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ORIGINAL PAPER

A novel inverse data envelopment analysis model with negative ratio data

Mehdi Soltanifar[1] · Madjid Tavana[2,3] · Vincent Charles[4,5] · Mojtaba Ghiyasi[6] · Hamid Sharafi[7]

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Abstract
Data envelopment analysis (DEA) is a mathematical programming method for evaluating the efficiency of a homogeneous set of decision-making units (DMUs) using multiple inputs and outputs. Inverse DEA estimates a DMU's input (or output) when some or all DMU outputs (or inputs) are changed. Ratio DEA (DEA-R) combines DEA with ratio analysis to handle ratio data. Real-world DEA-R models often involve negative values for the inputs or outputs. This study presents a novel model for solving inverse DEA problems with negative ratio data for the first time. We present a real-life case study to demonstrate the applicability and efficacy of the DEA models proposed in this study.

Keyword Data envelopment analysis · Inverse model · Ratio model · Negative data

1 Introduction

Data envelopment analysis (DEA) is a mathematical programming approach for measuring the relative efficiency of a set of homogenous decision-making units (DMUs). The origin of this approach in the current mathematical form goes back to Charnes et al. (1978), which relayed the constant returns-to-scale (CRS) assumption (see also Ghazi and Hosseinzadeh Lotfi 2023). Banker et al. (1984) extended this model to the case of variable returns-to-scale (VRS). Extensive theoretical developments and applications of this approach in different sectors have emerged over the past 40 years (see Emrouznejad & Yang 2018.)

1.1 Inverse DEA

Inverse DEA, as a branch of the classical DEA model, emerged in its current form through the work of Zhang and Cui (1999) and Wei et al. (2000a, b). The inverse

Extended author information available on the last page of the article

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DEA model addresses the following questions: If we increase specific inputs to a particular unit within a group of DMUs while assuming that the DMU maintains its current efficiency level compared to other units, how much more output can the unit produce? Alternatively, if the outputs need to be increased to a certain level and the unit's efficiency remains unchanged, how much more input should be allocated? Multi-objective linear programming (MOLP) is employed in the inverse DEA literature to provide answers to these questions.

Hadi-Vencheh and Foroughi (2006) extended the work of Wei et al. (2000a, b) by allowing arbitrary changes in input and output levels. They did not consider any restriction on changing input–output levels. Lertworasirikul et al. (2011) proposed an inverse DEA model assuming VRS. This model is called the inverse BBC model, following the fact that the relative DEA model with VRS properties is known as the BCC model introduced by Banker et al. (1984). Ghiyasi (2015) pointed out some drawbacks of Lertworasirikul et al. (2011) and revised the VRS case of the inverse DEA with a more straightforward proof based on the characteristics of production technology. Ghobadi and Jahangiri (2015) reviewed the inverse DEA model and its application. Ghiyasi (2017) incorporated the price information in input–output estimation. This yielded inverse DEA models based on cost and revenue efficiency. Ghiyasi (2019) considered the criteria models in the inverse DEA problem and proposed new models for this case. Ghobadi et al. (2023) proposed a novel model for unit restructuring in the presence of negative data.

Lu et al. (2022) introduced an inverse DEA framework with frontier change to study the achievement path of the CO_2 emissions target of China. Younesi et al. (2023) proposed an input estimation procedure using the slack-based inverse DEA models while some data are integer intervals. Sayar et al. (2023) proposed an inverse 1-median problem dealing with weights on the tree and their vertices transformation. Lin and Lu (2024) used inverse DEA as a performance analysis method and a machine learning algorithm as a prediction method for analyzing the suppliers. Ghiyasi (2024) incorporated the efficiency measure in the time substitution problem via an inverse DEA framework that yields a more realistic estimation of inputs and output and optimal allocation time. Jahani et al. (2024) dealt with the flexible measures in the inverse DEA models. Recently, Emrouznejad et al. (2023) reviewed the inverse DEA from its origin through theoretical developments and applications and then suggested future research lines.

1.2 DEA-R

Despić et al. (2007) proposed ratio DEA (DEA-R) models to handle performance evaluation problems with ratio data since traditional DEA models cannot use ratio data in the evaluation process. DEA-R models are primarily used to combine DEA with ratio analysis Mozaffari et al. (2014a); Mozaffari et al. (2014b) and Wei et al. (2011). In evaluating the performance of some organizations, some financial data are ratio-based (e.g., quick ratio, return on investments). Traditional DEA models cannot process ratio data to evaluate the DMUs in such cases. To solve this problem, Despic et al. (2007) proposed DEA-R models by combining DEA and ratio analysis, and since then, these models have been studied and used by many other researchers (Mozaffari et al. 2014a, 2014b; Wei et al. 2011; and Tohidnia and Tohidi, 2019). Ghiyasi et al. (2022) proposed inverse DEA models for the ratio data case.

1.3 Negative data in DEA

Despite these developments, Paradi and Zhu (2013) warned that no reliable DEA model can effectively handle problems with both positive and negative data. Portela et al. (2004) suggested handling negative and positive data by considering unrestricted data. Sharp et al. (2007) proposed a modified slack-based measure to deal with the negative inputs and outputs. Portela and Thanassoulis (2010) proposed a directional distance approach for handling negative data in DEA. Kerstens and Van de Woestyne (2011) developed a proportional distance function. Cheng et al. (2013) proposed a radial measure for inputs and outputs. Non-radial models for evaluating DMU performance with negative data can also be found in Soltanifar and Sharafi (2021) and Tavana et al. (2021). Emrouznejad et al. (2010) introduced a semi-oriented radial measure (SORM) for handling negative and positive data. Matin et al. (2014) modified the SORM model by removing some problematic target-setting issues in SORM. Kaffash et al. (2018) proposed the directional SORM model to handle negative data in DEA. This model is based on a directional distance function to find a relevant direction capable of dealing with both positive and negative data. Ghiyasi and Zhu (2020) dealt with the negative and positive data in the inverse models. We encounter negative ratio data in many real-world applications, including our case study. Unfortunately, the inverse DEA literature lacks theoretical advancements and appropriate models to address such cases.

1.4 Motivation and contribution

Banker et al. (1984) laid the foundational groundwork for DEA by introducing the concept of VRS into the model. This innovation opened up new possibilities for handling various data types, including ratio data and cases where inputs or outputs assume negative values. Our study extends the DEA framework to tackle inverse DEA problems with negative ratio data. This challenge has not been addressed in the DEA literature until now. Our models not only accommodate the complexities of real-world performance evaluation problems but also introduce the concept of inverse DEA-R models under similar conditions. By adapting the assumption of VRS to scenarios featuring negative inputs or outputs, our research provides a more comprehensive and flexible approach to performance evaluation. In this way, we continue to build on the pioneering work of Banker et al. (1984) to enhance the applicability of DEA in contemporary contexts. Based on the authors' best knowledge, there is no methodology or theoretical development for dealing with negative or positive inputs or outputs within the inverse DEA-R literature. Thus, the current paper addressed this methodological gap from a theoretical standpoint. The theoretical contributions of this study to the literature are twofold. We propose (1) DEA-R models capable of handling both positive and negative data and (2) inverse DEA-R models for estimating inputs and outputs when there are changes in both output and input levels involving both positive and negative data. The practical contribution of this study lies in the real-world application of the proposed model in healthcare management, where we encountered negative ratio data.

The remainder of the paper is organized as follows: Sect. 2 reviews the basic 2.DEA-R models. Section 3 introduces the DEA-R models for handling negative data. Section 4 proposes inverse DEA-R models for dealing with negative data. Section 5 presents an application in healthcare management to demonstrate the applicability and efficacy of the models proposed in this study. Section 6 concludes with our final thoughts and outlines future research directions.

2 DEA-R models

Consider an m-dimension positive input vector of $(x_{1j}, x_{2j}, ..., x_{mj}) > 0, 1 \leq j \leq n$ for DMU_j, $1 \leq j \leq n$ that is used for producing s-dimension positive output vector of $(y_{1j}, y_{2j}, ..., y_{sj}) > 0$, $1 \leq j \leq n$. Despić et al. (2007) and Despić et al. (2007) introduced the following max–min models for DMU_p, $1 \leq p \leq n$ assuming CRS, called the DEA-R efficiency (i, $(1 \leq i \leq m)$ is the index of inputs, r, $(1 \leq r \leq s)$ is the index of outputs and j, $(1 \leq j \leq n)$ is the index of DMUs.)

$$e_p = \max_{\sum_{r=1}^{s}\sum_{i=1}^{m} w_{ri}=1, w_{ri}\geq 0} \min_{1\leq j\leq n} \left(\sum_{r=1}^{s}\sum_{i=1}^{m} w_{ri} \frac{x_{ij}}{x_{ip}} \frac{y_{rp}}{y_{rj}} \right); \ 1 \leq p \leq n \tag{1}$$

$$e'_p = \max_{\sum_{r=1}^{s}\sum_{i=1}^{m} w_{ri}=1, w_{ri}\geq 0} \min_{1\leq j\leq n} \frac{1}{\left(\sum_{r=1}^{s}\sum_{i=1}^{m} w_{ri} \frac{y_{rj}}{y_{rp}} \frac{x_{ip}}{x_{ij}} \right)}; 1 \leq p \leq n \tag{2}$$

Models (3) and (4) give the DEA-R max–min models for DMU_p assuming VRS.

$$\bar{e}_p = \max_{\sum_{i=1}^{m} w_{i0} + \sum_{r=1}^{s}\sum_{i=1}^{m} w_{ri}=1, w_{ri}\geq 0} \min_{1\leq j\leq n} \frac{1}{\left(\sum_{i=1}^{m} w_{i0} \frac{x_{ip}}{x_{ij}} + \sum_{r=1}^{s}\sum_{i=1}^{m} w_{ri} \frac{y_{rj}}{y_{rp}} \frac{x_{ip}}{x_{ij}} \right)}; \ 1 \leq p \leq n \tag{3}$$

$$\bar{e}'_p = \max_{\sum_{r=1}^{s} w_{r0} + \sum_{r=1}^{s}\sum_{i=1}^{m} w_{ri}=1, w_{ri}\geq 0} \min_{1\leq j\leq n} \left(\sum_{r=1}^{s} w_{r0} \frac{y_{rp}}{y_{rj}} + \sum_{r=1}^{s}\sum_{i=1}^{m} w_{ri} \frac{x_{ij}}{x_{ip}} \frac{y_{rp}}{y_{rj}} \right); 1 \leq p \leq n \tag{4}$$

Models (1) to (4) are presented as a means to convey the idea introduced by Despić et al. (2007) (see also Despić (2013) and Hosseinzadeh Lotfi et al. (2023)). Similarly, by discussing the sign of variables w_{i0}, $(\forall i)$ and w_{r0}, $(\forall r)$, the BCC-CCR and CCR-BCC versions of the above models can be presented. Models (3) and (4) can be converted into linear programming models (5) and (6), respectively. To transform Model (3) into Model (5), it suffices to apply the variable substitution

$\frac{1}{\varphi} = \min_{1\leq j\leq n} \frac{1}{\left(\sum_{i=1}^{m} w_{i0}\frac{x_{ip}}{x_{ij}} + \sum_{r=1}^{s}\sum_{i=1}^{m} w_{ri}\frac{y_{rj}}{y_{rp}}\frac{x_{ip}}{x_{ij}}\right)}$. Thus, maximizing $\frac{1}{\varphi}$ becomes equivalent to minimizing φ. The first set of constraints in model (5) directly results from the variable substitution, while the other constraints are directly derived from model (3). A similar explanation applies to the process of converting model (4) into model (6).

$$
\begin{aligned}
&\overline{e}_p = \min \ \varphi; \ \ 1 \leq p \leq n \\
&s.t. \\
&\sum_{i=1}^{m} w_{i0}\frac{x_{ip}}{x_{ij}} + \sum_{r=1}^{s}\sum_{i=1}^{m} w_{ri}\left(\frac{\frac{y_{rj}}{x_{ij}}}{\frac{y_{rp}}{x_{ip}}}\right) \leq \varphi, \ \ 1 \leq j \leq n \\
&\sum_{i=1}^{m} w_{i0} + \sum_{r=1}^{s}\sum_{i=1}^{m} w_{ri} = 1 \\
&w_{ri} \geq 0, 1 \leq r \leq s, 1 \leq i \leq m \\
&w_{i0} \ \ is \ \ free, \ 1 \leq i \leq m.
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
&\overline{e}'_p = \max \ \varphi; \ \ 1 \leq p \leq n \\
&s.t. \\
&\sum_{r=1}^{s} w_{r0}\frac{y_{rp}}{y_{rj}} + \sum_{r=1}^{s}\sum_{i=1}^{m} w_{ri}\left(\frac{\frac{x_{ij}}{y_{rj}}}{\frac{x_{ip}}{y_{rp}}}\right) \geq \varphi, 1 \leq j \leq n \\
&\sum_{r=1}^{s} w_{r0} + \sum_{r=1}^{s}\sum_{i=1}^{m} w_{ri} = 1 \\
&w_{ri} \geq 0, \ \ 1 \leq r \leq s, \ 1 \leq i \leq m \\
&w_{r0} \ \ is \ \ free, \ 1 \leq r \leq s,
\end{aligned}
\tag{6}
$$

The above models are in the multiplier form, unlike traditional DEA models. However, the inverse DEA models are in the envelopment form. This is because, when dealing with the inverse DEA problem, the multiplier form leads to a non-linear problem. Thus, we retain this format and derive models (7) and (8) by taking the dual of models (5) and (6), respectively.

$$
\begin{aligned}
&\bar{e}_p = \max\ \theta;\quad 1 \le p \le n \\
&s.t. \\
&\sum_{j=1}^{n} \lambda_j \left(\frac{\frac{y_{rj}}{x_{ij}}}{\frac{y_{rp}}{x_{ip}}} \right) \ge \theta, 1 \le r \le s, 1 \le i \le m \\
&\sum_{j=1}^{n} \lambda_j \left(\frac{x_{ip}}{x_{ij}} \right) = \theta,\quad 1 \le i \le m \\
&\sum_{j=1}^{n} \lambda_j = 1 \\
&\lambda_j \ge 0, 1 \le j \le n \\
&\theta\ is\ free.
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
&\bar{e}'_p = \min \theta; 1 \le p \le n\ s.t.\ \sum_{j=1}^{n} \lambda_j \left(\frac{\frac{x_{ij}}{y_{rj}}}{\frac{x_{ip}}{y_{rp}}} \right) \le \theta,\ 1 \le r \le s, 1 \le i \le m \\
&\sum_{j=1}^{n} \lambda_j \left(\frac{y_{rp}}{y_{rj}} \right) = \theta,\ 1 \le r \le s\ \sum_{j=1}^{n} \lambda_j = 1\ \lambda_j \ge 0,\ 1 \le j \le n\ \theta\, is\, free.
\end{aligned}
\tag{8}
$$

Despić (2013) and Mozaffari et al. (2014a) presented valuable studies on the above models. The construction of the models mentioned is discussed through the application of foundational principles and the formulation of the Production Possibility Set (PPS) in Mozaffari et al. (2020). Here, too, by discussing the second category of constraints of the above models, the BCC-CCR and CCR-BCC versions can be presented. The relationship between DEA and ratio analysis has been the subject of research by several researchers. DEA-R models are the only modeling tools available for applying DEA methodology to performance evaluation problems with ratio data. The models presented so far can only be used for positive data. In the next section, we develop appropriate models for handling negative data.

3 DEA-R models with negative data

In this section, we redesign DEA-R models to handle negative data for two cases: the case with positive input data and free outputs and the case with positive outputs and free inputs. We know it is not permissible to accept the CRS assumption in the presence of negative data. Therefore, we must design the desired model using the appropriate production technology in each case.

As mentioned earlier, upon a review of the literature on DEA models capable of handling negative data, we find that accepting the CRS assumption is not feasible in these models. For instance, if inputs are negative and outputs are positive, the simultaneous multiplication of inputs and outputs by a constant does not necessarily result in a constant efficiency value. Therefore, in all models proposed thus far, including the model proposed by Soltanifar et al. (2022), the VRS assumption has been utilized. We typically encounter negative inputs or outputs in ratio data, not both. Hence, it can be assumed that either inputs or outputs may be negative. Moving forward, we aim to demonstrate how, under such conditions, we can adjust the VRS assumption to Increasing Returns-to-Scale (IRS) and Decreasing Returns-to-Scale (DRS) assumptions.

For a more detailed explanation, in DEA modeling, one of the fundamental principles is the return to scale, which can be accepted or rejected either fully or partially, leading to various model formulations. The different scenarios for accepting or rejecting the return to scale assumption are defined by specific mathematical formulations as follows:

Constant Return to Scale (CRS): $\begin{pmatrix} X \\ Y \end{pmatrix} \in PPS \& \lambda \geq 0 \Rightarrow \begin{pmatrix} \lambda X \\ \lambda Y \end{pmatrix} \in PPS.$

Increase Return to Scale (IRS): $\begin{pmatrix} X \\ Y \end{pmatrix} \in PPS \& \lambda \geq 1 \Rightarrow \begin{pmatrix} \lambda X \\ \lambda Y \end{pmatrix} \in PPS.$

Decrease Return to Scale (DRS): $\begin{pmatrix} X \\ Y \end{pmatrix} \in PPS \& 0 \leq \lambda \leq 1 \Rightarrow \begin{pmatrix} \lambda X \\ \lambda Y \end{pmatrix} \in PPS.$

We assume that the IRS assumption is appropriate for cases where the problem involves only negative outputs. This is because when $\lambda \geq 1$, multiplying it by the corresponding DMU$=\begin{pmatrix} X \\ Y \end{pmatrix}$ leads to an increase in inputs and a decrease in outputs, thereby ensuring that the DMU remains within the PPS. Other assumptions, such as CRS and DRS, would not be suitable for such scenarios. Similarly, the DRS assumption is appropriate for problems that involve only negative inputs. This approach clearly illustrates why the IRS assumption provides a suitable model for handling negative outputs and the DRS assumption provides a suitable model for handling negative inputs.

3.1 Negative outputs

Suppose the inputs are positive, and the outputs are divided into positive and positive/negative groups. Let us assume an output variable Y_r is positive for some DMUs and negative for others. Emrouznejad et al. (2010) defined the following two variables Y_r^P and Y_r^N :

$$y_{rj}^P = \begin{cases} y_{rj} & if\ y_{rj} \geq 0 \\ 0 & Otherwise \end{cases} \qquad y_{rj}^N = \begin{cases} -y_{rj} & if\ y_{rj} < 0 \\ 0 & Otherwise \end{cases}, 1 \leq j \leq n$$

where $y_{rj} = y_{rj}^P - y_{rj}^N$ and $y_{rj}^P, y_{rj}^N \geq 0$. Using this idea, Soltanifar et al. (2022), assuming VRS, presented DEA-R models in the presence of negative data. Here, the DRS assumption is not acceptable. We assume the IRS provides a suitable model for handling negative outputs. To do this, the second constraint category in Model 7 must be $\sum_{j=1}^{n} \lambda_j \left(\frac{x_{ip}}{x_{ij}} \right) \geq \theta, 1 \leq i \leq m$. Thus, our proposed DEA-R model for handling negative outputs is Model 9, where O' is the index set associated with positive outputs, and O'' is the index set associated with negative outputs.

$$
\begin{aligned}
&\sum_{j=1}^{n} \lambda_j \frac{y_{rj}}{x_{ij}} \geq \theta \frac{y_{rp}}{x_{ip}}, \quad 1 \leq i \leq m, \forall r \in O' \\
&\sum_{j=1}^{n} \lambda_j \frac{y_{rj}^P}{x_{ij}} \geq \theta \frac{y_{rp}^P}{x_{ip}}, \quad 1 \leq i \leq m, \forall r \in O'' \\
&\sum_{j=1}^{n} \lambda_j \frac{y_{rj}^N}{x_{ij}} \leq \theta \frac{y_{rp}^N}{x_{ip}}, \quad 1 \leq i \leq m, \forall r \in O'' \\
&\sum_{j=1}^{n} \lambda_j \left(\frac{x_{ip}}{x_{ij}} \right) \geq \theta, \qquad 1 \leq i \leq m \\
&\sum_{j=1}^{n} \lambda_j = 1 \\
&\lambda_j \geq 0, \qquad 1 \leq j \leq n \\
&\theta isfree.
\end{aligned}
\tag{9}
$$

Theorem 1 *Model* 9 *is always feasible*.

Proof By setting $\theta = 1$, $\lambda_j = 1$ for $j = p$, and $\lambda_j = 0$ for $j \neq p$, a feasible solution is obtained for Model 9, and this completes the proof. □

The third category of constraints specifies dealing with the negative part of the outputs in Model 9. To read more about dealing with negative data, we refer readers to Emrouznejad et al. (2010) and Kazemi Matin et al. (2014). Another way of dealing with negative data can be found in Kaffash et al. (2018). They recently reviewed DEA models in the presence of negative data and tackled the drawbacks of the previous model by proposing a directional semi-oriented radial DEA model capable of handling negative data. The approach for dealing with negative outputs in DEA-R results in Model 10.

$$
\begin{aligned}
&\max \theta \\
&s.t. \\
&\sum_{j=1}^{n} \lambda_j \frac{y_{rj}}{x_{ij}} \geq (1+\theta)\frac{y_{rp}}{x_{ip}}, \qquad 1 \leq i \leq m, \forall r \in O' \\
&\sum_{j=1}^{n} \lambda_j \frac{y_{rj}^{P}}{x_{ij}} \geq (1+\theta)\frac{y_{rp}^{P}}{x_{ip}}, \qquad 1 \leq i \leq m, \forall r \in O'' \\
&\sum_{j=1}^{n} \lambda_j \frac{y_{rj}^{N}}{x_{ij}} \leq (1-\theta)\frac{y_{rp}^{N}}{x_{ip}}, \ 1 \leq i \leq m, \forall r \in O1 \leq j \leq n \\
&\sum_{j=1}^{n} \lambda_j \left(\frac{x_{ip}}{x_{ij}}\right) \geq (1+\theta), \qquad 1 \leq i \leq m \\
&\sum_{j=1}^{n} \lambda_j = 1 \\
&\lambda_j \geq 0, \qquad 1 \leq j \leq n \\
&\theta isfree.
\end{aligned}
\tag{10}
$$

Theorem 2 *Model* 10 *is always feasible.*

Proof By setting $\theta = 0$ and $\lambda_p = 1$, a feasible solution is obtained for Model 10, and this completes the proof. □

3.2 Negative inputs

In this case, we assume that the outputs are necessarily positive, and some inputs can be negative or positive. As in the previous section, the same decomposition is used for an input variable that is positive for some DMUs and negative for others. That is:

$$
x_{ij}^{P} = \begin{cases} x_{ij} & if\, x_{ij} \geq 0 \\ 0 & Otherwise \end{cases} \qquad x_{ij}^{N} = \begin{cases} -x_{ij} & if\, x_{ij} < 0 \\ 0 & Otherwise \end{cases}, \ 1 \leq j \leq n
$$

where $x_{ij} = x_{ij}^{P} - x_{ij}^{N}$ and $x_{ij}^{P}, x_{ij}^{N} \geq 0$. Here, the IRS assumption is not acceptable. We assume the DRS will provide a suitable model for handling negative outputs. To do this, the second constraint category in Model 8 must be $\sum_{j=1}^{n} \lambda_j \left(\frac{y_{rp}}{y_{rj}}\right) \leq \theta, \ \ 1 \leq r \leq s$. Thus, our proposed DEA-R model for handling negative outputs is Model 11, where I' is the index set associated with positive inputs, and I'' is the index set associated with negative inputs.

$$
\min \theta s.t. \sum_{j=1}^{n} \lambda_j \frac{x_{ij}}{y_{rj}} \leq \theta \frac{x_{ip}}{y_{rp}}, 1 \leq r \leq s, \forall i \in I' \sum_{j=1}^{n} \lambda_j \frac{x_{ij}^{P}}{y_{rj}} \leq \theta \frac{x_{ip}^{P}}{y_{rp}}, 1 \leq r \leq s, \forall i \in I''
$$
$$
\sum_{j=1}^{n} \lambda_j \frac{x_{ij}^{N}}{y_{rj}} \geq \theta \frac{x_{ip}^{N}}{y_{rp}}, 1 \leq r \leq s, \forall i \in I'' \sum_{j=1}^{n} \lambda_j \left(\frac{y_{rp}}{y_{rj}}\right) \leq \theta, \ 1 \leq r \leq s \sum_{j=1}^{n} \lambda_j = 1 \lambda_j \geq 0, \ 1 \leq j \leq n \theta \,\text{is free.}
\tag{11}
$$

Similar to Model 10 for handling negative outputs in the previous section, Model 12 can be used to handle negative inputs in DEA-R.

$$\min \theta \sum_{j=1}^{n} \lambda_j \frac{x_{ij}}{y_{rj}} \leq (1+\theta)\frac{x_{ip}}{y_{rp}}, 1 \leq r \leq s, \forall i \in I' \sum_{j=1}^{n} \lambda_j \frac{x_{ij}^P}{y_{rj}} \leq (1+\theta)\frac{x_{ip}^P}{y_{rp}}, 1 \leq r \leq s, \forall i \in I''$$
$$\sum_{j=1}^{n} \lambda_j \frac{x_{ij}^N}{y_{rj}} \geq (1-\theta)\frac{x_{ip}^N}{y_{rp}}, 1 \leq r \leq s, \forall i \in I'' \tag{12}$$
$$\sum_{j=1}^{n} \lambda_j \left(\frac{y_{rp}}{y_{rj}}\right) \leq (1+\theta), 1 \leq r \leq s \sum_{j=1}^{n} \lambda_j = 1 \lambda_j \geq 0,\ 1 \leq j \leq n \theta \,\text{is free.}$$

Theorem 2 *Models* (11) *and* (12) *are always feasible.*

Proof The proof is similar to Theorems (1) and (2). ☐

4 Inverse DEA-R models with negative data

Inverse DEA was introduced by Zhang and Cui (1999) and Wei et al. (2000a, b) to address two fundamental questions in real-world problems:

Question 1. If DMU_p aims to increase its inputs in the next period while maintaining or improving its efficiency, how much should it increase its outputs?

Question 2. If DMU_p aims to increase its output in the next period while maintaining or improving its efficiency, how much should it decrease its inputs?

The first question is usually used to predict organizational performance, and the second is commonly used for resource allocation. Concepts such as "extra input" were introduced by Jahanshahloo et al. (2004) as a response to the first question. Subsequently, researchers delved into this concept by framing questions similar to the ones mentioned above (Hadi-Vencheh and Foroughi 2006; Hadi-Vencheh et al. 2008; Lertworasirikul et al. 2011). Jahanshahloo et al. (2014) introduced a third question and developed the necessary modeling based on the ERM model.

Question 3. What steps should DMU_p take to simultaneously increase its inputs and outputs in the next period and at the same time while maintaining or improving its efficiency?

This question typically pertains to resizing a DMU within an organization. Ghobadi and Jahangiri (2015) extended the concept of inverse DEA to fuzzy data, while Ghiyasi (2017) studied inverse DEA based on cost and revenue efficiency, ensuring both technical and cost-effective efficiency in the proposed model. Hu et al. (2020) examined some disadvantages of inverse DEA radial models and introduced a modified model to tackle these issues. Further applications of inverse DEA in various industries can be found in Emrouznejad et al. (2018) and Wegener and Amin (2019). In this paper, we intend to study the concept of inverse DEA in the presence of negative data. Recently, Ghiyasi and Zhu (2020 presented an inverse DEA model using the framework established by Kaffash et al. (2018), specifically for situations involving negative outputs. In the subsequent sections, we delve into inverse DEA-R in the presence of negative data.

4.1 Inverse DEA-R model with negative outputs

Let us assume that an organization is faced with Question 1. Also, the assumptions outlined in Sect. 3.1, which pertain to the feasibility of negative outputs in some cases, hold true within this organization. In other words, suppose that DMU_p aims to change its input vector to $\alpha_p = x_p + \Delta x_p$ so that its efficiency remains unchanged while maximizing output. For this purpose, according to Model 10, multi-objective programming (13) is suggested.

$$\max \left\{ \underset{r\in O'}{\beta_{rp}}, \underset{r\in O''}{\beta^P_{rp}}, \underset{r\in O''}{-\beta^N_{rp}} \right\} s.t. \sum_{j=1}^{n} \lambda_j \frac{y_{rj}}{x_{ij}} \geq \left(1+\theta^*\right) \frac{\beta_{rp}}{\alpha_{ip}}, 1 \leq i \leq m, \forall r \in O'$$
$$\sum_{j=1}^{n} \lambda_j \frac{y^P_{rj}}{x_{ij}} \geq \left(1+\theta^*\right) \frac{\beta^P_{rp}}{\alpha_{ip}}, 1 \leq i \leq m, \forall r \in O'' \sum_{j=1}^{n} \lambda_j \frac{y^N_{rj}}{x_{ij}} \leq \left(1-\theta^*\right) \frac{\beta^N_{rp}}{\alpha_{ip}}, 1 \leq i \leq m, \forall r \in O'' \quad (13)$$
$$\sum_{j=1}^{n} \lambda_j \left(\frac{\alpha_{ip}}{x_{ij}}\right) \geq \left(1+\theta^*\right), 1 \leq i \leq m \sum_{j=1}^{n} \lambda_j = 1 \lambda_j \geq 0, 1 \leq j \leq n$$

where θ^* is the optimal solution of Model 10 for DMU_p before increasing the input.

Definition 1 Suppose $\left(\lambda, \beta_p, \beta^P_p, \beta^N_p\right)$ is a feasible solution of Model 13. In this case, we call this solution a weak efficient solution of Model 13 if there is no feasible solution, like $\left(\overline{\lambda}, \overline{\beta}_p, \overline{\beta}^P_p, \overline{\beta}^N_p\right)$ such that $\left(\overline{\beta}_p, \overline{\beta}^P_p, -\overline{\beta}^N_p\right) > \left(\beta_p, \beta^P_p, -\beta^N_p\right)$.

Weak efficient solutions of Model 13 specify the level of DMU_p inputs and outputs that, according to Theorem 3, will answer Question 1.

Theorem 3 *Suppose DMU_p changes its input vector as X_p to $\alpha_p = X_p + \Delta X_p$. In this case, if $\left(\lambda^*, \beta^*_p, \beta^{P*}_p, \beta^{N*}_p\right)$ is a weak efficient solution of Model* 13, *then the efficiency of the DMU $\left(\alpha_p, \beta^*_p, \beta^{P*}_p, \beta^{N*}_p\right)$ will be equal to the efficiency of DMU_p.*

Proof Suppose $\left(\lambda^*, \beta^*_p, \beta^{P*}_p, \beta^{N*}_p\right)$ is a weak efficient solution of Model 13. In this case, to evaluate the efficiency of the modified unit $\left(\alpha_p, \beta^*_p, \beta^{P*}_p, \beta^{N*}_p\right)$ based on Model 10, we must solve Model 14.

$$\begin{aligned}
&\max\ \theta s.t. \sum_{j=1}^{n} \lambda_j \frac{y_{rj}}{x_{ij}} \geq (1+\theta)\frac{\beta^*_{rp}}{\alpha_{ip}},\ 1 \leq i \leq m, \forall r \in O' \\
&\sum_{j=1}^{n} \lambda_j \frac{y^P_{rj}}{x_{ij}} \geq (1+\theta)\frac{\beta^{P*}_{rp}}{\alpha_{ip}},\ 1 \leq i \leq m, \forall r \in O'' \\
&\sum_{j=1}^{n} \lambda_j \frac{y^N_{rj}}{x_{ij}} \leq (1-\theta)\frac{\beta^{N*}_{rp}}{\alpha_{ip}},\ 1 \leq i \leq m, \forall r \in O'' \\
&\sum_{j=1}^{n} \lambda_j \left(\frac{\alpha_{ip}}{x_{ij}}\right) \geq (1+\theta),\ 1 \leq i \leq m \sum_{j=1}^{n} \lambda_j = 1 \lambda_j \geq 0, 1 \leq j \leq n \theta \text{ is free}
\end{aligned} \tag{14}$$

Given the constraints of Model 13, $\left(\lambda^*, \theta^*\right)$ will be a feasible solution to Model 14. That is, if the optimal solution of Model 14 is $\left(\lambda^+, \theta^+\right)$, then we have $\theta^* \leq \theta^+$. To prove the theorem, it is sufficient to show that $\theta^* = \theta^+$. In other words, we need to demonstrate that the optimal value of Model 14 is equal to θ^*. To achieve this, it suffices to prove $\theta^* \not< \theta^+$. With proof by contradiction, assume $\theta^* < \theta^+$, in which case $k > 1, 0 < h < 1$ exists such that $\left(1+\theta^+\right) = k(1+\theta^*)$,$\left(1-\theta^+\right) = h(1-\theta^*)$ and also constraints 15 are confirmed.

$$\begin{aligned}
&\sum_{j=1}^{n} \lambda^+_j \frac{y_{rj}}{x_{ij}} \geq \left(1+\theta^+\right)\frac{\beta^*_{rp}}{\alpha_{ip}} = (1+\theta^*)\frac{k\beta^*_{rp}}{\alpha_{ip}}, 1 \leq i \leq m, \forall r \in O' \\
&\sum_{j=1}^{n} \lambda^+_j \frac{y^P_{rj}}{x_{ij}} \geq \left(1+\theta^+\right)\frac{\beta^{P*}_{rp}}{\alpha_{ip}} = (1+\theta^*)\frac{k\beta^{P*}_{rp}}{\alpha_{ip}}, 1 \leq i \leq m, \forall r \in O'' \\
&\sum_{j=1}^{n} \lambda^+_j \frac{y^N_{rj}}{x_{ij}} \leq \left(1-\theta^+\right)\frac{\beta^{N*}_{rp}}{\alpha_{ip}} = (1-\theta^*)\frac{h\beta^{N*}_{rp}}{\alpha_{ip}}, 1 \leq i \leq m, \forall r \in O'' \\
&\sum_{j=1}^{n} \lambda^+_j \left(\frac{\alpha_{ip}}{x_{ij}}\right) \geq \left(1+\theta^+\right) = k(1+\theta^*) \geq (1+\theta^*), 1 \leq i \leq m \\
&\sum_{j=1}^{n} \lambda^+_j = 1\ \lambda^+_j \geq 0, 1 \leq j \leq n
\end{aligned} \tag{15}$$

This shows that $\left(\lambda^+, k\beta^*_p, k\beta^{P*}_p, h\beta^{N*}_p\right)$ is a feasible solution for Model 13 and $\left(k\beta^*_p, k\beta^{P*}_p, -h\beta^{N*}_p\right) > \left(\beta^*_p, \beta^{P*}_p, -\beta^{N*}_p\right)$. That is contradictory to the weak efficiency of $\left(\lambda^*, \beta^*_p, \beta^{P*}_p, \beta^{N*}_p\right)$, and therefore, the proof is complete. □

4.2 Inverse DEA-R model with negative inputs

This time, let us assume that an organization is faced with Question 2, and we also assume the conditions outlined in Sect. 3.2. Given the second question and Model 12 assumptions, suppose DMUp aims to change its output vector $\beta_p = Y_p + \Delta Y_p$ so that its performance efficiency does not change. Based on Model 12, a multi-objective

Model 16 can be presented in this case. Theorem 4 shows that each weak efficient solution of Model 16 will estimate the inputs and outputs for the DMUp, which maintains the unit performance efficiency.

$$
\begin{aligned}
&\min \left\{ \underset{i\in I'}{\alpha_{ip}}, \underset{i\in I''}{\alpha^P_{ip}}, \underset{i\in I''}{-\alpha^N_{ip}} \right\} \\
&s.t. \\
&\sum_{j=1}^{n} \lambda_j \frac{x_{ij}}{y_{rj}} \le (1+\theta^*)\frac{\alpha_{ip}}{\beta_{rp}}, 1 \le r \le s, \forall i \in I' \\
&\sum_{j=1}^{n} \lambda_j \frac{x^P_{ij}}{y_{rj}} \le (1+\theta^*)\frac{\alpha^P_{ip}}{\beta_{rp}}, 1 \le r \le s, \forall i \in I'' \\
&\sum_{j=1}^{n} \lambda_j \frac{x^N_{ij}}{y_{rj}} \ge (1-\theta^*)\frac{\alpha^N_{ip}}{\beta_{rp}}, 1 \le r \le s, \forall i \in I'' \\
&\sum_{j=1}^{n} \lambda_j \left(\frac{\beta_{rp}}{y_{rj}}\right) \le (1+\theta^*), \quad 1 \le r \le s \\
&\sum_{j=1}^{n} \lambda_j = 1 \\
&\lambda_j \ge 0, 1 \le j \le n
\end{aligned}
\tag{16}
$$

Theorem 4 *Suppose DMU$_p$ changes its output vector as Y_p to $\beta_p = Y_p + \Delta Y_p$. In this case, if* $\left(\lambda^*, \alpha^*_p, \alpha^{P*}_p, \alpha^{N*}_p\right)$ *is a weak efficient solution of Model* 16, *then the efficiency of the DMU* $\left(\alpha^*_p, \alpha^{P*}_p, \alpha^{N*}_p, \beta_p\right)$ *will be equal to the efficiency of DMU$_p$.*

Proof The proof is essentially the same as the proof of Theorem 3. □

In the next section, we apply the provided models to the healthcare management context, where we evaluate performance and establish targets for 118 hospitals.

5 Case study

The hospital industry plays a crucial role in public health, necessitating the efficient use of resources to provide high-quality care. Evaluating hospitals' performance is essential for identifying best practices, optimizing resource allocation, and enhancing patient outcomes. DEA-R is a powerful analytical tool used to assess the relative efficiency of decision-making units, such as hospitals. By comparing the inputs (e.g., medical staff, equipment) to outputs (e.g., patient satisfaction, treatment success rates), DEA-R helps pinpoint areas of inefficiency and provides actionable insights for improvement. This study employs DEA-R to evaluate hospital performance, offering a comprehensive view of efficiency across various operational dimensions.

This section presents a case study illustrating the applicability and effectiveness of the proposed method, utilizing DEA-R to rank 118 hospitals in Iran. The study aims to analyze the operational efficiency of these hospitals by investigating key inputs and outputs, such as the number of nurses per bed and patient satisfaction scores. Table 1 briefly explains the criteria, all of which are based on annual data. The subsequent sections will detail the data collection process, the methodology used for analysis, and the interpretation of the results.

Iran's Ministry of Health and Medical Education has allocated resources to hospitals to enhance their inputs. We aim to use the proposed method to estimate the outputs while maintaining hospital performance efficiency. After allocating these resources, hospitals must determine how much their outputs should increase to maintain efficiency. It is important to note that these changes require guidelines from hospital managers and are not immediate. According to the policy, resources such as nurses, physicians, specialist physicians, and hospital equipment will be allocated to hospitals over four months. The higher authority expects increased outputs from the hospitals following this allocation. The question is how these changes can be communicated as an overarching document to the hospitals to meet the Ministry's expectations and satisfy their requirements.

As previously mentioned, one of the benchmarking approaches for these outputs is to determine output levels that preserve previous efficiency. This is achieved using the DEA-R method proposed in this study. The method proposed in this study allows for projecting the necessary hospital output increases while maintaining efficiency scores. Additional explanation is provided below:

1. *Allocation and Gradual Implementation*: The resources will be distributed in four stages over four months, with 25% of the total increase allocated each month. This phased approach allows hospitals to adapt gradually to the new inputs without compromising efficiency.
2. *Efficiency Calculation with DEA-R Model*: The DEA-R model will be used to calculate the efficiency of hospitals with the newly allocated resources. The goal is to maintain or improve the efficiency scores while increasing output levels.
3. *Benchmarking Outputs*: For each hospital, the model will provide output targets that need to be achieved to maintain efficiency. These targets will serve as benchmarks for hospital managers.
4. Management Guidelines: Hospital managers will receive comprehensive guidelines to implement these changes effectively. These guidelines will detail steps to increase outputs in line with the allocated resources while maintaining efficiency. Key aspects include optimal resource allocation, staff training programs, performance monitoring systems, and process optimization techniques. Additionally, managers will be provided with risk management strategies and documentation protocols to ensure transparency and accountability throughout the implementation process. This structured approach aims to facilitate a smooth transition and achieve the desired improvements in hospital performance. Table 2

5. *Monitoring and Evaluation*: Hospitals' performance will be monitored throughout the four months. Tables 3, 4, and 5 in our study provide detailed output estimates for each stage based on the DEA-R model. Figures 1a, b, c and d present a comprehensive analysis of output changes across the stages for the first 10 hospitals. This methodology can be extended to all 188 hospitals to gain insights into output trends.
6. *Outcome Expectations*: Hospitals can increase their output levels by following this policy while preserving their newly achieved efficiency scores. This approach ensures that the Ministry's expectations are met and that hospitals are better equipped to handle the increased service demand.

This strategy aligns with the Ministry's goals and provides a structured approach to managing the allocated resources effectively. The gradual implementation and detailed guidelines will facilitate smoother transitions and better outcomes for hospitals and patients. By adopting this approach, we aim to provide a clear and actionable plan that satisfies the Ministry's requirements and enhances hospital services' overall efficiency and effectiveness.

A hospital is an integral and critical component of the health system responsible for providing health care, treatment, and prevention measures. A hospital is also a center for health education and research. Measuring hospital efficiency and productivity is useful for policymaking, efficiency evaluation, resource allocation, and target setting. Table 1 presents the inputs and outputs in ratio form. The criterion O_1 represents the risk of bed occupancy rate and can be negative. The negative score suggests a high risk of nosocomial infections. The Bed Occupancy Rate Risk is calculated using the following formula:

$$85 - \left(\frac{\text{Utilized bed - days}}{\text{Available bed - days}} \times 100 \right)$$

This formula is designed to assess the risk associated with the occupancy rate of hospital beds. A high bed occupancy rate (typically above 85%) can indicate a shortage of available beds, leading to increased operational risks, such as hospital-acquired infections. When the calculated value of the Bed Occupancy Rate Risk is negative, the actual bed occupancy rate is significantly higher than the 85% threshold. For example, when Output 1 is 12, the occupancy rate is below 73%. This suggests that the hospital has a relatively lower risk of bed occupancy and is operating below the critical threshold. However, when Output 1 is -2.5, the actual bed occupancy rate is 87.5%. This indicates that the hospital operates above the 85% threshold, which is considered a high-risk zone for bed occupancy. The negative value highlights the extent to which the occupancy rate exceeds the safe limit, implying increased risk and potential operational challenges. Table 6

The negative values in the Bed Occupancy Rate Risk metric are an important indicator of the overutilization of hospital beds. These values provide a quantifiable measure of the degree to which bed occupancy exceeds the optimal threshold. Hospital managers can use this information to (1) identify periods of high demand and strain on hospital resources, (2) implement strategies to manage patient flow more

Table 1 Criteria in the study of hospitals

Criterion	Title	Criterion formula	Type of criterion	Description	Unit
I_1	Number of nurses to hospital beds	$\frac{\text{Nursing staff number}}{\text{Total hospital beds}}$	Input	This criterion measures the ratio of nursing staff to total hospital beds and guides human resource allocation planning. Total hospital beds include curative (or acute) care, rehabilitative, long-term, and other hospital beds	Number per bed
I_2	Number of generalist physicians in hospital beds	$\frac{\text{Generalist physicians number}}{\text{Total hospital beds}}$	Input	General physicians are medical doctors who treat acute and chronic illnesses and provide preventive care and health education to patients of all ages. They are responsible for the provision of continuing care to individuals and families. The ratio of generalist physicians to total hospital beds is a criterion for allocating hospital resources	Number per bed
I_3	Number of specialists				

Table 1 (continued)

Criterion	Title	Criterion formula	Type of criterion	Description	Unit
physicians to hospital beds	$\frac{\text{Specialist physicians number}}{\text{Total hospital beds}}$	Input	Specialist physicians include pediatricians, anesthesiologists, cardiologists, dermatologists, hematologists, internists, pathologists, orthopedists, ophthalmologists, neurologists, obstetricians/gynecologists, psychiatrists, medical specialists, surgical specialists, and more	Number per bed	
I_4	Equipment for the population covered by the hospital	$\frac{\text{Equipment score}}{\text{Total population covered by the hospital}}$	Input	This criterion assesses access to hospital equipment, such as Gamma Cameras, Radiotherapy, Angiography, CT scans, MRI machines, etc. Geographical distribution and waiting times, particularly for these devices, are essential. In essence, it scores each hospital based on the presence or absence of these devices	Count per population

Table 1 (continued)

Criterion	Title	Criterion formula	Type of criterion	Description	Unit
O_1	Bed occupancy rate risk	$85 - \left(\frac{\text{Utilized bed - days}}{\text{Available bed - days}} \times 100\right)$	Output	Bed occupancy rate is the percentage of days a hospital bed is occupied out of the available 365 days. It serves as a hospital productivity indicator and aids in resource planning and management of hospital beds. Studies have shown that a high occupancy rate (usually above 85%) may indicate a bed shortage and is generally associated with an increased risk of hospital-acquired infections	Percentage (%)
O_2	Bed turnover interval	$\frac{\text{Available bed days} - \text{Utilized bed days}}{\text{Inpatient discharges}}$	Output	This criterion reflects the utilization level of hospital beds. Short turnover intervals have been linked to an increased risk of methicillin-resistant Staphylococcus aureus (MRSA) infections	Days

Table 1 (continued)

Criterion	Title	Criterion formula	Type of criterion	Description	Unit
O_3	Inpatient admission rate	$\frac{\text{The number of hospital admissions}}{\text{Total population covered by the hospital}} \times 1000$	Output	This criterion shows the number of hospital admissions per person per year, representing inpatient care and utilization. Hospital records are the basis for statistics related to inpatient activities, including bed count, admissions, discharges, deaths, and stay duration	Admissions per person
O_4	Bed turnover rate	$\frac{\text{Number of discharges (including deaths)}}{\text{Total hospital beds}}$	Output	The hospital bed turnover rate measures the extent of hospital utilization. It is the number of times there is a change of occupant for a bed during a given period	Turns per year

Table 2 Statistical summary of the input–output data

Statistical Parameters	Input 1	Input 2	Input 3	Input 4	Output 1	Output 2	Output 3	Output 4
Max	2.291	0.226	0.526	0.752	65.477	2.374	272.020	173.594
Min	0.502	0.009	0.022	0.031	– 14.303	0.073	2.566	1.763
Mean	1.377	0.085	0.199	0.284	9.454	1.200	140.831	91.939
Median	1.351	0.080	0.186	0.266	8.497	1.199	136.570	96.418
Variance	0.131	0.002	0.009	0.019	165.751	0.311	4266.304	1349.881
Standard Deviation	0.362	0.041	0.097	0.138	12.874	0.557	65.317	36.741
Scattering Coefficient	380.305	205.544	205.544	205.544	73.432	215.264	215.611	250.236

effectively, or (3) allocate additional resources or adjust operational policies to mitigate risks associated with high bed occupancy rates. By understanding and monitoring these negative outputs, hospitals can proactively address issues related to bed shortages and enhance patient care quality and safety. Table 2 reports the statistical summary of the data set for 118 hospitals in Iran. The data set used to evaluate the hospitals' performance with Model 10 is given in Table 7 in the Appendix.

This information has the potential to revolutionize the services provided by enabling the establishment of an upstream policy for incrementing input levels. According to the upstream policy, the increase in hospital inputs is considered to a certain extent in each input. In other words, we have: $\text{Input}_1^{new} = 1.1 \times \text{Input}_1^{old}$, $\text{Input}_2^{new} = 1.07 \times \text{Input}_2^{old}$, $\text{Input}_3^{new} = 1.03 \times \text{Input}_3^{old}$ and $\text{Input}_4^{new} = 1.015 \times \text{Input}_4^{old}$. With this policy in place, hospitals can increase their outputs without affecting their newly calculated efficiency scores according to Model 13, which we proposed in this study. As outlined in Theorem 3, any weak efficient solution of this model can serve as a reasonable estimate for the outputs of each hospital when implementing the upstream policy. Under this proposed upstream policy, the increase in hospital inputs will be carried out over four stages spanning four months. In each monthly stage, 25% of the input increase will be allocated to each hospital, ensuring their efficiency levels remain unchanged. Through the implementation of this policy, hospitals can increase their output levels without compromising their newly achieved efficiency scores, as outlined in Model 13 proposed in this study. In each stage, corresponding to one month, 25% of the additional input is allocated to each hospital while preserving its efficiency. For a more detailed understanding of how this policy impacts hospital output levels while maintaining efficiency, please refer to Tables 3, 4, 5, and 6, which provide the output estimates for each stage based on Model 13. The change in Output 3, when a 75% pre-determined input increase is achieved, ranges from 250 to 350, reflecting a 40% increase. It is important to note that such changes do not occur instantaneously; in this specific case, they are expected to materialize over three months, during which inputs have been increased to 75% of the pre-determined level.

A key aspect to consider is the flexibility of the proposed method, which allows decision-makers to set minimum and maximum limits on changes based on the

Table 3 Output estimation for the first period (25%)

DMU	Output 1	Output 2	Output 3	Output 4	DMU	Output 1	Output 2	Output 3	Output 4
DMU1	13.58254	1.96472	283.52928	125.53225	DMU60	10.51043	1.79691	133.68428	36.2045
DMU2	15.03017	1.3742	83.8096	98.6371	DMU61	19.79472	0.54034	56.92497	54.64078
DMU3	−0.14963	2.43427	161.76204	57.10093	DMU62	30.76144	1.93298	137.58446	115.45724
DMU4	2.71465	1.68914	147.82591	98.3975	DMU63	9.67552	1.33841	178.35932	113.18647
DMU5	51.72707	2.22617	228.88554	59.95905	DMU64	−0.51534	0.19244	125.64804	51.26867
DMU6	33.08346	1.15285	248.0075	109.67052	DMU65	18.68175	1.65612	151.45514	131.50587
DMU7	49.58847	1.66969	56.20904	56.53046	DMU66	8.81864	1.36485	65.58514	85.7927
DMU8	8.77719	0.87273	30.86245	93.16305	DMU67	24.22092	0.21495	44.45273	113.23616
DMU9	9.43064	0.44807	187.09367	66.29255	DMU68	3.61606	0.86969	119.79809	157.99359
DMU10	32.85004	1.59112	277.39008	77.94202	DMU69	12.36093	1.34016	226.25756	130.58349
DMU11	11.52446	2.05633	160.46207	108.76127	DMU70	10.61498	1.54922	262.55686	108.5279
DMU12	13.34307	2.08171	150.27046	202.8658	DMU71	27.65558	1.306	198.3156	30.25272
DMU13	67.11417	1.8294	87.67287	177.06642	DMU72	12.15658	1.33111	224.33413	37.64344
DMU14	22.96115	1.4031	256.8235	110.95065	DMU73	0.49451	0.36908	240.8417	105.09026
DMU15	−5.6853	1.94785	50.29742	142.53229	DMU74	−0.7631	0.68596	249.16259	83.32943
DMU16	−0.10093	1.05998	120.71459	162.40114	DMU75	−1.7448	1.23643	87.74212	121.83404
DMU17	12.25293	0.45477	278.4629	140.28357	DMU76	7.56803	0.69555	64.02754	133.07802
DMU18	15.24075	2.06437	144.70552	94.75901	DMU77	15.69592	1.96978	49.16538	118.95098
DMU19	46.08318	0.84638	155.98897	126.07145	DMU78	10.9674	1.53384	127.83729	105.27827
DMU20	35.3326	1.86506	103.74812	121.91769	DMU79	8.11782	2.11667	139.73562	128.50685
DMU21	10.58438	1.15122	169.15032	92.09245	DMU80	10.36079	0.99518	208.34898	146.24579
DMU22	10.87664	1.23316	119.51594	71.94404	DMU81	1.02037	1.09369	110.64001	84.55865
DMU23	−0.05825	1.47809	210.41609	119.00168	DMU82	7.92964	1.2494	214.11247	125.01251
DMU24	4.0232	2.27607	56.42113	135.46864	DMU83	−0.83773	2.33967	259.2765	66.82483
DMU25	−0.00835	2.37297	39.82206	31.05346	DMU84	26.49284	2.02205	3.01448	31.89105
DMU26	20.9463	2.45914	229.28107	47.72236	DMU85	7.94543	1.49071	119.54912	97.93773
DMU27	13.8725	1.07208	58.46574	140.70869	DMU86	15.91351	1.54039	82.63964	128.82109
DMU28	22.32128	1.34178	181.81681	162.16708	DMU87	9.0728	1.63526	230.86478	124.73156
DMU29	−1.48637	1.13637	87.90538	109.79921	DMU88	10.13736	0.83191	116.70044	110.93794
DMU30	−1.55678	0.08281	164.30034	80.01342	DMU89	1.18597	1.24791	158.00876	40.61019
DMU31	−1.52008	1.73775	93.25479	108.63246	DMU90	17.33621	2.2065	114.06486	174.55822
DMU32	10.98894	1.81995	172.43542	133.90893	DMU91	28.16965	0.92332	215.005	162.55921
DMU33	−0.50944	0.97975	139.9802	100.72601	DMU92	4.95618	0.9961	60.47363	97.62971
DMU34	−0.08914	0.66736	268.41351	118.25789	DMU93	21.05947	0.45921	145.2684	122.21921
DMU35	−0.13809	0.32734	172.4089	79.94378	DMU94	26.99984	0.7315	179.47184	78.88224
DMU36	9.4905	2.39021	111.58772	113.27903	DMU95	1.80174	2.16192	171.11435	60.61585
DMU37	−0.81478	1.38977	155.8179	42.45573	DMU96	5.5486	0.24605	104.55749	125.14749
DMU38	−1.78574	0.40077	145.36455	54.78415	DMU97	29.14606	1.01095	118.74071	157.04613
DMU39	−0.74556	1.61211	21.67142	46.903	DMU98	11.0173	0.83908	119.76927	109.29858
DMU40	1.56209	1.39619	109.14823	127.38515	DMU99	3.11146	0.75946	236.64669	136.09002
DMU41	17.36269	1.01438	157.02342	1.9923	DMU100	-0.51419	1.81867	167.06844	133.7143
DMU42	−0.33087	0.38134	208.9621	105.13025	DMU101	12.35954	1.44311	273.65209	49.90383
DMU43	24.48907	0.91187	198.09812	139.94857	DMU102	18.7613	1.4092	140.02801	139.97522
DMU44	3.09965	1.33628	130.71897	65.6956	DMU103	7.68488	1.55418	315.32295	174.20556

Table 3 (continued)

DMU	Output 1	Output 2	Output 3	Output 4	DMU	Output 1	Output 2	Output 3	Output 4
DMU45	– 0.06611	0.72888	194.06664	167.90352	DMU104	3.31411	2.01266	298.21186	76.34614
DMU46	– 1.25239	1.41793	201.89311	132.25536	DMU105	13.02263	1.16835	98.18223	162.82304
DMU47	– 0.00683	1.07441	269.88419	69.27038	DMU106	9.69656	0.77919	75.41338	52.49814
DMU48	3.1142	1.14433	121.71194	108.9914	DMU107	3.0623	0.90536	74.51232	151.82765
DMU49	– 1.15874	1.65903	195.69489	74.84529	DMU108	43.27471	1.57177	274.01639	142.26352
DMU50	22.82431	1.56181	182.21543	64.98614	DMU109	27.03738	1.63645	73.14396	120.78208
DMU51	16.9466	1.57497	190.83743	116.17945	DMU110	10.62739	0.71168	140.12117	121.56421
DMU52	– 0.18166	2.31148	222.42694	160.76676	DMU111	17.87689	1.37712	180.94799	113.08934
DMU53	20.47355	1.87369	209.35347	68.00143	DMU112	2.74139	1.09857	83.45635	151.91266
DMU54	1.87228	0.59038	151.41947	67.4879	DMU113	11.75503	1.13678	182.33038	122.8333
DMU55	– 0.10199	0.47684	194.71553	131.88909	DMU114	21.38792	0.91573	201.08913	73.58493
DMU56	17.25501	1.08482	225.64153	13.47792	DMU115	– 0.32071	1.83932	134.05206	50.00313
DMU57	7.59138	0.60578	76.48449	116.98576	DMU116	7.62157	0.75405	165.61693	61.47819
DMU58	1.04489	0.53471	127.15855	57.8132	DMU117	10.71078	0.95567	271.28792	94.16592
DMU59	13.207	1.01472	195.84541	84.53577	DMU118	7.13488	1.30538	121.89719	114.16946

characteristics of the DMUs. A constraint like the following can be added to Model (13) to control the extent of changes, $\beta_{rp} \leq (1 + \kappa)y_{rp}$, the κ parameter determines the maximum percentage increase.

The variability in output changes, or the level of control over outputs, is a significant factor in real-world applications. While inputs can often be adjusted relatively quickly, output changes usually require more time due to the need for changes in methods and practices within an organization. This factor has been considered in our study, where the changes are distributed over three months, and previous two-month periods have been used to incorporate output patterns as part of a longer-term plan.

This approach acknowledges that input adjustments can be implemented swiftly, achieving desired output changes requires careful planning and time. It also reflects the reality that organizations may need to undertake systematic reforms to achieve these changes; thus, the time frame and gradual implementation strategy considered here align with practical management considerations.

Additionally, Fig. 1a, b, c and d present a comprehensive analysis of output changes across the four stages for the first 10 hospitals. This framework equips hospitals with output targets without compromising their efficiencies, all while considering input changes following the upstream policy. The reason for implementing input changes over four months, with monthly adjustments, is to facilitate a better understanding of the variations in hospital outputs and allow top-level managers to adapt to the evolving conditions gradually. This gradual approach enhances the operational feasibility of the plan. In Fig. 1, we illustrate the changes in output for each DMU across different input levels. The Y-axis represents the output values for each DMU. The chart comprises five bars for each DMU:

1. *Initial Output*: The first bar on the left represents the initial output value of each DMU without any increase in input.

Table 4 Output estimation for the second period (50%)

DMU	Output 1	Output 2	Output 3	Output 4	DMU	Output 1	Output 2	Output 3	Output 4
DMU1	15.18317	2.19625	316.94173	140.32558	DMU60	12.06641	1.88658	153.47513	41.56427
DMU2	16.87664	1.54302	94.10568	110.75475	DMU61	22.01338	0.6009	63.30531	60.7651
DMU3	− 0.29926	2.49448	181.93699	64.22256	DMU62	34.19959	2.14903	152.96202	128.36168
DMU4	3.06875	1.74747	167.10873	111.23274	DMU63	10.76089	1.43935	198.36716	125.8834
DMU5	59.41448	2.37394	262.90135	68.86986	DMU64	− 1.03069	0.21634	141.25005	57.63482
DMU6	37.5724	1.30927	258.17058	124.55119	DMU65	21.21723	1.64681	172.01055	144.37639
DMU7	50.79795	1.84502	63.89703	64.26241	DMU66	9.93317	1.53735	73.87402	96.63548
DMU8	9.87243	0.91081	34.71351	104.78809	DMU67	27.27963	0.2421	50.06638	127.53602
DMU9	10.69237	0.50802	212.1251	75.16189	DMU68	4.06691	0.97812	134.73442	168.93276
DMU10	37.33919	1.62993	284.15569	88.59326	DMU69	14.00673	1.38548	235.89015	132.50886
DMU11	12.85501	2.10277	178.98821	121.31829	DMU70	12.10525	1.57538	274.64633	123.7645
DMU12	15.26837	1.97592	171.95331	223.41705	DMU71	31.34652	1.33726	224.783	34.29028
DMU13	68.7511	2.07449	99.41862	200.78845	DMU72	13.67018	1.37621	233.91981	42.33037
DMU14	25.9246	1.53282	267.2016	125.27038	DMU73	0.54977	0.41033	250.83	116.83441
DMU15	− 5.7277	1.94785	56.50178	142.53229	DMU74	− 1.5262	0.77784	280.96276	94.49148
DMU16	− 0.20186	1.18532	134.98878	166.36215	DMU75	− 3.48959	1.23643	100.59365	129.46412
DMU17	13.97475	0.51867	285.25468	148.0982	DMU76	8.5542	0.78619	72.37078	136.32382
DMU18	17.37446	2.10849	164.96433	108.0253	DMU77	17.76275	2.01187	55.63945	134.61439
DMU19	47.20716	0.9433	173.85163	140.50819	DMU78	12.45219	1.56997	145.14415	119.53105
DMU20	40.00161	1.90002	117.45786	138.02844	DMU79	9.23955	2.21014	159.0445	146.26412
DMU21	11.79075	1.28243	188.4295	102.58883	DMU80	11.75952	1.12954	236.47651	142.75693
DMU22	12.39711	1.2757	136.22339	82.00129	DMU81	1.16454	1.23841	126.27214	96.5058
DMU23	− 0.11651	1.46697	238.5364	134.90524	DMU82	8.95949	1.32694	223.44884	130.02208
DMU24	4.52634	2.35961	63.47716	152.41037	DMU83	− 1.67545	2.33967	319.58843	74.75962
DMU25	− 0.0167	2.46606	44.45086	34.66303	DMU84	30.43064	2.05973	3.46254	36.63122
DMU26	23.90013	2.51048	261.61405	54.45212	DMU85	8.82845	1.55695	132.83514	108.82198
DMU27	13.8725	1.0947	66.63594	144.49312	DMU86	18.1788	1.54039	94.40337	139.79177
DMU28	21.60438	1.36095	202.05506	167.47293	DMU87	10.24136	1.63526	256.01252	135.52828
DMU29	− 1.88571	1.13637	98.19452	112.5244	DMU88	12.80505	0.87885	128.22118	110.98795
DMU30	− 3.11356	0.09303	184.58437	89.89163	DMU89	1.35442	1.30006	180.45143	46.37823
DMU31	− 3.04016	1.73775	107.12219	108.85802	DMU90	19.57483	2.16286	128.79403	197.09888
DMU32	12.53396	1.79922	196.67938	149.83729	DMU91	31.27691	1.02826	239.43996	164.00258
DMU33	− 0.5961	0.99858	153.66589	112.3911	DMU92	5.59669	1.03534	68.28888	110.2468
DMU34	− 0.17827	0.75633	274.96018	134.02284	DMU93	21.05947	0.51626	163.31821	125.73531
DMU35	− 0.27618	0.36881	194.25042	90.0714	DMU94	28.86737	0.819	200.93948	88.31779
DMU36	10.6288	2.49631	124.97167	126.86583	DMU95	2.0083	2.21465	190.73258	67.56545
DMU37	− 1.62957	1.5525	174.06266	47.42688	DMU96	6.25214	0.27724	117.81512	129.90513
DMU38	− 3.57149	0.44742	162.28491	61.16099	DMU97	33.27769	1.15425	135.57295	179.30841
DMU39	− 1.49111	1.6833	24.30651	52.60606	DMU98	12.36122	0.87219	126.80459	110.7048
DMU40	1.75378	1.38199	122.54194	137.51996	DMU99	3.53844	0.86368	253.26	136.09002
DMU41	19.36131	1.05176	164.96737	2.22163	DMU100	− 1.02838	1.9547	191.33604	153.13702
DMU42	− 0.66173	0.42658	218.17285	117.60262	DMU101	13.97363	1.60797	280.32653	56.42098
DMU43	27.69389	1.0312	214.07861	139.94857	DMU102	20.90663	1.4092	156.04	146.81464
DMU44	3.5552	1.38924	149.93034	75.35068	DMU103	8.74023	1.76761	358.62551	196.38553
DMU45	− 0.13223	0.82014	218.36483	188.92595	DMU104	3.72313	2.0014	335.01615	85.76852
DMU46	− 2.50478	1.60323	228.27726	149.53899	DMU105	14.55526	1.27005	109.73725	170.01546
DMU47	− 0.01366	1.21552	276.46673	78.36813	DMU106	10.98919	0.88307	85.46655	59.49654
DMU48	3.488	1.12921	136.32138	122.07396	DMU107	3.40496	1.00667	82.84984	155.60311

Table 4 (continued)

DMU	Output 1	Output 2	Output 3	Output 4	DMU	Output 1	Output 2	Output 3	Output 4
DMU49	−2.31747	1.67791	224.36083	85.80884	DMU108	44.3302	1.61011	280.69972	145.73336
DMU50	25.53664	1.6012	203.86903	72.70877	DMU109	30.24737	1.66734	81.82791	135.12182
DMU51	19.30606	1.56961	217.40746	132.35495	DMU110	12.10561	0.81068	153.22302	121.56421
DMU52	−0.36332	2.34786	250.88413	181.33518	DMU111	19.87204	1.39735	201.14277	125.71073
DMU53	23.34527	1.91101	238.71837	77.53963	DMU112	3.14736	1.21198	95.81547	161.15669
DMU54	2.13021	0.67172	159.22675	76.78511	DMU113	13.12649	1.18355	192.97659	123.93128
DMU55	−0.20398	0.54744	209.97147	131.88909	DMU114	23.91859	1.02409	210.10786	82.29168
DMU56	19.2314	1.20907	235.25909	15.02168	DMU115	−0.64143	1.90612	150.5466	56.1558
DMU57	8.64804	0.6901	87.13056	119.83907	DMU116	8.57209	0.84809	173.77048	69.1454
DMU58	1.19271	0.61036	134.37409	65.99196	DMU117	12.08573	1.07835	277.9047	106.25404
DMU59	14.67999	1.12789	204.73624	93.9641	DMU118	7.98734	1.28606	136.46115	127.81013

2. *Estimated output with 25% Input Increase*: The second bar shows the estimated output when the input is increased by 25%.
3. *Estimated output with 50% Input Increase*: The third bar displays the projected output corresponding to a 50% increase in input.
4. *Estimated output with 75% Input Increase*: The fourth bar indicates the output estimate when the input is increased by 75%.
5. *Estimated output with 100% Input Increase*: Finally, the fifth bar on the right illustrates the estimated output for a 100% increase in input.

This progression helps visualize the impact of varying input levels on the output performance of each DMU, providing a comprehensive view of their efficiency and potential productivity gains under different scenarios. It is worth noting that the analysis of output increase and the examination of this process focus on the first 10 hospitals among the 188 hospitals discussed in this case study. However, this methodology can be extended to encompass all 188 hospitals, providing valuable insights into the upward trend of output changes across the entire dataset.

6 Conclusion

Real-world performance evaluation problems often involve ratio data without precise input variables. Sometimes, the data, such as losses or risks, may assume negative values. Traditional DEA models are ill-equipped to handle ratio data with both positive and negative values. The DEA literature has explored the concept of inverse models and the utilization of ratio inputs and outputs. Models for handling negative inputs and outputs are also not new in the DEA literature. All these models are typically presented under the assumption of variable returns-to-scale. However, it is important to acknowledge that accepting the constant returns-to-scale assumption is unfeasible when dealing with negative inputs and outputs. To address this issue, we adapted this assumption in cases where the problem involves negative inputs or outputs. Our study showed that the IRS

Table 5 Output estimation for the third period (75%)

DMU	Output 1	Output 2	Output 3	Output 4	DMU	Output 1	Output 2	Output 3	Output 4
DMU1	16.7838	2.31967	350.35417	155.1189	DMU60	13.62239	1.94646	173.26598	46.92405
DMU2	18.7231	1.63805	104.40176	122.87239	DMU61	24.23203	0.66147	69.68565	66.88942
DMU3	−0.44889	2.5542	202.11195	71.34419	DMU62	37.63774	2.33283	168.33959	141.26612
DMU4	3.42286	1.78907	186.39155	124.06798	DMU63	11.84627	1.41185	218.375	138.58034
DMU5	66.37655	2.41085	296.91716	77.78067	DMU64	−1.54603	0.24023	156.85205	64.00096
DMU6	42.06133	1.4657	267.94656	139.11219	DMU65	23.75271	1.64681	192.56596	153.31468
DMU7	52.00742	1.88895	71.58501	71.99436	DMU66	11.0477	1.63946	82.1629	107.47826
DMU8	10.96766	0.89733	38.56458	116.41313	DMU67	30.33833	0.26924	55.68003	141.83589
DMU9	11.95411	0.56796	237.15653	84.03123	DMU68	4.51776	1.08655	149.67075	177.13063
DMU10	41.82834	1.66874	290.9213	99.2445	DMU69	15.65254	1.43081	245.52275	134.43422
DMU11	14.18557	2.14921	197.51434	133.87531	DMU70	13.59552	1.61289	281.18552	139.00111
DMU12	17.19367	1.97592	193.63615	233.42507	DMU71	35.03747	1.38011	234.74884	38.32784
DMU13	70.38804	2.31958	111.16438	224.51049	DMU72	15.18378	1.42132	243.50549	47.01729
DMU14	28.88806	1.56947	273.61561	139.5901	DMU73	0.60503	0.45158	260.81831	128.57856
DMU15	−8.59156	1.94785	62.70615	142.53229	DMU74	−2.28931	0.86973	287.65235	105.65354
DMU16	−0.30279	1.27235	149.26297	170.32315	DMU75	−5.23439	1.23643	113.44517	135.49812
DMU17	15.69657	0.58258	292.04646	151.62435	DMU76	9.54037	0.87682	80.71402	139.56963
DMU18	19.50817	2.15261	185.22314	121.29158	DMU77	19.82958	2.05396	62.11352	150.27779
DMU19	48.33114	1.04023	191.7143	154.94493	DMU78	13.93698	1.55314	162.45101	133.78382
DMU20	44.67062	1.93212	131.1676	154.13918	DMU79	10.36128	2.23052	178.35338	164.02139
DMU21	12.99712	1.41364	207.70867	113.08521	DMU80	13.15825	1.26389	263.28454	139.67205
DMU22	13.91759	1.31824	152.93083	92.05853	DMU81	1.3087	1.29747	141.90428	108.45294
DMU23	−0.17476	1.50877	262.09141	137.74597	DMU82	9.98934	1.37087	232.78521	131.88824
DMU24	5.02948	2.41047	70.5332	169.35209	DMU83	−2.51318	2.33967	354.5864	82.69442
DMU25	−0.02504	2.52479	49.07966	38.27259	DMU84	34.36844	2.09741	3.91061	41.3714
DMU26	26.85395	2.56182	293.94702	61.18188	DMU85	9.71146	1.58376	146.12117	119.70624
DMU27	13.8725	1.11731	74.80613	148.27756	DMU86	20.44408	1.54039	106.1671	146.77526
DMU28	26.65237	1.44075	222.2933	166.73705	DMU87	11.40991	1.63526	281.16026	146.325
DMU29	−2.82857	1.13637	108.48366	121.44304	DMU88	14.25967	0.91211	135.29023	112.40091
DMU30	−4.67035	0.10326	204.86839	99.76985	DMU89	1.52287	1.31198	202.8941	52.14627
DMU31	−4.56025	1.73775	120.98959	109.12402	DMU90	21.81345	2.11702	143.5232	211.49284
DMU32	14.07898	1.79922	220.92334	161.73369	DMU91	37.52689	1.13319	263.87492	162.17171
DMU33	−0.89415	1.03469	161.34076	117.6079	DMU92	6.23719	0.9892	76.10414	122.86389
DMU34	−0.26741	0.84529	281.50685	146.1524	DMU93	23.23737	0.57332	181.36802	126.98237
DMU35	−0.41426	0.41028	216.09193	100.19903	DMU94	30.7349	0.9065	212.78027	97.75334
DMU36	11.7671	2.55777	138.35562	140.45263	DMU95	2.21487	2.26738	210.3508	74.51504
DMU37	−2.44435	1.68429	192.30742	52.39804	DMU96	6.95569	0.30844	131.07275	132.99811
DMU38	−5.35723	0.49407	179.20527	67.53784	DMU97	35.28898	1.29756	152.4052	201.57069
DMU39	−2.23667	1.72364	26.94159	58.30912	DMU98	13.70513	0.90529	133.83991	112.11102
DMU40	1.94546	1.38199	135.93565	144.0735	DMU99	3.96541	0.9679	263.52725	138.03297
DMU41	21.35994	1.08914	172.91133	2.45097	DMU100	-1.54257	1.9201	215.60363	172.55975
DMU42	−0.9926	0.47182	227.3836	130.07499	DMU101	15.58771	1.64625	287.00097	62.93812
DMU43	30.89871	1.15053	230.0591	139.94857	DMU102	23.05195	1.4092	172.05199	153.65406
DMU44	4.01075	1.44028	169.1417	85.00576	DMU103	9.79558	1.98104	393.23638	206.19455
DMU45	−0.19834	0.9114	242.66302	209.94838	DMU104	4.13214	2.02761	353.48604	95.1909
DMU46	−3.75718	1.62069	254.66142	166.82262	DMU105	15.60613	1.30029	121.29227	174.06345
DMU47	−0.02049	1.35663	283.04927	87.46587	DMU106	12.28181	0.98694	95.51972	66.49494
DMU48	3.86181	1.06612	150.93082	133.94706	DMU107	3.74761	1.10797	91.18737	159.30794

Table 5 (continued)

DMU	Output 1	Output 2	Output 3	Output 4	DMU	Output 1	Output 2	Output 3	Output 4
DMU49	–3.47621	1.67019	253.02677	96.77239	DMU108	45.38568	1.64844	287.38305	149.2032
DMU50	28.24897	1.63188	225.52262	80.4314	DMU109	33.45736	1.6573	90.51185	149.46155
DMU51	21.66551	1.50948	243.97748	148.53045	DMU110	13.58384	0.90967	166.32488	121.56421
DMU52	–0.54498	2.32061	279.34132	201.9036	DMU111	21.8672	1.36313	221.33754	138.33211
DMU53	26.21699	1.94832	268.08326	87.07783	DMU112	3.55334	1.25743	108.17458	168.21769
DMU54	2.38813	0.75305	167.03402	86.08232	DMU113	14.49796	1.22407	201.58743	125.65242
DMU55	–0.30596	0.61804	225.22741	131.88909	DMU114	26.44926	1.13244	219.12659	90.99843
DMU56	21.20778	1.33333	244.87666	16.56544	DMU115	-0.96214	1.9521	167.04113	62.30848
DMU57	9.70471	0.77442	97.77663	122.69238	DMU116	9.52261	0.94214	181.92403	76.81262
DMU58	1.34053	0.686	141.58964	74.17073	DMU117	13.46068	1.20103	284.52147	118.34215
DMU59	16.15298	1.24106	213.62708	103.39243	DMU118	8.8398	1.2682	151.0251	135.87421

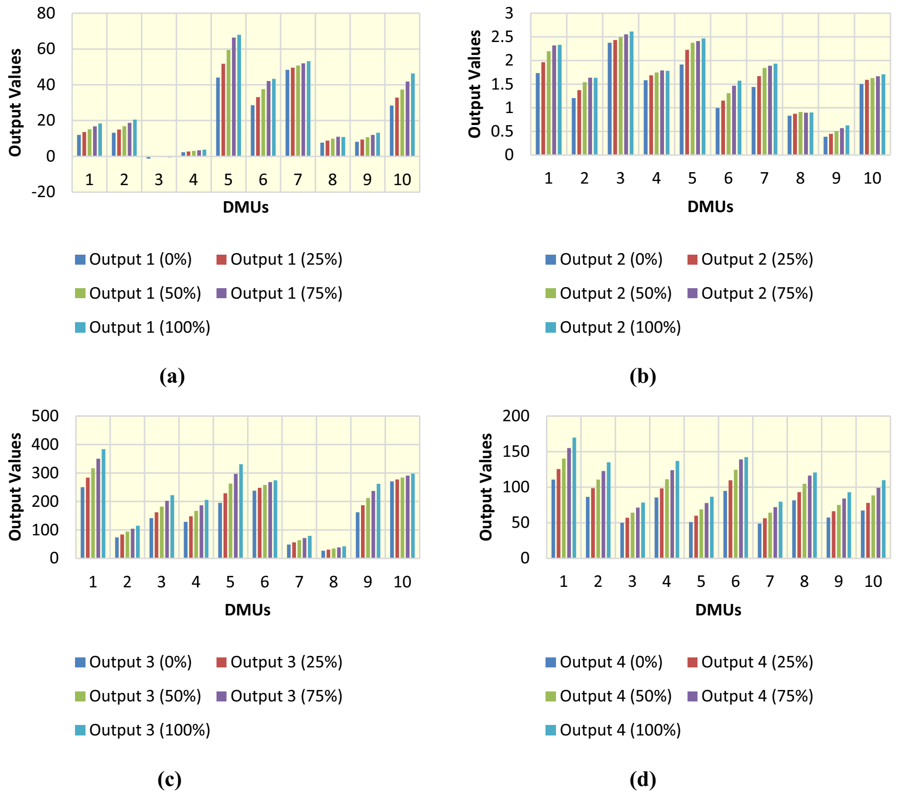

Fig. 1 Changes in the outputs of the first 10 hospitals in four periods

assumption is sufficient for problems with a limited number of negative outputs, while the DRS assumption is sufficient for scenarios featuring a few negative inputs.

Subsequently, we proposed DEA-R models tailored for handling negative data. These proposed models offer greater flexibility compared to their predecessors. The primary

Table 6 Output estimation for the fourth period (100%)

DMU	Output 1	Output 2	Output 3	Output 4	DMU	Output 1	Output 2	Output 3	Output 4
DMU1	18.38443	2.33176	383.76662	169.91222	DMU60	15.17838	1.98734	193.05684	52.28383
DMU2	20.56957	1.63471	114.69783	134.99004	DMU61	26.02485	0.72203	76.06598	73.01373
DMU3	– 0.59852	2.61378	222.28691	78.46581	DMU62	41.07588	2.40624	183.71715	154.17056
DMU4	3.77697	1.78257	205.67436	136.90322	DMU63	12.93164	1.43025	238.38284	143.24215
DMU5	67.92019	2.46691	330.93297	86.69148	DMU64	– 2.06138	0.26413	172.45406	70.36711
DMU6	43.30022	1.5727	274.17788	142.34736	DMU65	26.28818	1.64681	213.12136	160.6969
DMU7	53.21689	1.93288	79.273	79.72631	DMU66	12.16223	1.68641	90.45178	118.32104
DMU8	10.82525	0.90195	42.41565	120.73968	DMU67	33.39703	0.29639	61.29368	156.13575
DMU9	13.21584	0.62791	262.18797	92.90057	DMU68	4.9686	1.19498	164.60708	184.0268
DMU10	46.3175	1.70755	297.68692	109.89574	DMU69	17.29834	1.47613	255.15534	136.35959
DMU11	15.51612	2.19566	216.04048	146.43233	DMU70	15.0858	1.6504	287.72472	149.38059
DMU12	19.11897	1.97592	215.319	243.43309	DMU71	37.33476	1.42426	244.13088	42.3654
DMU13	65.47724	2.54755	122.91014	243.60033	DMU72	16.69738	1.46642	253.09117	51.70422
DMU14	31.85151	1.60597	279.97877	145.35906	DMU73	0.6603	0.49282	269.46281	139.89939
DMU15	– 11.45541	1.94785	68.91051	153.01861	DMU74	– 3.05241	0.96161	294.34194	116.8156
DMU16	– 0.40372	1.30194	163.53716	174.28415	DMU75	– 6.97919	1.23643	126.2967	141.52448
DMU17	17.41838	0.64649	298.83824	155.15049	DMU76	10.52654	0.96746	89.05727	142.81543
DMU18	21.64188	2.19674	205.48196	134.55787	DMU77	21.89642	2.04925	68.58759	165.9412
DMU19	44.9592	1.13715	209.57696	167.26536	DMU78	15.42176	1.50078	179.75787	148.0366
DMU20	49.33963	1.96422	144.87733	170.24992	DMU79	11.48302	2.21068	197.66226	181.77867
DMU21	14.20349	1.54485	226.98785	123.58159	DMU80	14.55698	1.39824	273.10238	141.78898
DMU22	15.43806	1.36078	169.63828	102.11578	DMU81	1.45287	1.30002	157.53641	120.40008
DMU23	– 0.23301	1.55023	270.26155	140.31408	DMU82	11.0192	1.4148	242.12158	133.7544
DMU24	5.53263	2.39965	77.58923	186.29382	DMU83	– 3.3509	2.33967	389.52831	90.62921
DMU25	– 0.03339	2.58352	53.70846	41.88215	DMU84	38.30625	2.13509	4.35867	46.11158
DMU26	29.80778	2.61316	326.28	67.91164	DMU85	10.59447	1.58192	159.4072	130.5905
DMU27	13.8725	1.13992	82.97632	152.06199	DMU86	22.70937	1.54039	117.93083	153.75874
DMU28	29.7751	1.52055	242.53155	166.00117	DMU87	12.57847	1.66724	290.65921	150.90411
DMU29	– 3.77143	1.13637	118.7728	126.91678	DMU88	15.71429	0.94538	142.35927	113.81388
DMU30	– 6.22713	0.11348	225.15241	109.64806	DMU89	1.69132	1.32772	223.61418	57.91431
DMU31	– 6.08033	1.73775	134.85699	109.40203	DMU90	24.05207	2.11702	158.25237	221.31782
DMU32	15.62399	1.79922	245.1673	170.04478	DMU91	42.61735	1.23813	288.30989	160.34083
DMU33	– 1.1922	1.07081	169.01564	119.14196	DMU92	6.8777	0.97033	83.9194	129.89391
DMU34	– 0.35654	0.93426	288.05352	149.5513	DMU93	27.84251	0.63038	199.41783	125.70059
DMU35	– 0.55235	0.45175	237.93344	110.32666	DMU94	32.60244	0.994	221.65141	107.18889
DMU36	12.9054	2.61635	151.73956	154.03944	DMU95	2.42144	2.32011	229.96902	81.46464
DMU37	– 3.25913	1.72346	210.55218	57.36919	DMU96	7.65924	0.33964	144.33038	135.43825
DMU38	– 7.14298	0.54072	196.12562	73.91469	DMU97	25.01442	1.44087	169.23744	217.81796
DMU39	– 2.98223	1.76373	29.57668	64.01218	DMU98	15.04905	0.93839	140.87522	113.51724
DMU40	2.13715	1.38199	149.32936	150.62704	DMU99	4.39238	1.07212	271.16238	140.78177
DMU41	23.35857	1.12652	180.85528	2.6803	DMU100	– 2.05676	1.81433	239.87123	191.98248
DMU42	– 1.32347	0.51706	236.59436	132.64961	DMU101	17.20179	1.68454	293.67542	69.45527
DMU43	34.10353	1.26987	246.03959	139.94857	DMU102	25.19728	1.4092	188.06399	160.49348
DMU44	4.46629	1.49132	188.35307	94.66084	DMU103	10.85092	2.19447	414.28367	216.26014
DMU45	– 0.26446	1.00266	266.96121	230.9708	DMU104	4.54116	2.07477	361.70665	104.61329
DMU46	– 5.00957	1.6839	281.04558	168.15944	DMU105	15.96907	1.33053	132.84728	178.11144
DMU47	– 0.02732	1.49774	289.63181	96.56361	DMU106	13.57443	1.09081	105.57289	73.49333
DMU48	4.23562	1.10879	165.54026	135.281	DMU107	4.09027	1.20928	99.52489	163.01278

Table 6 (continued)

DMU	Output 1	Output 2	Output 3	Output 4	DMU	Output 1	Output 2	Output 3	Output 4
DMU49	−4.63494	1.66248	281.69271	107.73594	DMU108	46.44116	1.68678	294.06637	152.67304
DMU50	30.9613	1.66257	247.17621	88.15404	DMU109	36.66735	1.60555	99.1958	162.12033
DMU51	24.02496	1.57824	270.54751	148.63053	DMU110	15.06206	1.00866	179.42673	121.56421
DMU52	−0.72664	2.23978	307.79851	222.47202	DMU111	23.86236	1.42717	241.53232	138.32667
DMU53	29.0887	1.97342	297.44816	96.61603	DMU112	3.95931	1.28853	120.5337	172.48962
DMU54	2.64606	0.83438	174.84129	95.37953	DMU113	15.86943	1.26459	210.19827	127.37356
DMU55	−0.40795	0.68864	237.44438	132.81952	DMU114	28.97993	1.24079	228.14531	99.70518
DMU56	23.18416	1.45758	254.49423	18.10919	DMU115	-1.28286	1.99782	183.53567	68.46115
DMU57	10.76137	0.85874	108.4227	125.54569	DMU116	10.47313	1.03618	190.07759	84.47983
DMU58	1.48835	0.76165	148.80518	82.34949	DMU117	14.83562	1.32371	291.13825	130.43027
DMU59	17.62596	1.32256	222.51791	112.82075	DMU118	9.69225	1.2682	165.58906	141.97334

innovations in this study encompass (1) the application of DEA-R models in the presence of negative data and (2) the introduction of inverse DEA-R models under similar conditions. Thus, the proposed models can deal with positive or negative ratio data in input–output estimation via the inverse DEA framework. This is a generalized inverse DEA approach in that ignoring negative or ration data yields some classical and existing inverse DEA models in the literature. We used these proposed models for target-setting purposes, estimating outputs or inputs, and establishing targets for managers by adjusting inputs or outputs while considering the pre-determined efficiency score. We further presented a case study in healthcare to illustrate the applicability and effectiveness of the performance evaluation models proposed in this study. Our case study analyzed several hospitals using ratio data, including an output variable that could assume negative values. The objective was to allocate inputs to the hospitals at different intervals to increase outputs while ensuring that the efficiency scores did not decrease. By designing appropriate models, we provided recommendations for the necessary input adjustments and proposed output increases to achieve this goal. This approach demonstrated the practical applicability of our methodology in real-world scenarios where data can include both positive and negative values. Future research could focus on developing new models that explicitly address cases where both inputs and outputs can take negative values by categorizing them into positive and negative groups and incorporating appropriate constraints to handle the resulting ratio scenarios.

Additionally, modifications to the inverse models proposed in this study could be explored to extend their applicability to scenarios relevant to Question 3. We assume available and crisp data in our models, which may not be the case in some real-world applications, and we may face different types of uncertainty for future developments. The proposed models are the first models in the literature to consider the internal structure of the production units. Many real-world problems may not have a straightforward structure. Thus, another fundamental future research path can be analyzing production units' more complex internal structure.

Appendix

See Table 7

Table 7 Input–output data

	Input 1	Input 2	Input 3	Input 4	Output 1	Output 2	Output 3	Output 4	Efficiency
DMU1	1.306	0.024	0.055	0.079	11.982	1.733	250.117	110.739	0.811
DMU2	1.540	0.041	0.096	0.137	13.184	1.205	73.514	86.519	0.476
DMU3	1.134	0.070	0.163	0.233	– 1.252	2.374	141.587	49.979	0.981
DMU4	1.309	0.066	0.155	0.222	2.361	1.583	128.543	85.562	0.597
DMU5	1.137	0.022	0.052	0.074	44.040	1.916	194.870	51.048	1.000
DMU6	1.296	0.058	0.134	0.192	28.595	0.996	237.844	94.790	0.559
DMU7	1.969	0.088	0.204	0.292	48.379	1.441	48.521	48.799	0.452
DMU8	2.157	0.179	0.417	0.596	7.682	0.833	27.011	81.538	0.237
DMU9	2.291	0.027	0.062	0.089	8.169	0.388	162.062	57.423	0.418
DMU10	1.724	0.173	0.404	0.578	28.361	1.502	270.624	67.291	0.456
DMU11	1.779	0.107	0.250	0.357	10.194	1.942	141.936	96.204	0.531
DMU12	2.041	0.059	0.138	0.197	11.418	1.976	128.588	173.594	0.523
DMU13	2.290	0.194	0.452	0.645	65.477	1.584	75.927	153.344	0.526
DMU14	1.157	0.091	0.212	0.303	19.998	1.222	246.445	96.631	0.639
DMU15	1.091	0.055	0.127	0.182	– 14.303	1.948	44.093	142.532	0.753
DMU16	0.952	0.112	0.262	0.374	– 0.751	0.935	106.440	158.440	0.774
DMU17	1.131	0.117	0.272	0.389	10.531	0.391	271.671	120.571	0.698
DMU18	0.972	0.077	0.179	0.256	13.107	2.001	124.447	81.493	0.982
DMU19	1.511	0.102	0.238	0.340	44.959	0.749	138.126	111.635	0.548
DMU20	1.639	0.049	0.115	0.164	30.664	1.681	90.038	105.807	0.548
DMU21	1.230	0.026	0.061	0.088	9.378	1.020	149.871	81.596	0.589
DMU22	0.715	0.044	0.102	0.146	9.356	1.148	102.808	61.887	0.875
DMU23	1.253	0.048	0.111	0.158	-0.451	1.312	182.296	103.098	0.570
DMU24	1.057	0.052	0.121	0.173	3.520	2.130	49.365	118.527	0.999
DMU25	1.346	0.064	0.148	0.212	-0.065	2.220	35.193	27.444	0.820
DMU26	1.423	0.076	0.177	0.254	17.992	2.322	196.948	40.993	0.803
DMU27	1.177	0.152	0.355	0.508	13.872	0.992	50.296	136.924	0.547
DMU28	1.356	0.061	0.141	0.202	19.837	1.342	161.579	148.743	0.568
DMU29	1.015	0.075	0.174	0.249	– 7.400	1.136	77.616	106.991	0.572
DMU30	1.369	0.121	0.282	0.402	– 10.632	0.073	144.016	70.135	0.500
DMU31	0.546	0.099	0.230	0.328	– 7.531	1.738	79.387	95.601	1.000
DMU32	0.881	0.109	0.255	0.364	9.444	1.799	148.191	115.082	0.961
DMU33	0.932	0.074	0.174	0.248	– 2.501	0.980	132.128	89.061	0.587
DMU34	1.093	0.063	0.147	0.211	– 0.757	0.578	258.418	102.493	0.696
DMU35	1.625	0.096	0.225	0.321	– 1.013	0.286	150.567	69.816	0.500
DMU36	2.041	0.088	0.205	0.293	8.352	2.225	98.204	99.692	0.548
DMU37	1.472	0.101	0.235	0.335	– 6.259	1.227	137.573	37.485	0.500
DMU38	1.417	0.123	0.287	0.410	– 13.871	0.354	128.444	48.407	0.500
DMU39	1.501	0.066	0.153	0.219	– 6.053	1.481	19.036	41.200	0.502
DMU40	1.235	0.084	0.196	0.280	1.370	1.382	95.755	111.754	0.566
DMU41	0.721	0.054	0.126	0.180	15.364	0.922	149.079	1.763	0.786
DMU42	1.110	0.089	0.207	0.296	– 2.444	0.336	199.751	92.658	0.592

Table 7 (continued)

	Input 1	Input 2	Input 3	Input 4	Output 1	Output 2	Output 3	Output 4	Efficiency
DMU43	1.473	0.096	0.225	0.321	21.284	0.793	181.574	139.949	0.468
DMU44	1.322	0.074	0.173	0.248	2.644	1.228	111.508	56.041	0.497
DMU45	2.206	0.133	0.310	0.443	−0.479	0.638	169.768	146.881	0.500
DMU46	1.628	0.066	0.155	0.221	−9.128	1.233	175.509	114.972	0.500
DMU47	1.500	0.118	0.275	0.393	−0.046	0.933	260.758	60.173	0.510
DMU48	1.786	0.118	0.275	0.393	2.740	1.066	107.102	95.909	0.333
DMU49	1.521	0.097	0.226	0.323	−9.166	1.534	167.029	63.882	0.500
DMU50	1.421	0.071	0.167	0.238	20.112	1.476	160.562	57.264	0.524
DMU51	1.676	0.103	0.241	0.345	14.587	1.498	164.267	100.004	0.442
DMU52	1.315	0.057	0.133	0.190	−1.314	2.202	193.970	140.198	0.819
DMU53	1.002	0.074	0.172	0.245	17.602	1.829	179.989	58.463	0.876
DMU54	1.018	0.102	0.238	0.339	1.614	0.509	143.612	58.191	0.547
DMU55	1.245	0.133	0.310	0.443	−0.828	0.406	179.460	131.889	0.529
DMU56	1.456	0.078	0.181	0.259	15.279	0.961	216.024	11.934	0.471
DMU57	1.394	0.081	0.189	0.270	6.535	0.521	65.838	101.368	0.381
DMU58	1.207	0.085	0.198	0.283	0.897	0.459	116.517	49.634	0.426
DMU59	1.383	0.110	0.257	0.367	11.734	0.902	186.955	75.107	0.458
DMU60	1.030	0.034	0.080	0.114	8.954	1.639	113.893	30.845	0.832
DMU61	1.250	0.071	0.166	0.237	17.576	0.480	50.545	48.516	0.466
DMU62	1.420	0.024	0.056	0.080	27.323	1.717	122.207	102.553	0.755
DMU63	2.000	0.068	0.158	0.225	8.590	1.188	158.351	100.490	0.346
DMU64	1.494	0.019	0.043	0.062	−4.114	0.169	110.046	44.903	0.601
DMU65	1.161	0.134	0.314	0.448	16.146	1.647	130.900	113.658	0.678
DMU66	1.445	0.039	0.091	0.130	7.704	1.192	57.296	74.950	0.503
DMU67	1.609	0.017	0.039	0.055	21.162	0.188	38.839	98.936	0.863
DMU68	0.873	0.018	0.042	0.061	3.165	0.761	104.862	139.065	0.896
DMU69	1.172	0.165	0.384	0.548	10.715	1.254	216.625	113.771	0.586
DMU70	1.455	0.087	0.203	0.290	9.125	1.549	230.895	93.291	0.523
DMU71	1.733	0.094	0.220	0.314	23.965	1.261	171.848	26.215	0.386
DMU72	1.039	0.084	0.195	0.279	10.643	1.173	214.748	32.957	0.658
DMU73	1.774	0.147	0.344	0.491	0.439	0.328	230.853	93.346	0.402
DMU74	1.296	0.028	0.066	0.094	−5.346	0.594	215.787	72.167	0.600
DMU75	1.235	0.072	0.169	0.241	−13.794	1.236	74.891	104.046	0.519
DMU76	1.024	0.171	0.400	0.571	6.582	0.605	55.684	129.832	0.590
DMU77	1.244	0.124	0.289	0.412	13.629	1.928	42.691	103.288	0.734
DMU78	1.185	0.127	0.296	0.423	9.483	1.484	110.530	91.025	0.609
DMU79	1.395	0.061	0.142	0.203	6.996	1.972	120.427	110.750	0.709
DMU80	2.035	0.073	0.171	0.245	8.962	0.861	180.221	139.672	0.355
DMU81	1.533	0.053	0.124	0.178	0.876	0.939	95.008	72.612	0.400
DMU82	1.579	0.099	0.230	0.329	6.900	1.087	204.776	108.777	0.422
DMU83	1.655	0.162	0.377	0.539	-4.165	2.340	259.276	58.890	0.642
DMU84	1.441	0.153	0.357	0.510	22.555	1.984	2.566	27.151	0.662

Table 7 (continued)

	Input 1	Input 2	Input 3	Input 4	Output 1	Output 2	Output 3	Output 4	Efficiency
DMU85	1.460	0.074	0.173	0.247	7.062	1.370	106.263	87.053	0.485
DMU86	1.308	0.126	0.295	0.421	13.648	1.540	70.876	113.563	0.569
DMU87	0.768	0.080	0.187	0.267	7.904	1.635	205.717	113.935	1.000
DMU88	0.915	0.094	0.219	0.313	9.896	0.768	101.745	108.226	0.551
DMU89	1.455	0.062	0.145	0.207	1.018	1.089	135.566	34.842	0.437
DMU90	1.589	0.079	0.185	0.264	15.098	2.117	99.336	152.018	0.659
DMU91	1.246	0.088	0.205	0.292	28.170	0.818	190.570	157.879	0.639
DMU92	1.088	0.099	0.231	0.330	4.316	0.953	52.658	85.013	0.505
DMU93	1.468	0.100	0.233	0.333	21.059	0.402	127.219	109.639	0.407
DMU94	1.327	0.130	0.304	0.434	25.132	0.644	158.004	69.447	0.477
DMU95	1.407	0.087	0.203	0.289	1.595	2.041	151.496	53.666	0.705
DMU96	1.467	0.059	0.138	0.197	4.845	0.215	91.300	109.650	0.392
DMU97	1.556	0.032	0.074	0.105	25.014	0.868	101.908	134.784	0.606
DMU98	0.502	0.041	0.096	0.137	9.673	0.806	112.734	107.892	1.000
DMU99	1.645	0.068	0.158	0.226	2.684	0.655	206.031	136.090	0.436
DMU100	1.807	0.163	0.380	0.542	−3.979	1.554	142.801	114.292	0.500
DMU101	1.410	0.062	0.144	0.206	10.745	1.255	266.753	43.387	0.551
DMU102	2.076	0.117	0.273	0.390	16.616	1.409	124.016	132.894	0.351
DMU103	1.136	0.009	0.022	0.031	6.630	1.341	272.020	150.282	1.000
DMU104	2.116	0.101	0.236	0.337	2.905	1.924	261.408	66.924	0.452
DMU105	1.131	0.043	0.099	0.142	11.490	1.031	86.627	148.434	0.666
DMU106	1.218	0.029	0.068	0.097	8.404	0.675	65.360	45.500	0.563
DMU107	1.315	0.062	0.146	0.208	2.720	0.804	66.175	141.049	0.524
DMU108	0.777	0.031	0.072	0.103	42.219	1.533	267.333	138.794	1.000
DMU109	1.253	0.152	0.355	0.507	23.827	1.606	64.460	106.442	0.623
DMU110	0.911	0.093	0.218	0.311	9.149	0.613	123.605	121.564	0.620
DMU111	1.202	0.072	0.168	0.239	15.882	1.308	160.753	100.468	0.566
DMU112	1.207	0.032	0.074	0.105	2.335	0.936	71.097	136.307	0.604
DMU113	1.157	0.076	0.177	0.253	10.384	1.121	161.058	114.716	0.531
DMU114	1.439	0.132	0.307	0.439	18.857	0.807	192.070	64.878	0.447
DMU115	1.021	0.051	0.120	0.171	−2.459	1.823	117.558	43.850	0.830
DMU116	2.033	0.107	0.250	0.358	6.671	0.660	150.057	53.811	0.286
DMU117	1.450	0.093	0.217	0.310	9.336	0.833	262.992	82.078	0.531
DMU118	2.113	0.226	0.526	0.752	6.282	1.268	107.333	100.529	0.308

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Authors and Affiliations

Mehdi Soltanifar[1] · **Madjid Tavana**[2,3] · **Vincent Charles**[4,5] · **Mojtaba Ghiyasi**[6] · **Hamid Sharafi**[7]

✉ Madjid Tavana
tavana@lasalle.edu

Mehdi Soltanifar
mehdi.soltanifar@iau.ac.ir

Vincent Charles
vcharles@pucp.pe

Mojtaba Ghiyasi
mog@shahroodut.ac.ir

Hamid Sharafi
hamid.sharafi@srbiau.ac.ir

1 Department of Mathematics, Semnan Branch, Islamic Azad University, Semnan, Iran

2 Business Systems and Analytics Department, Distinguished Chair of Business Analytics, La Salle University, Philadelphia, USA

3 Business Information Systems Department, Faculty of Business Administration and Economics, University of Paderborn, Paderborn, Germany

4 CENTRUM Católica Graduate Business School, Lima, Peru

5 Pontifical Catholic University of Peru, Lima, Peru

6 Faculty of Industrial Engineering and Management Science, Shahrood University of Technology, Shahrood, Iran

7 Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran