \{0, 1\} & j = 1, \ldots, n \\
\beta_j \ge 1 & j = 1, \ldots, n \\
d_j \ge 0 & j = 1, \ldots, n \\
u_r \ge \varepsilon^* & r = 1, \ldots, s \\
v_i \ge \varepsilon^* & i = 1, \ldots, m \\
u_0 \text{ free in sign} &
\end{array} \tag{6}
$$

where u_0 is a free variable. Analogously, the model of Toloo (2014) can be reformulated in the same way as has been done with model (5) in order to find a suitable ε^* for model (6) under the VRS assumption:

$$
\begin{array}{l}
\max \varepsilon \\
s.t. \\
\vdots \\
w_i \ge \varepsilon, i = 1, \ldots, m \\
u_r \ge \varepsilon, r = 1, \ldots, s.
\end{array} \tag{7}
$$

The following properties follow unambiguously from Toloo (2014):

1. Model (7) is always feasible.
2. $0 < \varepsilon^* < +\infty$.
3. Model (6) is always feasible under the non-Archimedean value obtained from model (7).
4. Both models (6) and (7) are solvable (see Toloo 2014, pg. 5339).

A similar analysis to the one performed in Sect. 3 can be applied to verify that the following MILP model is equivalent to the MINLP model (6) under the VRS assumption:

$$
\begin{array}{ll}
\min d_{max} & \\
\text{s.t.} & \\
\sum_{i=1}^{m} v_i x_{ij} \leq 1 & j = 1, ..., n \\
\sum_{r=1}^{s} u_r y_{rj} - u_0 - \sum_{i=1}^{m} v_i x_{ij} + d_j = 0 & j = 1, ..., n \\
d_{max} - d_j \geq 0 & j = 1, ..., n \\
\sum_{j=1}^{n} \theta_j = n - 1 & \\
d_j \leq M\theta_j & j = 1, ..., n \\
\theta_j \leq M d_j & j = 1, ..., n \\
\theta_j \in \{0, 1\} & j = 1, ..., n \\
d_j \geq 0 & j = 1, ..., n \\
u_r \geq \varepsilon^* & r = 1, ..., s \\
v_i \geq \varepsilon^* & i = 1, ..., m \\
u_0 \text{ free in sign} &
\end{array}
\tag{8}
$$

Similar to model (5), a model can be formulated with the aim of determining a suitable epsilon value for model (8). In addition, the feasibility of models (7) and (8) can be addressed using theorems analogous to those introduced in Sect. 3.

5 Application

In this section, we demonstrate the equivalence relationship between Model (2) and Model (4) using a real data set originally provided by Ertay et al. (2006) that involves 19 facility layout alternatives.[1] Each facility layout uses two inputs (cost and adjacency score) to produce four outputs: shape ratio, flexibility, quality, and hand-carry utility. The input and output data are presented in Table 2.

To identify the most efficient DMU, we first exploit Model (5) and find the value of ε^*. Then, we consider $\varepsilon^* = 0.000026$ as the optimal objective value of Model (5) and apply Model (4) to the data set of Table 2. The optimal solution is described in Table 3. The following results are also obtained by solving Model (4):

$$
\theta_j^* = \begin{cases} 0 & j = 14 \\ 1 & otherwise \end{cases}, \begin{cases} d_j = 0, & j = 14 \\ d_j > 0, & otherwise \end{cases}
$$

Hence, Model (4) defines a common set of optimal weights (u^*, w^*) that identify DMU_{14} as the single efficient alternative. Note that if we let $\beta_{14}^* = 1$ and $\beta_j^* = \left(d_j^*\right)^{-1}, \forall j \neq 14$, then $(d_{\max}^*, \mathbf{d}^*, \mathbf{u}^*, \mathbf{w}^*, \boldsymbol{\theta}^*, \boldsymbol{\beta}^*)$ is the optimal solution of Model (2), which emphasizes the validity of Theorem 1.

At the same time, since the model of Ertay et al. (2006) delivers a complete ranking of the DMUs based on their efficiency scores, we can now compare their results with those obtained using the current model. As already stated, one of the main advantages of the current model is the elimination of the trial and error choice of the parameter k in the objective function. In addition, its lower computational burden when compared

[1] A similar demonstration can be provided for models (6) and (8) under the VRS assumption.

Table 2 Inputs and outputs of the 19 facility layouts

lDMUs	DEA inputs		DEA outputs			
	Cost $	Adjacency score	Shape ratio	Flexibility	Quality	Hand-carry utility
1	20309.56	6405	0.4697	0.0113	0.0410	30.89
2	20411.22	5393	0.4380	0.0337	0.0484	31.34
3	20280.28	5294	0.4392	0.0308	0.0653	30.26
4	20053.20	4450	0.3776	0.0245	0.0638	28.03
5	19998.75	4370	0.3526	0.0856	0.0484	25.43
6	20193.68	4393	0.3674	0.0717	0.0361	29.11
7	19779.73	2862	0.2854	0.0245	0.0846	25.29
8	19831.00	5473	0.4398	0.0113	0.0125	24.80
9	19608.43	5161	0.2868	0.0674	0.0724	24.45
10	20038.10	6078	0.6624	0.0856	0.0653	26.45
11	20330.68	4516	0.3437	0.0856	0.0638	29.46
12	20155.09	3702	0.3526	0.0856	0.0846	28.07
13	19641.86	5726	0.2690	0.0337	0.0361	24.58
14	20575.67	4639	0.3441	0.0856	0.0638	32.20
15	20687.50	5646	0.4326	0.0337	0.0452	33.21
16	20779.75	5507	0.3312	0.0856	0.0653	33.60
17	19853.38	3912	0.2847	0.0245	0.0638	31.29
18	19853.38	5974	0.4398	0.0337	0.0179	25.12
19	20355.00	17402	0.4421	0.0856	0.0217	30.02

Table 3 The optimal solutions of model (4) for the data of Table 2

w_1^*	w_2^*	u_1^*	u_2^*	u_3^*	u_4^*	$d_{\max}^*$
0.000027	0.000026	0.238251	3.392240	0.000026	0.009371	0.32298

to that of Ertay et al. (2006) has been emphasized by Amin and Toloo (2007), whose model finds the most efficient DMU without having to solve a linear programming problem per DMU.

More importantly, the subjectivity inherent to the trial and error choice of the parameter k implies that the results obtained will differ from ours, particularly when defining a complete ranking of alternatives. In this regard, it should be emphasized that even though a decision maker may be initially interested in the most efficient DMU, a relative evaluation of the other alternatives could be required if an additional DMU must be considered or potential improvements suggested to a given subset of them.

The rankings obtained using the standard CCR method and that of Ertay et al. (2006) for three different values of the parameter k are presented in Table 4. It should be emphasized that Cook et al. (1996) suggested the use of a maximum non-Archimedean

Table 4 Efficiency scores of Ertay et al. (2006) for different ε and k values

DMUs	$\varepsilon^* = 0$			$\varepsilon^* = 0.000026$		
	CCR	Minimax ($k = 0$)	Minimax ($k = 0.3$)	CCR	Minimax ($k = 0$)	Minimax ($k = 0.4$)
1	0.985	0.952	0.932	0.966	0.741	0.741
2	0.988	0.959	0.952	0.988	0.837	0.837
3	0.997	0.933	0.926	0.995	0.817	0.817
4	0.949	0.872	0.872	0.949	0.765	0.765
5	1	0.794	0.794	0.989	0.905	0.868
6	0.973	0.897	0.897	0.973	0.929	0.929
7	1	0.794	0.793	1	0.732	0.732
8	0.857	0.787	0.776	0.857	0.668	0.668
9	0.889	0.775	0.775	0.853	0.799	0.799
10	1	0.847	0.811	1	1	0.854
11	0.998	0.900	0.900	0.985	0.962	0.947
12	1	0.868	0.868	1	0.966	0.944
13	0.776	0.776	0.776	0.764	0.672	0.672
14	1	0.970	0.970	1	1	1
15	1	1	0.994	1	0.847	0.847
16	1	1	1	1	0.988	0.988
17	1	0.973	0.924	1	0.818	0.818
18	0.852	0.796	0.785	0.845	0.733	0.733
19	1	0.806	0.806	0.671	0.671	0.671

ε value that helps discriminating among the DMUs. However, Ertay et al. (2006) chose $\varepsilon^* = 0$ as the lower bound for all the weights, which can lead to incorrect results, as illustrated in the left section of Table 4.

In particular, DMU_{16} is selected as the most efficient unit, though some of the weights obtained are equal to zero. On the other hand, their model selects DMU14 as the most efficient unit, unlike our model, which uses the optimal value of "epsilon" and allows us to consider all input/output weights. With this modification in mind, we will focus our analysis on the differences arising across the three main rankings obtained. The leftward half of Table 5 compares the efficiency scores obtained by applying the current model and the one of Ertay et al. (2006) with ($\varepsilon^* = 0, k = 0.3$) and ($\varepsilon^* = 0.000026, k = 0.4$), so that a unique optimum is selected. Moreover, in order to provide further insight, we have applied two additional and well-known ranking methods to the data set, i.e. the cross efficiency and super efficiency models developed by Sexton et al. (1986) and Andersen and Petersen (1993), respectively.

Cross efficiency uses the optimal set of weights chosen for a particular DMU to weight the inputs and outputs of each of the other DMUs. The super efficiency model is formulated by modifying the CCR one and excluding the DMU under evaluation from the reference set. Cross efficiency scores provide a peer-evaluation across DMUs, while super efficiency scores focus on self-evaluation. Thus, these

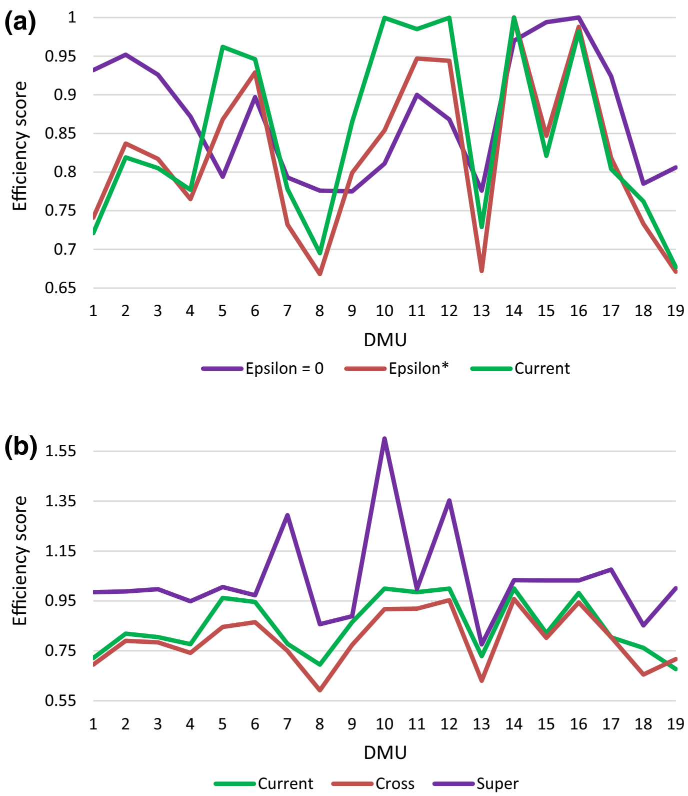

Fig. 1 Comparison of efficiency scores across models. **a** Efficiency scores of Ertay et al. (2006) and the current model. *Notation* Epsilon = 0 refers to $\varepsilon^* = 0$ while Epsilon* corresponds to $\varepsilon^* = 0.000026$. **b** Efficiency scores of the current, cross and super efficiency models

methods have been included to have both a peer- and self-evaluation benchmark for comparison.

Note once again that Ertay et al. (2006) deliver two potential efficient solutions, i.e. DMU_{14} and DMU_{16}, while the analysis performed in the current paper identifies DMU_{14} as the most efficient one. In addition, DMU_{14}, with an efficiency score of 0.958, is at the top of the ranking when applying the cross-efficiency method, while super efficiency ranks DMU_{10} as the most efficient unit with an efficiency score of 1.601. The difference between both rankings is understandable since, similarly to the common weights approach, the cross-efficiency method runs a peer-evaluation across DMUs, while the super efficiency method focuses on the self-evaluation of the DMUs, as is the case with traditional DEA models.

Figure 1 illustrates the efficiency scores obtained by the different models, which allows us to observe the similarity between the current approach and those of Ertay et al. (2006) with ($\varepsilon^* = 0.000026$, $k = 0.4$) and the cross efficiency one. The analysis of the correlation among the scores assigned by the different ranking methods

Table 5 Comparing efficiency scores and rankings: Ertay et al. (2006) versus the current, cross and super efficiency models

Method	Efficiency scores					Rankings				
	Ertay et al. (2006)		Current	Cross efficiency	Super efficiency	Ertay et al. (2006)		Current	Cross efficiency	Super efficiency
DMU	$\varepsilon^* = 0\ (k = 0.3)$	$\varepsilon^* = 0.000026\ (k = 0.4)$	$1 - d^*$			$\varepsilon^* = 0\ (k = 0.3)$	$\varepsilon^* = 0.000026.\ (k = 0.4)$.	$1 - d^*$		
1	0.932	0.741	0.721	0.695	0.985	5	14	17	16	13
2	0.952	0.837	0.819	0.790	0.988	4	9	10	10	12
3	0.926	0.817	0.805	0.784	0.997	6	11	11	11	11
4	0.872	0.765	0.777	0.742	0.949	10	13	14	14	15
5	0.794	0.868	0.962	0.846	1.006	14	6	6	7	8
6	0.897	0.929	0.946	0.865	0.973	9	5	7	6	14
7	0.793	0.732	0.778	0.750	1.294	15	16	13	13	3
8	0.776	0.668	0.695	0.592	0.857	17	19	18	19	17
9	0.775	0.799	0.865	0.774	0.889	19	12	8	12	16
10	0.811	0.854	0.9996	0.917	1.601	12	7	3	5	1
11	0.900	0.947	0.985	0.919	0.998	8	3	4	4	10
12	0.868	0.944	0.9997	0.953	1.353	11	4	2	2	2
13	0.776	0.672	0.729	0.630	0.776	17	17	16	18	19
14	0.970	1	1	0.958	1.033	3	1	1	1	5
15	0.994	0.847	0.821	0.802	1.032	2	8	9	9	6
16	1	0.988	0.982	0.944	1.032	1	2	5	3	7
17	0.924	0.818	0.804	0.805	1.076	7	10	12	8	4
18	0.785	0.733	0.762	0.655	0.852	16	15	15	17	18
19	0.806	0.671	0.677	0.717	1.001	13	18	19	15	9

Table 6 Correlation of efficiency scores: Ertay et al. (2006) and the current, cross and super efficiency models

		$\varepsilon^* = 0$	$\varepsilon^* = 0.000026$	Current	Cross	Super
$\varepsilon^* = 0$	Pearson correlation	1	0.622**	0.309	.510*	0.037
	Sig. (2-tailed)		0.004	0.197	0.026	0.881
	N	19	19	19	19	19
$\varepsilon^* = 0.000026$	Pearson correlation	0.622**	1	0.922**	.941**	0.327
	Sig. (2-tailed)	0.004		0.000	0.000	0.172
	N	19	19	19	19	19
Current	Pearson correlation	0.309	0.922**	1	0.932**	0.486*
	Sig. (2-tailed)	0.197	0.000		0.000	0.035
	N	19	19	19	19	19
Cross	Pearson correlation	0.510*	0.941**	0.932**	1	0.567*
	Sig. (2-tailed)	0.026	0.000	0.000		0.011
	N	19	19	19	19	19
Super	Pearson correlation	0.037	0.327	0.486*	0.567*	1
	Sig. (2-tailed)	0.881	0.172	0.035	0.011	
	N	19	19	19	19	19

**Correlation is significant at the 0.01 level (2-tailed)
*Correlation is significant at the 0.05 level (2-tailed)

validates this conclusion, as can inferred from Table 6, which presents the Pearson correlation coefficients, the corresponding (2-tailed) significance values, and the number of observations analyzed per model (N). The significance level of a correlation coefficient (or p-value) accounts for the probability of concluding that a correlation exists when it does not. Note, in particular, that the choice of the optimal value of ε^* gives rise to a significant difference between the rankings of Ertay et al. (2006), leading $\varepsilon^* = 0.000026$ to a much higher correlation with the current model in the assignment of efficiency scores than the alternative $\varepsilon^* = 0$ framework.

The same conclusion applies to the ranking positions assigned to each DMU. The rightward half of Table 5 describes the different rankings derived from each model using the efficiency scores provided in its leftward half. As can be directly observed, the method of Ertay et al. (2006) with $\varepsilon^* = 0.000026$, the current model, and the cross-efficiency approach all coincide in the most efficient DMU. However, differences occur already when considering the second best DMU, and the overall rankings are similar, but differ to a certain extent. These differences are exacerbated when examining the rankings obtained from Ertay et al. (2006) with $\varepsilon^* = 0$ and the super efficiency method, both selecting different best DMUs.

Figure 2 illustrates the different ranked derived from each model, as well as a similar trend followed by the current approach, that of Ertay et al. (2006) with "epsilon" =0.000026 and cross efficiency. As before, we validate the graphical intu-

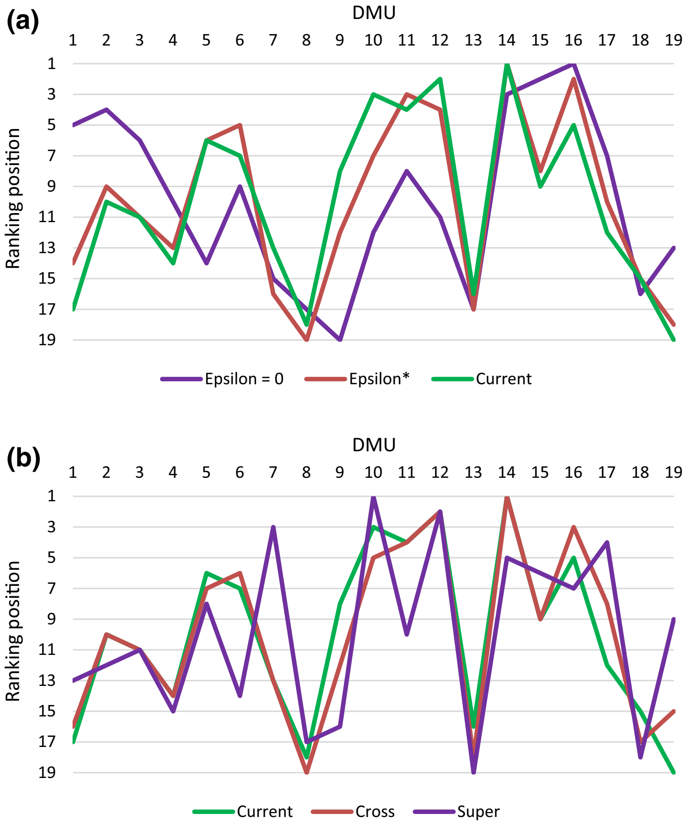

Fig. 2 Ranking comparison across models. **a** Rankings obtained by Ertay et al. (2006) and the current model. *Notation* Epsilon = 0 refers to $\varepsilon^* = 0$ while Epsilon* corresponds to $\varepsilon^* = 0.000026$. **b** Rankings obtained by the current, cross and super efficiency models

ition by analyzing the correlation among the different rankings. The corresponding values can be found in Table 7, which shows a considerably high (and significant) correlation between the ($\varepsilon^* = 0.000026, k = 0.4$) model and the current and cross efficiency ones. Note, however, that both rankings differ already in the second alternative, which constitutes a problem if a decision maker wants to consider which DMU should be potentially improved.

The reliance of the model of Ertay et al. (2006) on a subjective parameter validated via trial and error suffices to select a single most efficient DMU. However, the current model is not subject to the constraint of choosing a subjective parameter to determine the resulting ranking and perform comparisons among the DMUs.

Table 7 Ranking correlations: Ertay et al. (2006) and the current, cross and super efficiency models

		$\varepsilon^* = 0$	$\varepsilon^* = 0.000026$	Current	Cross	Super
$\varepsilon^* = 0$	Pearson correlation	1	0.601**	0.356	0.535*	0.389
	Sig. (2-tailed)		0.007	0.135	0.018	0.099
	N	19	19	19	19	19
$\varepsilon^* = 0.000026$	Pearson correlation	0.601**	1	0.932**	0.960**	0.516*
	Sig. (2-tailed)	0.007		0.000	0.000	0.024
	N	19	19	19	19	19
Current	Pearson correlation	0.356	0.932**	1	0.940**	0.570*
	Sig. (2-tailed)	0.135	0.000		0.000	0.011
	N	19	19	19	19	19
Cross	Pearson correlation	0.535*	0.960**	0.940**	1	0.696**
	Sig. (2-tailed)	0.018	0.000	0.000		0.001
	N	19	19	19	19	19
Super	Pearson correlation	0.389	.516*	0.570*	0.696**	1
	Sig. (2-tailed)	0.099	0.024	0.011	0.001	
	N	19	19	19	19	19

**Correlation is significant at the 0.01 level (2-tailed)
*Correlation is significant at the 0.05 level (2-tailed)

6 Conclusion

We conclude by emphasizing that non-linear programming models are very difficult (if not impossible) to solve. We have linearized the MINLP model of Amin (2009) by substituting the non-linear constraint with a linear one. Just as the model of Amin (2009), the proposed model uses a common set of weights to identify a unique DMU as the most efficient one. However, the new formulated MILP model is simpler and more practical than Amin's (2009) MINLP model.

We have also illustrated how the efficiency scores assigned by Ertay et al. (2006) are like those of the current model, which improves upon the former one, while being independent of the subjectively chosen parameter k and bearing a lower computational burden. In this regard, the proposed approach can be extended to other DEA models by adjusting the corresponding assumptions and constraints. Moreover, the development of novel approaches that take imprecise data sets into consideration are suggested as future potential research directions.

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