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A fuzzy group linear programming technique for multidimentional analysis of preference

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Abstract. Although crisp data are fundamentally indispensable in the conventional linear programming technique for multidimensional analysis of preference (LINMAP), the observed values in the real-world problems are often imprecise or vague. These imprecise or vague data can be suitably characterized by linguistic terms which are fuzzy in nature. LINMAP has been widely used to solve multi-attribute decision making (MADM) problems. This paper extends the conventional LINMAP model to a fuzzy group decision making framework using trapezoidal fuzzy numbers. The ranking approach is used to transform the fuzzy model into a crisp model. The fuzzy LINMAP method proposed in this paper is a simple and effective tool for tackling the uncertainty and imprecision associated with the group MADM problems. A case study in fast food industry is presented to demonstrate the applicability of the proposed framework and exhibit the efficacy of the procedures.

Keywords: LINMAP, fuzzy linear programming, multi-attribute decision making, group decision making, fast food industry

1. Introduction

The linear programming technique for multidimensional analysis of preference (LINMAP) developed by Srinivasan and Shocker [20] is a multi-attribute decision making (MADM) method commonly used to solve decision problems involving multiple attributes of a conflicting nature. In this method *m* alternatives composed of *n* attributes are represented as *m* points in an *n*-dimensional space. The decision maker (DM) is assumed to have his ideal point representing his most preferred alternative location in this space. Those alternatives which are closer to this ideal point are preferred. A linear programming model is proposed to estimate the coordinates of this ideal point and the weights (involved in the Euclidean distance measure) by analyzing paired comparison preference judgments on a set of stimuli, pre-specified by their coordinate locations in the multidimensional space. In the conventional LINMAP, all the decision data are known precisely or given as crisp values. However, the decision data in the real-world problems are often imprecise or vague [3, 14, 16].

Recently, many researchers have used the theory of the fuzzy set introduced by Zadeh [30] to suitably characterize these imprecise or vague linguistic terms which are fuzzy in nature [7, 9, 21–23, 29]. Li and Yang [16] and Xia et al. [28] have extended the conventional LINMAP to solve group MADM problems with fuzzy information. They used triangular fuzzy numbers to assess alternatives with respect to qualitative attributes

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in their fuzzy linear programming model. Li and Sun [17] developed a fuzzy LINMAP method to handle the group decision making problems involving linguistic variables and incomplete preference information. They used the fuzzy numbers and the vertex method to capture the uncertainty in the DMs' preferences and to calculate the distances between the alternatives and the ideal point. Sadi-Nezhad and Akhtari [19] proposed a LINMAP method in group decision making environments by formulating the problem as a possibilistic programming with multiple objectives. Li [15] extended the LINMAP method and developed a methodology for solving MADM problems under Atanassov's intuitionistic fuzzy environments.

The aim of this paper is to further extend the LINMAP method by proposing a new methodology for solving ill-structured group MADM problems in complex environments. In this methodology, linguistic variables are used to capture the fuzziness associated with the decision information by means of trapezoidal fuzzy numbers. The fuzzy ranking approach is used to transform the fuzzy model into a crisp model. The contribution of this paper is fourfold: (1) we consider ambiguous, imprecise and uncertain data in the decision-making problems; (2) we develop an alternative fuzzy version of the LINMAP method for problems with multiple DMs; (3) we use a ranking method to transform the fuzzy linear programming model into a crisp model; and (4) we present the practical aspects of the proposed method in a real-life problem using a case study in the fast food industry.

The remainder of this paper is organized as follows. The next section presents an introduction to the basic fuzzy sets definitions and preliminaries. We then discuss the LINMAP method with precise data and the LINMAP method with fuzzy data. Next, we present a case study in the food industry to exhibit the simplicity and efficacy of the procedures. Finally, we present our conclusions and future research directions.

2. Definitions and preliminaries

This section introduces some basic definitions for fuzzy sets [4, 5, 12, 13, 30, 31].

Definition 1. Let U be a universe set. A fuzzy set $\tilde{A}$ in U is defined by a set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x) \,|x \in X\}$ where $\mu_{\tilde{A}}(x), \forall x \in U$, indicates the degree of membership of $\tilde{A}$ to U.

Definition 2. A fuzzy subset $\tilde{A}$ of real number U is convex if and only if

$$\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq (\mu_{\tilde{A}}(x) \wedge \mu_{\tilde{A}}(y)),$$
$$\forall x, y \in U, \forall \lambda \in [0,\ 1],$$

where " $\wedge$ " denotes the minimum operator.

Definition 3. The α -level of fuzzy set $\tilde{A}$, $\tilde{A}_\alpha$, is the crisp set $\tilde{A}_\alpha = \{x\,|\mu_{\tilde{A}}(x) \geq \alpha\}$. The support of $\tilde{A}$ is the crisp set $Sup(\tilde{A}) = \{x\,|\mu_{\tilde{A}}(x) > 0\}$. $\tilde{A}$ is normal if and only if $Sup_{x\in U}\mu_{\tilde{A}}(x) = 1$, where U is the universal set.

Definition 4. $\tilde{A}$ is a fuzzy number if $\tilde{A}$ is a normal and convex fuzzy subset of U.

Definition 5. A fuzzy number $\tilde{A} = (a^{m1}, a^{m2}, a^l, a^u)$, is called a generalized trapezoidal fuzzy number with membership function $\mu_{\tilde{A}}(x)$ and the following properties:

a) $\mu_{\tilde{A}}(x)$ is a continuous mapping from R to the closed interval [0, 1],
b) $\mu_{\tilde{A}}(x) = 0$ for all $x \in (-\infty, a^l]$,
c) $\mu_{\tilde{A}}(x)$ is strictly increasing on $x \in [a^l, a^{m1}]$,
d) $\mu_{\tilde{A}}(x) = 1$ for all $x \in [a^{m1}, a^{m2}]$,
e) $\mu_{\tilde{A}}(x)$ is strictly decreasing on $x \in [a^{m2}, a^u]$, and
f) $\mu_{\tilde{A}}(x) = 0$ for all $x \in [a^u, +\infty)$.

The membership function $\mu_{\tilde{A}}(x)$ of $\tilde{A}$ can be defined as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} f_a(x), & a^l \leq x \leq a^{m1}, \\ 1, & a^{m1} \leq x \leq a^{m2}, \\ g_a(x), & a^{m2} \leq x \leq a^u, \\ 0, & Otherwise. \end{cases} \quad (1)$$

where $f_a : [a^l, a^{m1}] \to [0, 1]$ and $g_a : [a^{m2}, a^u] \to [0, 1]$.

The inverse functions of f_a and g_a, are denoted as f_a^{-1} and g_a^{-1}, respectively. Since $f_a : [a^l, a^{m1}] \to [0, 1]$ is continuous and strictly increasing, $f_a^{-1} : [0, 1] \to [a^l, a^{m1}]$ is also continuous and strictly increasing. Similarly, since $g_a : [a^{m2}, a^u] \to [0, 1]$ is continuous and strictly decreasing, $g_a^{-1} : [0, 1] \to [a^{m2}, a^u]$ is also continuous and strictly decreasing. That is, both $\int_0^1 f_a^{-1}$ and $\int_0^1 g_a^{-1}$ exist [18].

Moreover, the parametric form of a fuzzy number $\tilde{A}$ can be denoted by $(\underline{a}(r), \bar{a}(r))$, $0 \leq r \leq 1$, which satisfies the following requirements:

1) $\underline{a}(r)$ is a bounded increasing left continuous function,
2) $\bar{a}(r)$ is a bounded decreasing right continuous function,
3) $\underline{a}(r) \leq \bar{a}(r)$, where $0 \leq r \leq 1$.

Particularly, we are working with a special type of trapezoidal fuzzy number with a membership function $\mu_{\tilde{A}}(x)$ expressed by:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a^l}{a^{m1}-a^l}, & a^l \leq x \leq a^{m1}, \\ 1, & a^{m1} \leq x \leq a^{m2}, \\ \frac{a^u-x}{a^u-a^{m2}}, & a^{m2} \leq x \leq a^u, \\ 0, & Otherwise. \end{cases} \tag{2}$$

The trapezoidal fuzzy number $\tilde{A} = (a^{m1}, a^{m2}, a^l, a^u)$ is reduced to a real number A if $a^l = a^{m1} = a^{m2} = a^u$. Conversely, a real number A can be written as a trapezoidal fuzzy number $\tilde{A} = (a, a, a, a)$. If $a^m = a^{m1} = a^{m2}$, then, $\tilde{A} = (a^m, a^l, a^u)$ is called a triangular fuzzy number. A triangular fuzzy number has the following membership function:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a^l}{a^m-a^l}, & a^l \leq x \leq a^m, \\ 1, & x = a^m, \\ \frac{a^u-x}{a^u-a^m}, & a^m \leq x \leq a^u, \\ 0, & Otherwise. \end{cases} \tag{3}$$

For the sake of simplicity and without loss of generality, we assume that all fuzzy numbers used throughout the paper are trapezoidal fuzzy numbers.

Definition 6. A fuzzy number $\tilde{A}$ is called positive if its membership function is $\mu_{\tilde{A}}(x) = 0,\ \forall x < 0$.

Definition 7. Given two positive trapezoidal fuzzy number$\tilde{A} = (a^{m1}, a^{m2}, a^l, a^u)$ and $\tilde{B} = (b^{m1}, b^{m2}, b^l, b^u)$, the arithmetic operations of these two trapezoidal fuzzy numbers are defined as follows:

Addition:

$$\tilde{A}(+)\tilde{B} = (a^{m1} + b^{m1}, a^{m2} + b^{m2}, a^l + b^l, a^u + b^u),$$

Subtraction:

$$\tilde{A}(-)\tilde{B} = (a^{m1} - b^{m2}, a^{m2} - b^{m1}, a^l - b^u, a^u - b^l),$$

Multiplication:

$$\tilde{A}(\times)\tilde{B} = (a^{m1}b^{m1}, a^{m2}b^{m2}, a^l b^l, a^u b^u),$$

$$k\tilde{A} = (ka^l, ka^{m1}, ka^{m2}, ka^u),\ \ \forall k \in R^+,$$

Square:

$$\tilde{A}^2 = \tilde{A}(\times)\tilde{A} = (a^{m1}a^{m1}, a^{m2}a^{m2}, a^l a^l, a^u a^u),$$

Inverse:

$$(\tilde{B})^{-1} = \left(\frac{1}{b^{m2}}, \frac{1}{b^{m1}}, \frac{1}{b^u}, \frac{1}{b^l}\right),$$

Division:

$$\tilde{A}(\div)\tilde{B} = \tilde{A}(\times)\tilde{B}^{-1} = \left(\frac{a^{m2}}{b^{m2}}, \frac{a^{m1}}{b^{m1}}, \frac{a^l}{b^u}, \frac{a^u}{b^l}\right).$$

Definition 8. The minimum *t*-norm is usually applied in fuzzy linear programming to assess a linear combination of fuzzy quantities. Therefore, for a given set of trapezoidal fuzzy numbers $\tilde{A}_j = (a_j^{m1}, a_j^{m2}, a_j^l, a_j^u)$, $j = 1, 2, \ldots, n$ and $\lambda_j \geq 0$, $\sum_{j=1}^{n} \lambda_j \tilde{A}_j$ is defined as follows:

$$\sum_{j=1}^{n} \lambda_j \tilde{A}_j = \left(\sum_{j=1}^{n} \lambda_j a_j^{m1}, \sum_{j=1}^{n} \lambda_j a_j^{m2}, \sum_{j=1}^{n} \lambda_j a_j^{l}, \sum_{j=1}^{n} \lambda_j a_j^{u}\right) \tag{4}$$

where $\sum_{j=1}^{n} \lambda_j \tilde{A}_j$ denotes the combination $\lambda_1 \tilde{A}_1 \oplus \lambda_2 \tilde{A}_2 \oplus \ldots \oplus \lambda_n \tilde{A}_n$.

Definition 9. Linguistic variables are those variables whose values are not numbers but phrases or sentences expressed in a natural or artificial language. For example, “very low”, “low”, “medium”, “high”, or “very high” are linguistic variables because their values are linguistic rather than numerical. The concept of linguistic variable is useful in dealing with situations which are too complex or too ill-defined to be reasonably described with quantitative values. Linguistic values can also be represented by fuzzy numbers.

Ranking fuzzy numbers play an important role in decision-making, data analysis, artificial intelligence and fuzzy linear programming. Jain [11] proposed the first method for ranking fuzzy numbers. Since 1976, many researchers have developed various methods for ranking fuzzy numbers [1, 2, 10, 24–27]. Recently, Asady and Zendehnam [2] proposed a method for minimizing distance to order the fuzzy numbers. The interval $EI(\tilde{A})$ of a fuzzy number $\tilde{A}$ was introduced independently by Dubois and Prade [6] and Heilpern [8] as follows:

$$EI(\tilde{A}) = \left[\int_0^1 \underline{a}\,(r)\,dr,\ \int_0^1 \bar{a}\,(r)\,dr\right] \tag{5}$$

The middle point of interval $EI(\tilde{A})$ can be defined as follows [2]:

$$M(\tilde{A}) = \frac{1}{2}\int_0^1 (\underline{a}\,(r) + \bar{a}\,(r))\,dr \tag{6}$$

$M(\tilde{A})$ can be used for ranking the fuzzy numbers. Therefore, for any two fuzzy numbers $\tilde{A}_i$ and $\tilde{A}_j$, if $M(\tilde{A}_i) > M(\tilde{A}_j)$, then $\tilde{A}_i > \tilde{A}_j$. If $M(\tilde{A}_i) < M(\tilde{A}_j)$, then $\tilde{A}_i < \tilde{A}_j$. Finally, if $M(\tilde{A}_i) = M(\tilde{A}_j)$, then $\tilde{A}_i \cong \tilde{A}_j$. For a trapezoidal fuzzy number $\tilde{A} = (a^{m1}, a^{m2}, a^l, a^u)$, Equation (6) can be simplified as follows:

$$M(\tilde{A}) = \frac{1}{2}(a^{m1} + a^{m2}) + \frac{(a^u - a^{m2}) - (a^{m1} - a^l)}{4} \tag{7}$$

More details about the properties of this ranking method can be found in Asady and Zendehnam [2].

3. LINMAP method with precise data

In this section, we review the LINMAP method developed by Srinivasan and Shocker [20] with precise data. A MADM problem with m alternatives and n attributes can be expressed in matrix format as follows:

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{bmatrix} \end{array} \quad W = [W_1, W_2, \ldots, W_n] \tag{8}$$

where $A_i\,(i = 1, 2, \ldots, m)$ is a set of m feasible alternatives, $C_j\,(j = 1, 2, \ldots, n)$ are attributes with which the performance of the alternatives are measured, r_{ij} is the rating of alternative A_i under attribute C_j, and W_j is the weight of attribute C_j. In addition, let us assume that $R^* = (r_1^*, \ldots, r_j^*)$ is the ideal point.

In the LINMAP method where A_k and A_l are two arbitrary alternatives, and the symbol "$\geq$" represents a preference relation, the DM gives a set of the preference relations, denoted by Ω, for the alternatives with regards to his/her knowledge and experience as follows:

$$\Omega = \{(k, l)\,|\,A_k \geq A_l\ ,\ (k, l = 1, \ldots, m)\} \tag{9}$$

Next, the square of the weighted Euclidean distance, $S_i\,(i = 1, 2, \ldots, m)$, between r_{ij} and r_j^* is computed as follows:

$$S_i = \sum_{j=1}^{n} w_j\,(r_{ij} - r_j^*)^2, \quad \forall i. \tag{10}$$

The aim of LINMAP is to specify the ideal solution by using the consistency and inconsistency measurements and to rank the alternatives without any knowledge of the weights. As a matter of fact, the consistency and inconsistency measurements are computed by the preference relations expressed by the DM. Hence, Sirinivasan and Shocker [20] defined an index $(S_l - S_k)^-$ for $(k, l) \in \Omega$ to measure the inconsistency (error) between the ranking order of alternatives and the preferences of alternatives (A_k and A_l). In other words, if $S_l \geq S_k$, there is no error because it is based on the preferences given by the DM but if $S_l < S_k$, we have $S_k - S_l$ error because of the inconsistency between the ranking order of the alternatives and the preferences. In short, the inconsistency index for $(k, l) \in \Omega$ can be expressed as follows:

$$(S_l - S_k)^- = \begin{cases} S_k - S_l, & S_l < S_k, \\ 0, & S_l \geq S_k. \end{cases} \tag{11}$$

Equation (11) is for the set Ω indicated in Equation (9) ($A_k \geq A_l$). The above index, $(S_l - S_k)^-$, is simply equal to $max\,\{0, S_k - S_l\}$. Then, a total inconsistency index of the DM, which is also called total poorness of fit, based on the set of Ω is formulated as:

$$P = \sum_{(k,l)\in\Omega} (S_l - S_k)^- = \sum_{(k,l)\in\Omega} \max\,\{0, S_k - S_l\} \tag{12}$$

It is clear that P is non-negative because the value of $(S_l - S_k)^-$ is always non-negative. In order to measure the consistency between the ranking order of the alternatives and the preferences of the alternatives (A_k and A_l), similarly, an index $(S_l - S_k)^+$ for $(k, l) \in \Omega$ can be defined as:

$$(S_l - S_k)^+ = \begin{cases} S_l - S_k, & S_l \geq S_k, \\ 0\,, & S_l < S_k. \end{cases} \tag{13}$$

Obviously, the index $(S_l - S_k)^+$ is equal to $\max\,\{0, S_l - S_k\}$. Next, a total consistency index of the DM, which is also called total goodness of fit, based on the set of Ω is formulated as:

$$G = \sum_{(k,l)\in\Omega} (S_l - S_k)^+ = \sum_{(k,l)\in\Omega} \max\,\{0, S_l - S_k\} \tag{14}$$

The objective of the LINMAP method is to minimize the sum of the errors for all pairs in Ω with subject to $G-P=h$ in which h is an arbitrary non-negative number. Therefore, the mathematical programming is constructed as follows:

$$\begin{aligned} &\min \ P \\ &s.t.\ G-P=h \end{aligned} \tag{15}$$

By applying (12) and (14) to $G–P$ in (15), we have

$$\begin{aligned} G-P &= \sum_{(k,l)\in\Omega} (S_l-S_k)^+ - \sum_{(k,l)\in\Omega} (S_l-S_k)^- \\ &= \sum_{(k,l)\in\Omega} (S_l-S_k) \end{aligned} \tag{16}$$

Model (15) is rewritten by using (12) and (16):

$$\begin{aligned} &\min \sum_{(k,l)\in\Omega} \max\{0, S_k-S_l\} \\ &s.t. \sum_{(k,l)\in\Omega} (S_l-S_k)=h \end{aligned} \tag{17}$$

Next, we can transform (15) into the following mathematical programming:

$$\begin{aligned} &\min \sum_{(k,l)\in\Omega} \phi_{kl} \\ &s.t.\ (\mathrm{S}_l-\mathrm{S}_k)+\phi_{kl}\ge 0, \quad (k,l)\in\Omega, \\ &\sum_{(k,l)\in\Omega} (\mathrm{S}_l-\mathrm{S}_k)=h, \\ &\phi_{kl}\ge 0, \quad (k,l)\in\Omega. \end{aligned} \tag{18}$$

Using (10), Model (18) can be formulated as follows:

$$\begin{aligned} &\min \sum_{(k,l)\in\Omega} \phi_{kl} \\ &s.t. \sum_{j=1}^{n} w_j\,(r_{lj}^2-r_{kj}^2) - 2\sum_{j=1}^{n} v_j\,(r_{lj}-r_{kj}) \\ &+\phi_{kl}\ge 0,\ (k,l)\in\Omega, \\ &\sum_{j=1}^{n} w_j \sum_{(k,l)\in\Omega} (r_{lj}^2-r_{kj}^2) - 2\sum_{j=1}^{n} v_j \\ &\sum_{(k,l)\in\Omega} (r_{lj}-r_{kj})=h, \\ &\sum_{j=1}^{n} w_j=1, \\ &\phi_{kl}\ge 0, \quad (k,l)\in\Omega, \\ &w_j\ge 0, \quad \forall j. \end{aligned} \tag{19}$$

where $\phi_{kl}\,((k,l)\in\Omega)$, $w_j\,(j=1,\ldots,n)$ and $v_j\,(j=1,\ldots,n)$ are the decision variables while r_{lj} and r_{kj} $((k,l)\in\Omega; j=1,\ldots,\ n)$ are the parameters. After solving model (19), the subsequent four cases proposed by Srinivasan and Shocker [20] are used with regards to the optimum values of w_j^* and v_j^* .

$$\begin{aligned} &\text{I) If } w_j^*>0, \text{ then } r_j^*=\frac{\mathrm{v}_j^*}{w_j^*}. \\ &\text{II) If } w_j^*=0 \text{ and } v_j^*=0, \text{ define } r_j^*=0. \\ &\text{III) If } w_j^*=0 \text{ and } v_j^*>0, \text{ then } r_j^*=+\infty. \\ &\text{IV) If } w_j^*=0 \text{ and } v_j^*<0, \text{ then } r_j^*=-\infty \end{aligned} \tag{20}$$

In the first and second cases, the square of the weighted Euclidean distance (S_i) between the alternatives and the ideal point is computed by Equation (10) but the following formula is used in the third and fourth cases:

$$S_i=-2\sum_{j=1}^{n} v_j r_{ij}, \quad \forall i. \tag{21}$$

4. LINMAP method with fuzzy data

In the conventional LINMAP method, all the decision data are known precisely or given as crisp values. However, often in real-world problems, crisp data are inadequate or insufficient to model the decision problems. Moreover, most real-world problems involve multiple DMs. Suppose that there are S DMs and the fuzzy performance ratings of each DM can be represented with linguistic terms. We use trapezoidal fuzzy numbers to capture the individual DM's fuzzy judgments. Then, in order to consider all DM's preferences, we define the aggregated fuzzy performance ratings.

Let the fuzzy performance ratings of the sth DM be $\tilde{x}_{ijs}=(x_{ijs}^{m1}, x_{ijs}^{m2}, x_{ijs}^{l}, x_{ijs}^{u})\,(i=1,2,...,m;\ j=1,2,...,n;\ s=1,2,...,S)$. The aggregation of the fuzzy performance ratings A_i on attribute C_j can be defined as $\tilde{r}_{ij}=(r_{ij}^{m1}, r_{ij}^{m2}, r_{ij}^{l}, r_{ij}^{u})$ where $r_{ij}^{m1}=\frac{1}{S}\sum_{s=1}^{S} r_{ijs}^{m1}$, $r_{ij}^{m2}=\frac{1}{S}\sum_{s=1}^{S} r_{ijs}^{m2}$, $r_{ij}^{l}=\frac{1}{S}\sum_{s=1}^{S} r_{ijs}^{l}$ and $r_{ij}^{u}=\frac{1}{S}\sum_{s=1}^{S} r_{ijs}^{u}$.

Linear transformation scales are often used to provide comparable scales for dealing with complex operations in the decision process. Suppose that Φ_B and Φ_C are the benefit and cost attribute index sets, respectively. The transformation scale can be expressed as follows:

$$\tilde{r}_{ij} = (r_{ij}^{m1}, r_{ij}^{m2}, r_{ij}^{l}, r_{ij}^{u}) = \left(\frac{x_{ij}^{m1}}{I_j^*}, \frac{x_{ij}^{m2}}{I_j^*}, \frac{x_{ij}^{l}}{I_j^*}, \frac{x_{ij}^{u}}{I_j^*}\right),$$
$$i = 1, 2, ..., m;\ j \in \Phi_B$$
$$\tilde{r}_{ij} = (r_{ij}^{m1}, r_{ij}^{m2}, r_{ij}^{l}, r_{ij}^{u}) = \left(\frac{A_j^-}{x_{ij}^{m2}}, \frac{A_j^-}{x_{ij}^{m1}}, \frac{A_j^-}{x_{ij}^{u}}, \frac{A_j^-}{x_{ij}^{l}}\right),$$
$$i = 1, 2, ..., m;\ j \in \Phi_C \tag{22}$$

where

$$I_j^* = \max_i \left\{x_{ij}^u\right\},\quad j \in \Phi_B$$
$$A_j^- = \min_i \left\{x_{ij}^l\right\},\quad j \in \Phi_C$$

One of the disadvantages of the classical LINMAP method is to use one DM for creating pair-wise comparison between the alternatives with either crisp degree 0 or 1. However, the real-world decision problems involve a group of DMs and they may express their opinions with a different degree. To address this shortcoming in the literature, we assume that the sth DM determines a set of preference relations for each pair of alternatives. The following preference matrix can be used for the sth DM:

$$P^s = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array}\begin{array}{c} A_1 \quad A_2 \ \cdots \ A_m \\ \begin{bmatrix} - & p_{12}^s & \cdots & p_{1m}^s \\ p_{21}^s & - & \cdots & p_{2m}^s \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1}^s & p_{m2}^s & \cdots & - \end{bmatrix}\end{array}$$

where $p_{kl}^s\,(k, l = 1, ..., n, k \neq l)$ can take either 1 or 0 values. Note that in the preference matrix, 1 and 0 represent the preference and no preference, respectively, between two given alternatives. The aggregated preference matrix can be defined as follows:

$$P = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array}\begin{array}{c} A_1 \quad A_2 \ \cdots \ A_m \\ \begin{bmatrix} - & p_{12} & \cdots & p_{1m} \\ p_{21} & - & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & - \end{bmatrix}\end{array}$$

where

$$p_{kl} = \sum_{s=1}^{S} p_{kl}^s \tag{23}$$

After the abovementioned preparation is considered for the decision-making matrix with regards to a group of DMs, the LINMAP model (19) can be expressed with the following fuzzy linear programming model:

$$\begin{aligned}
&\min \sum_{(k,l)\in\Omega} p_{kl}\phi_{kl} \\
&s.t. \sum_{j=1}^{n} w_j(\tilde{r}_{lj}^2 - \tilde{r}_{kj}^2) - 2\sum_{j=1}^{n} v_j(\tilde{r}_{lj} - \tilde{r}_{kj}) + \\
&\quad p_{kl}\phi_{kl} \geq 0, \quad (k, l) \in \Omega, \\
&\quad \sum_{j=1}^{n} w_j \sum_{(k,l)\in\Omega} (\tilde{r}_{lj}^2 - \tilde{r}_{kj}^2) - 2\sum_{j=1}^{n} v_j \\
&\sum_{(k,l)\in\Omega} p_{kl}(\tilde{r}_{lj} - \tilde{r}_{kj}) = h, \\
&\quad \sum_{j=1}^{n} w_j = 1, \\
&\quad \phi_{kl} \geq 0, \quad (k, l) \in \Omega, \\
&\quad w_j \geq 0, \quad \forall j.
\end{aligned} \tag{24}$$

where $\tilde{r}_{lj} = (r_{lj}^{m1}, r_{lj}^{m2}, r_{lj}^{l}, r_{lj}^{u})$ and $\tilde{r}_{kj} = (r_{kj}^{m1}, r_{kj}^{m2}, r_{kj}^{l}, r_{kj}^{u})$ are as the aggregated trapezoidal fuzzy numbers. Therefore, Model (24) can be rewritten as follows:

$$\begin{aligned}
&\min \sum_{(k,l)\in\Omega} p_{kl}\phi_{kl} \\
&s.t. \quad \sum_{j=1}^{n} w_j\,[(r_{lj}^{m1}, r_{lj}^{m2}, r_{lj}^{l}, r_{lj}^{u})^2 - (r_{kj}^{m1}, r_{kj}^{m2}, r_{kj}^{l}, r_{kj}^{u})^2] \\
&\quad - 2\sum_{j=1}^{n} v_j\,[(r_{lj}^{m1}, r_{lj}^{m2}, r_{lj}^{l}, r_{lj}^{u}) - (r_{kj}^{m1}, r_{kj}^{m2}, r_{kj}^{l}, r_{kj}^{u})] \\
&\quad + p_{kl}\phi_{kl} \geq 0, \quad (k, l) \in \Omega \\
&\quad \sum_{j=1}^{n} w_j \sum_{(k,l)\in\Omega} [(r_{lj}^{m1}, r_{lj}^{m2}, r_{lj}^{l}, r_{lj}^{u})^2 \\
&\quad - (r_{kj}^{m1}, r_{kj}^{m2}, r_{kj}^{l}, r_{kj}^{u})^2] - \\
&\quad 2\sum_{j=1}^{n} [v_j \sum_{(k,l)\in\Omega} p_{kl}((r_{lj}^{m1}, r_{lj}^{m2}, r_{lj}^{l}, r_{lj}^{u}) \\
&\quad - (r_{kj}^{m1}, r_{kj}^{m2}, r_{kj}^{l}, r_{kj}^{u}))] = h
\end{aligned} \tag{25}$$

$$\sum_{j=1}^{n} \mathrm{w}_j = 1,$$
$$\phi_{kl} \geq 0, \quad (k,l) \in \Omega,$$
$$\mathrm{w}_j \geq 0, \quad \forall j.$$

Equivalently, we have

$$\begin{aligned}
\min \quad & \sum_{(k,l)\in\Omega} p_{kl}\phi_{kl} \\
s.t. \quad & \sum_{j=1}^{n} \mathrm{w}_j \tilde{\xi}_{kl} - 2\sum_{j=1}^{n} \mathrm{v}_j \tilde{\psi}_{kl} + p_{kl}\phi_{kl} \geq 0, \ (k,l) \in \Omega \\
& \sum_{j=1}^{n} \mathrm{w}_j \sum_{(k,l)\in\Omega} \tilde{\xi}_{kl} - 2\sum_{j=1}^{n} [\mathrm{v}_j \sum_{(k,l)\in\Omega} p_{kl}\tilde{\psi}_{kl}] = h, \\
& \sum_{j=1}^{n} \mathrm{w}_j = 1, \\
& \phi_{kl} \geq 0, \quad (k,l) \in \Omega, \\
& \mathrm{w}_j \geq 0, \quad \forall j.
\end{aligned} \tag{26}$$

where

$$\tilde{\xi}_{kl} = [\underbrace{(\mathrm{r}_{lj}^{m1})^2 - (\mathrm{r}_{kj}^{m1})^2}_{\xi_{kl}^{m1}}, \underbrace{(\mathrm{r}_{lj}^{m2})^2 - (\mathrm{r}_{kj}^{m2})^2}_{\xi_{kl}^{m2}}, \underbrace{(\mathrm{r}_{lj}^{l})^2 - (\mathrm{r}_{kj}^{l})^2}_{\xi_{kl}^{l}}, \underbrace{(\mathrm{r}_{lj}^{u})^2 - (\mathrm{r}_{kj}^{u})^2}_{\xi_{kl}^{u}}]$$

$$\tilde{\psi}_{kl} = [\underbrace{\mathrm{r}_{lj}^{m1} - \mathrm{r}_{kj}^{m1}}_{\tilde{\psi}_{kl}^{m1}}, \underbrace{\mathrm{r}_{lj}^{m2} - \mathrm{r}_{kj}^{m2}}_{\tilde{\psi}_{kl}^{m2}}, \underbrace{\mathrm{r}_{lj}^{l} - \mathrm{r}_{kj}^{l}}_{\tilde{\psi}_{kl}^{l}}, \underbrace{\mathrm{r}_{lj}^{u} - \mathrm{r}_{kj}^{u}}_{\tilde{\psi}_{kl}^{u}}]$$

The above model can be transformed into the deterministic problem using one of the following three approaches proposed in the fuzzy linear programming literature: (1) the fuzzy ranking approach; (2) the defuzzification approach; and (3) the α-level based approach. Asady and Zendehnam [2] proposed a defuzzification method based on the fuzzy ranking approach. Although the α-level based method may provide better results than the competing methods in the literature, it is not computationally efficient. We use the fuzzy ranking approach proposed by Asady and Zendehnam [2] to attain a new fuzzy LINMAP model for solving the MADM problems in fuzzy environments. Our approach produces solutions as good as Asady and Zendehnam's [2] method while it is straight-forward and computationally efficient. Accordingly, Model (25) is transformed to:

$$\begin{aligned}
\min \quad & \sum_{(k,l)\in\Omega} p_{kl}\phi_{kl} \\
s.t. \quad & \sum_{j=1}^{n} \mathrm{w}_j M(\tilde{\xi}_{kl}) - 2\sum_{j=1}^{n} \mathrm{v}_j M(\tilde{\psi}_{kl}) + p_{kl}\phi_{kl} \geq 0, \ (k,l) \in \Omega \\
& \sum_{j=1}^{n} \mathrm{w}_j \sum_{(k,l)\in\Omega} M(\tilde{\xi}_{kl}) - 2\sum_{j=1}^{n} [\mathrm{v}_j \sum_{(k,l)\in\Omega} p_{kl} M(\tilde{\psi}_{kl})] = h, \\
& \sum_{j=1}^{n} \mathrm{w}_j = 1, \\
& \phi_{kl} \geq 0, \quad (k,l) \in \Omega, \\
& \mathrm{w}_j \geq 0, \quad \forall j.
\end{aligned} \tag{27}$$

where

$$M(\tilde{\xi}_{kl}) = \frac{1}{2}(\xi_{kl}^{m1} + \xi_{kl}^{m2}) + \frac{(\xi_{kl}^{u} - \xi_{kl}^{m2}) - (\xi_{kl}^{m1} - \xi_{kl}^{l})}{4}$$

$$M(\tilde{\psi}_{kl}) = \frac{1}{2}(\tilde{\psi}_{kl}^{m1} + \tilde{\psi}_{kl}^{m2}) + \frac{(\tilde{\psi}_{kl}^{u} - \tilde{\psi}_{kl}^{m2}) - (\tilde{\psi}_{kl}^{m1} - \tilde{\psi}_{kl}^{l})}{4}$$

Model (27) is a LP model and can be easily constructed with a common optimization software package. In this paper, we use the General Algebraic Modeling System (GAMS) software to solve Model (27). The proposed ranking method can be summarized in to the following four general steps:

1) Run model (27) to calculate the optimal solutions denoted by w_j^* and v_j^*.
2) Use the following four cases to obtain the ideal solution (R^*):

$$\begin{aligned}
&\text{I) If } w_j^* > 0, \text{ then } r_j^* = \frac{\mathrm{v}_j^*}{w_j^*}. \\
&\text{II) If } w_j^* = 0 \text{ and } v_j^* = 0, \text{ define } r_j^* = 0. \\
&\text{III) If } w_j^* = 0 \text{ and } v_j^* > 0, \text{ then } r_j^* = +\infty. \\
&\text{IV) If } w_j^* = 0 \text{ and } v_j^* < 0, \text{ then } r_j^* = -\infty
\end{aligned} \tag{28}$$

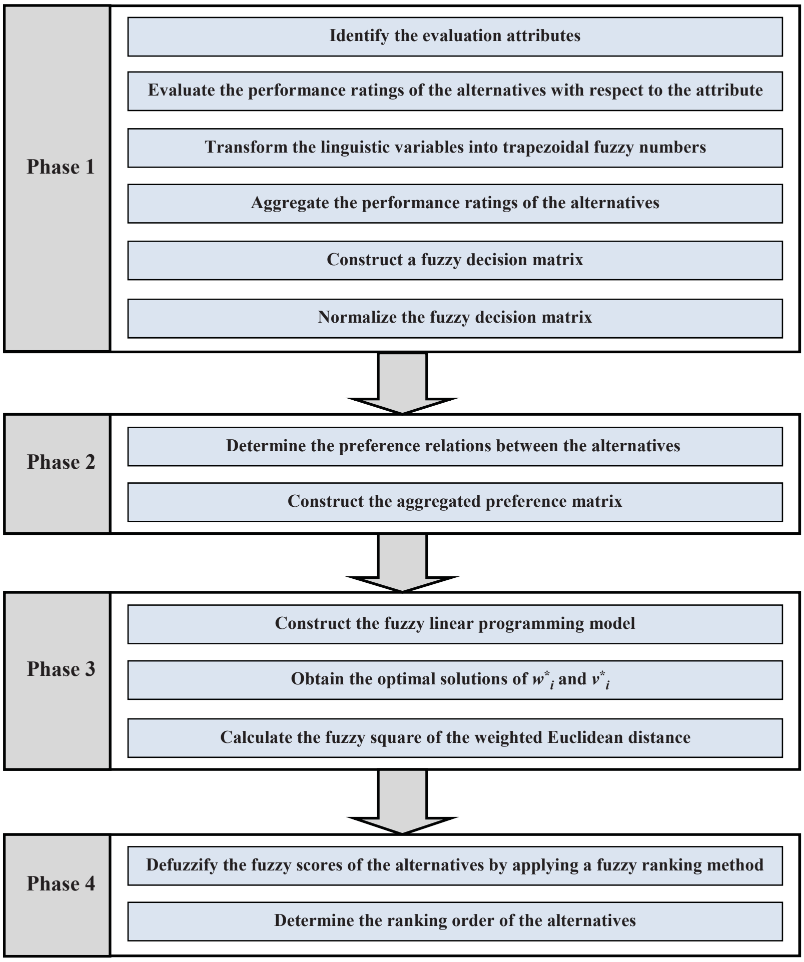

Fig. 1. The proposed framework.

3) Calculate the fuzzy square of the weighted Euclidean distance ($\tilde{S}_i$) between the alternatives and the ideal point using the following formulas:

$$\tilde{S}_i = \sum_{j=1}^{n} w_j (\tilde{r}_{ij} - r_j^*)^2, \quad \forall i.$$

$$\tilde{S}_i = -2 \sum_{j=1}^{n} v_j \tilde{r}_{ij}, \quad \forall i.$$

where the first formula is used for cases (I) and (II) of step 2, and the second one is for cases (III) and (IV) of step 2.

4) Apply a fuzzy ranking method in order to compare the fuzzy scores of the alternatives. Then, the priority order of the alternatives is determined according to the results of the fuzzy ranking method in increasing order.

Contrary to the conventional method, the main advantage of the proposed LINMAP method is to address the gap in LINMAP literature for problems not suitable or difficult to model with crisp values as well as ranking the alternatives in a fuzzy group decision- making environment without known weights. The trend of the proposed fuzzy group LINMAP is depicted in Fig. 1 schematically.

5. Case study

Burger Boys is an international fast food chain restaurant founded in 1979 in the United States. Burger Boys is the world's fifth largest hamburger fast food chain with approximately 3900 locations throughout the world. Burger Boys' management has decided to expand its presence in several geographic areas in the

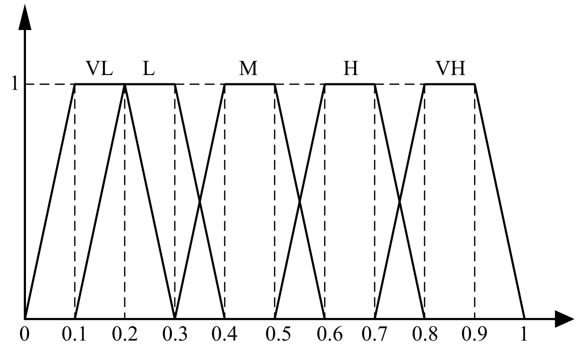

Fig. 2. Seven levels of the linguistic scale used for the ratings.

Southeastern Pennsylvania region. The franchise manager for this region had identified seven key towns in the Bucks County area where Burger Boys does not operate or lease any food franchise operations. The towns under consideration were: Bensalem (A1), Hatboro (A2), Horsham (A3), Newtown (A4), Richboro (A5), Warminster (A6) and Warwick (A7)[1]. These towns support a strong retail business base and offer varying degrees of growth in the areas of population and consumer spending.

A committee of three DMs used the following five attributes (established by the franchise manager) to evaluate the suitability of the seven towns for expanding Berger Boys' presence in Southeastern Pennsylvania: population (C1), number of retail outlets (C2), average family income (C3), start-up cost (C4), and taxes (C5). In this problem, higher population, higher number of retail outlets, lower average family income, lower start-up cost, and lower taxes were preferred. Due to the uncertainties in the real interest rates, preferences and income shocks, liquidity constraints, and the volatility of the household consumption in the region, the franchise manager had defined the average family income (C3) as a fuzzy variable. In addition, due the uncertainties in the economy, employment, consumer spending, and regulatory reforms in the region, the franchise manager had also defined the taxes (C5) as a fuzzy variable. Therefore, factors C3 and C5 were characterized by the following linguistic terms depicted in Fig. 2 and further defined in Table 1: very low (VL), low (L), medium (M), high (H), and very high (VH). The three DMs utilized these linguistic variables and specified the performance

[1]Please refer to the following map for the geographical locations of the seven towns in Southeastern Pennsylvania: https://maps.google.com/maps/ms?msid=216347618323923846552.0004c074050c74a472f35&msa=0&ll=40.193325,-75.106148&spn=0.315753,0.701065.

Table 1
Linguistic variables used for the performance ratings

Linguistic variable	Fuzzy number
Very Low (VL)	(0, 0.1, 0.2, 0.3)
Low (L)	(0.1, 0.2, 0.3, 0.4)
Medium (M)	(0.3, 0.4, 0.5, 0.6)
High (H)	(0.5, 0.6, 0.7, 0.8)
Very High (VH)	(0.7, 0.8, 0.9, 1)

ratings of the seven alternatives under C3 and C5. Furthermore, the real performance values for attributes C1, C2, and C4 were represented with trapezoidal fuzzy numbers.

The linguistic evaluations presented in Table 2 were converted to trapezoidal fuzzy numbers. The aggregated decision matrix presented in Table 3 was then constructed by integrating the fuzzy performance ratings for each attribute. Next, we normalized the elements of the aggregated decision matrix as shown in Table 4. This normalization guaranteed that all the values in the decision matrix had comparable scales.

The DMs then determined a set of preference relations between each pairs of alternative towns as presented in Table 5:

$$\Omega^1 = \{(2,3), (4,7), (6,7), (6,1), (6,2), (1,2)\},$$
$$\Omega^2 = \{(6,2), (6,1), (6,7), (4,5)\}, \text{ and}$$
$$\Omega^3 = \{(6,1), (3,5), (6,4), (3,4)\}.$$

Next, Equation (23) was used to develop the aggregated preference matrix presented in Table 6.

We then used the information provided in Tables 4 and 6 to construct a fuzzy linear programming model and convert the constructed model into the linear Model (27). Using the GAMS software, the optimal solutions for the resulting linear model were found in 0.078 seconds as follows:

$$w_1^* = 0.000,\ w_2^* = 0.520,\ w_3^* = 0.000,$$
$$w_4^* = 0.479,\ w_5^* = 0.000,\ v_1^* = 0.184,$$
$$v_2^* = 0.237,\ v_3^* = -0.227,\ v_4^* = 0.341,$$
$$v_5^* = -0.116$$

Based on the optimum values of w^* and v^*, the ideal points were calculated using the four cases defined in Equation 28 as follows:

$$w_1^* = 0.000,\ v_1^* = 0.184 \Rightarrow r_1^* = +\infty,$$
$$w_2^* = 0.520,\ v_2^* = 0.237 \Rightarrow r_2^* = \frac{0.237}{0.520} = 0.455$$
$$w_3^* = 0.000,\ v_3^* = -0.227 \Rightarrow r_3^* = -\infty$$
$$w_4^* = 0.479,\ v_4^* = 0.341 \Rightarrow r_4^* = \frac{0.341}{0.479} = 0.712$$
$$w_5^* = 0.000,\ v_5^* = -0.116 \Rightarrow r_5^* = -\infty$$

Table 2
Performance scores of the seven towns with respect to the five attributes provided by the three DMs

Attributes	Alternatives																				
	Bensalem (A1)			Hatboro (A2)			Horsham (A3)			Newtown (A4)			Richboro (A5)			Warminster (A6)			Warwick (A7)		
Population* (C1)	200			250			200			350			300			400			150		
Retail outlets (C2)	200			100			350			400			450			600			250		
Average family	DM1	DM2	DM3	DM1	DM2	DM3	DM1	DM2	DM3	DM1	DM2	DM3	DM1	DM2	DM3	DM1	DM2	DM3	DM1	DM2	DM3
Income (C3)	H	VH	VH	M	H	L	L	M	M	VL	L	VL	L	M	L	L	L	VL	VH	M	H
Start-up cost** (C4)	300			300			350			400			400			500			450		
Taxes (C5)	DM1	DM2	DM3	DM1	DM2	DM3	DM1	DM2	DM3	DM1	DM2	DM3	DM1	DM2	DM3	DM1	DM2	DM3	DM1	DM2	DM3
	M	L	M	H	H	M	M	VL	L	L	L	L	VL	L	VL	VH	VH	H	H	L	M

*Per 100 people - **Per $1000.

Table 3
Aggregated decision matrix

Alternatives	Attributes				
	C1	C2	C3	C4	C5
A1	200	200	(0.63, 0.73, 0.83, 0.93)	300	(0.23, 0.33, 0.43, 0.53)
A2	250	100	(0.30, 0.40, 0.50, 0.60)	300	(0.43, 0.53, 0.63, 0.73)
A3	200	350	(0.23, 0.33, 0.43, 0.53)	350	(0.13, 0.23, 0.33, 0.43)
A4	350	400	(0.03, 0.13, 0.23, 0.33)	400	(0.10, 0.20, 0.30, 0.40)
A5	300	450	(0.16, 0.26, 0.36, 0.46)	400	(0.03, 0.13, 0.23, 0.33)
A6	400	600	(0.06, 0.16, 0.26, 0.36)	500	(0.63, 0.73, 0.83, 0.93)
A7	150	250	(0.50, 0.60, 0.70, 0.80)	450	(0.30, 0.40, 0.50, 0.60)

Table 4
Normalized fuzzy decision matrix

Alternatives	Attributes				
	C1	C2	C3	C4	C5
A1	(0.50, 0.50, 0.50, 0.50)	(0.33, 0.33, 0.33, 0.33)	(0.032, 0.036, 0.041, 0.047)	(1.00, 1.00, 1.00, 1.00)	(0.05, 0.06, 0.09, 0.13)
A2	(0.62, 0.62, 0.62, 0.62)	(0.16, 0.16, 0.16, 0.16)	(0.05, 0.06, 0.07, 0.10)	(1.00, 1.00, 1.00, 1.00)	(0.041, 0.047, 0.05, 0.06)
A3	(0.50, 0.50, 0.50, 0.50)	(0.58, 0.58, 0.58, 0.58)	(0.05, 0.06, 0.09, 0.13)	(0.85, 0.85, 0.85, 0.85)	(0.06, 0.09, 0.13, 0.23)
A4	(0.87, 0.87, 0.87, 0.87)	(0.66, 0.66, 0.66, 0.66)	(0.09, 0.13, 0.23, 1.00)	(0.75, 0.75, 0.75, 0.75)	(0.07, 0.10, 0.15, 0.30)
A5	(0.75, 0.75, 0.75, 0.75)	(0.75, 0.75, 0.75, 0.75)	(0.06, 0.08, 0.11, 0.18)	(0.75, 0.75, 0.75, 0.75)	(0.09, 0.13, 0.23, 1.00)
A6	(1.00, 1.00, 1.00, 1.00)	(1.00, 1.00, 1.00, 1.00)	(0.08, 0.11, 0.18, 0.50)	(0.60, 0.60, 0.60, 0.60)	(0.032, 0.036, 0.041, 0.047)
A7	(0.37, 0.37, 0.37, 0.37)	(0.41, 0.41, 0.41, 0.41)	(0.03, 0.04, 0.05, 0.06)	(0.66, 0.66, 0.66, 0.66)	(0.05, 0.06, 0.07, 0.10)

Table 5
Preferences of the three DMs for each pair of alternatives

Alternatives	P1							P2							P3						
	A1	A2	A3	A4	A5	A6	A7	A1	A2	A3	A4	A5	A6	A7	A1	A2	A3	A4	A5	A6	A7
A1	–	1	0	0	0	0	0	–	0	0	0	0	0	0	–	0	0	0	0	0	0
A2	0	–	1	0	0	0	0	0	–	0	0	0	0	0	0	–	0	0	0	0	0
A3	0	0	–	0	0	0	0	0	0	–	0	0	0	0	0	0	–	1	1	0	0
A4	0	0	0	–	0	0	1	0	0	0	–	1	0	0	0	0	0	–	0	0	0
A5	0	0	0	0	–	0	0	0	0	0	0	–	0	0	0	0	0	0	–	0	0
A6	1	1	0	0	0	–	1	1	1	0	0	0	–	1	1	0	0	1	0	–	0
A7	0	0	0	0	0	0	–	0	0	0	0	0	0	–	0	0	0	0	0	0	–

Table 6
Aggregated preference matrix

Alternatives	A1	A2	A3	A4	A5	A6	A7
A1	–	1	0	0	0	0	0
A2	0	–	1	0	0	0	0
A3	0	0	–	1	1	0	0
A4	0	0	0	–	1	0	1
A5	0	0	0	0	–	0	0
A6	3	2	0	1	0	–	2
A7	0	0	0	0	0	0	–

Then, the fuzzy square of the weighted Euclidean distance, $\tilde{S}_i$ (i $= A1, A2, ..., A7$), between the alternatives and the ideal point, R^*, were calculated as:

$$\tilde{S}_{A1} = (-0.110, -0.106, -0.097, -0.084),$$
$$\tilde{S}_{A2} = (-0.111, -0.105, -0.099, -0.084),$$
$$\tilde{S}_{A3} = (-0.130, -0.118, -0.096, -0.054),$$
$$\tilde{S}_{A4} = (-0.240, -0.215, -0.158, 0.226),$$
$$\tilde{S}_{A5} = (-0.182, -0.163, -0.127, 0.083),$$
$$\tilde{S}_{A6} = (-0.164, -0.149, -0.116, -0.030),$$
$$\tilde{S}_{A7} = (-0.109, -0.102, -0.095, -0.083).$$

The detailed computations for the first three alternatives A_1, A_2 and A_3 is presented here to demonstrate the computational process. The fuzzy square of the weighted Euclidean distance between alternatives A_1, A_2 and A_3 and the ideal point was computed by using Equations (10) and (21) as follows:

$$\begin{aligned}\tilde{S}_{A1} = & -2\,[0.184\,(0.5,\ 0.5,\ 0.5,\ 0.5)] \\ & +0.520\,[(0.33, 0.33,\ 0.33,\ 0.33) \\ & -(0.455,\ 0.455,\ 0.455,\ 0.455)]^2 \\ & -2\,[-0.227\,(0.032, 0.036, 0.041, 0.047)] \\ & +0.480\,[(1.0, 1.0, 1.0, 1.0) \\ & -(0.712, 0.712, 0.712, 0.712)]^2 \\ & -2\,[-0.116\,(0.05,\ 0.06, 0.09, 0.13)] \\ & = (-0.110, -0.106, -0.097, -0.084)\end{aligned}$$

$$\begin{aligned}\tilde{S}_{A2} = & -2\,[0.184\,(0.62,\ 0.62,\ 0.62,\ 0.62)] \\ & +0.520\,[(0.16, 0.16, 0.16, 0.16) \\ & -(0.455, 0.455, 0.455, 0.455)]^2 \\ & -2\,[-0.227\,(0.05, 0.06, 0.07, 0.10)] \\ & +0.480\,[(1.0, 1.0, 1.0, 1.0) \\ & -(0.712, 0.712, 0.712,\ 0.712)]^2 - 2 \\ & [-0.116(0.041, 0.047, 0.05, 0.06)] \\ & = (-0.111, -0.105, -0.099, -0.084)\end{aligned}$$

$$\begin{aligned}\tilde{S}_{A3} = & -2\,[0.184\,(0.5,\ 0.5,\ 0.5,\ 0.5)] \\ & +0.520\,[(0.58, 0.58, 0.58, 0.58) \\ & -(0.455, 0.455, 0.455, 0.455)]^2 \\ & -2[-0.227\,(0.05,\ 0.06,\ 0.09,\ 0.13)] \\ & +0.480[(0.85, 0.85, 0.85, 0.85) \\ & -(0.712,\ 0.712, 0.712, 0.712)]^2 \\ & -2\,[-0.116(0.06, 0.09, 0.13, 0.23)] \\ & = (-0.130, -0.118, -0.096, -0.054)\end{aligned}$$

The corresponding defuzzification indices were then determined as follows:

$$M(S_{A1}) = -0.0992,\ M(S_{A2}) = -0.0997,$$
$$M(S_{A3}) = -0.0995,\ M(S_{A4}) = -0.0967,$$
$$M(S_{A5}) = -0.0972,\ M(S_{A6}) = -0.1147,$$
$$M(S_{A7}) = -0.0972.$$

For example, the defuzzification values of $\tilde{S}_{A1}$, $\tilde{S}_{A2}$, and $\tilde{S}_{A3}$ are calculated using Equation (7) as follows:

$$\begin{aligned}M(S_{A1}) = & 1/2\,(-0.106 - 0.097) \\ & + \frac{[-0.084 - (-0.097)] - [-0.106 - (-0.110)]}{4} \\ & = -0.0992\end{aligned}$$

$$\begin{aligned}M(S_{A2}) = & 1/2(-0.105 - 0.099) \\ & + \frac{[-0.084 - (-0.099)] - [-0.105 - (-0.111)]}{4} \\ & = -0.0997\end{aligned}$$

$$\begin{aligned}M(S_{A3}) = & 1/2\,(-0.118 - 0.096) \\ & + \frac{[-0.054 - (-0.096)] - [-0.118 - (-0.130)]}{4} \\ & = -0.0995\end{aligned}$$

Finally, we used the above defuzzification indices and determined the overall ranking of the alternative towns as: $A4 > A5 \cong A7 > A1 > A3 > A2 > A6$, where "$\cong$" means that $A5$ and $A7$ are indifferent. The franchise manager used this ranking order to select Newtown ($A4$) as the most attractive alternative town in Southeastern Pennsylvania for establishing a new fast food franchise operations.

6. Conclusions and future research directions

A large number of LINMAP models with different levels of sophistication have been proposed over the past decade. However, many of these models have limited real-world applications because the conventional LINMAP models are generally assume crisp data, single DM, and knowledge of the importance weights. Contrary to the conventional LINMAP methods, the LINMAP method in this study addresses these shortfalls in the existing models in the literature. The proposed method is capable of ranking the alternatives for a group of DMs in a fuzzy environment and without any knowledge of the importance weights. The proposed method is most useful for situations where the DMs cannot precisely state their opinions about the preference of one alternative over another one or for situations where definite assessment of alternatives based on attributes is impossible. We showed the simplicity and efficacy of our procedures with a location planning case study for a fast food franchise company.

Future research will concentrate on the comparison of results obtained with those that might be obtained with other methods. In addition, we plan to extend the fuzzy group LINMAP approach proposed here to deal with fuzzy non-linear optimization problems with multiple objectives where the vagueness or impreciseness appears in all the components of the optimization problem. Such an extension also implies the study of new practical experiments. Finally, we plan to focus on the use of co-evolutionary algorithms to solve fuzzy optimization problems. This approach would permit the search for solutions covering optimality, diversity and interpretability. We hope that the concepts introduced here will provide inspiration for future research.

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