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A stepwise fuzzy linear programming model with possibility and necessity relations

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Abstract. Linear programming (LP) is the most widely used optimization technique for solving real-life problems because of its simplicity and efficiency. Although conventional LP models require precise data, managers and decision makers dealing with real-world optimization problems often do not have access to exact values. Fuzzy sets have been used in the fuzzy LP (FLP) problems to deal with the imprecise data in the decision variables, objective function and/or the constraints. The imprecisions in the FLP problems could be related to (1) the decision variables; (2) the coefficients of the decision variables in the objective function; (3) the coefficients of the decision variables in the constraints; (4) the right-hand-side of the constraints; or (5) all of these parameters. In this paper, we develop a new stepwise FLP model where fuzzy numbers are considered for the coefficients of the decision variables in the objective function, the coefficients of the decision variables in the constraints and the right-hand-side of the constraints. In the first step, we use the possibility and necessity relations for fuzzy constraints without considering the fuzzy objective function. In the subsequent step, we extend our method to the fuzzy objective function. We use two numerical examples from the FLP literature for comparison purposes and to demonstrate the applicability of the proposed method and the computational efficiency of the procedures and algorithms.

Keywords: Fuzzy numbers, possibility, necessity, linear programming, multi-criteria

1. Introduction

Linear Programming (LP) is a mathematical technique for optimal allocation of scarce resources to several competing activities on the basis of given criteria of optimality. LP is by far the most widely used method by practitioners for constrained optimization [3, 12, 13, 39, 43]. The conventional LP relies on the existence of coefficients for the objective function and the constraints defined as crisp parameters. However, managerial decision making is often characterized by elements that are resulting from imprecise, vague, uncertain or incomplete information.

Fuzzy set theory has been proposed to handle such imprecision by generalizing the notion of membership in a set. Essentially, each element in a fuzzy set is associated with a point-value selected from the unit interval [0, 1]. Fuzzy set theory has been extensively employed in linear optimization. The main objective in fuzzy LP (FLP) is to find the best solution possible with imprecise, vague, uncertain or incomplete information. The sources of imprecision in FLP vary. For example, sometimes constraint satisfaction limits are vague and other times coefficient variables are not known precisely. The ultimate challenge in FLP is to construct an optimization model that can produce optimal solutions with

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subjective preference information. In this paper, we propose a stepwise LP model to solve FLP problems with fuzzy data in the objective function and constraints. In the first step, we use the possibility and necessity relations for fuzzy constraints without considering the fuzzy objective function. In the subsequent step, we extend our method to the fuzzy objective function. In summary, the proposed model addresses a general class of problems with potentially fuzzy parameters in the objective function and the technical coefficients for crisp decision variables. In spite the fact that the proposed model is more comprehensive in the sense that both objective and constraints are fuzzy but we will see through numerical examples that solution to the model characterized by very low computational complexity.

This paper is organized as follows: The next section presents a brief review of the existing literature followed by preliminary definitions and a description of the FLP problem. We then introduce the mathematical details of the proposed method followed by a discussion of the stability of the results. Following this discussion, we present the results from two numerical examples and finish the paper with our conclusions and future research directions.

2. Literature review

Tanaka et al. [45] proposed the theory of fuzzy mathematical programming based on the fuzzy decision framework of Bellman and Zadeh [8]. The seminal work on FLP was introduced by Zimmermann [57] to address the impreciseness and vagueness of the parameters in LP problems with fuzzy constraints and objective functions. He constructed a crisp model of the problem and obtained its crisp results with an existing algorithm. Zimmermann [57] used Bellman and Zadeh [8]'s interpretation that a fuzzy decision is a union of goals and constraints.

In the past decade, researchers have discussed various properties of FLP problems and proposed various models. Zhang et al. [56] proposed a FLP with fuzzy numbers for the objective function coefficients. They showed also how to convert the FLP problems into multi-objective optimization problems with four objective functions. Stanciulescu et al. [44] proposed a FLP model with fuzzy coefficients for the objectives and the constraints. Their model uses fuzzy decision variables with a joint membership function instead of crisp decision variables and linked the decision variables together to sum them up to a constant.

Ganesan and Veeramani [21] proposed a FLP model with symmetric trapezoidal fuzzy numbers. They proved fuzzy analogues of some important LP theorems and derived a solution for FLP problems without converting them into crisp LP problems. Ebrahimnejad [19] showed that the method proposed by Ganesan and Veermani [21] stops in a finite number of iterations and proposed a revised version of their method that was more efficient and robust in practice. He also proved the absence of degeneracy and showed that if an FLP problem has a fuzzy feasible solution, it also has a fuzzy basic feasible solution and if an FLP problem has an optimal fuzzy solution, it also has an optimal fuzzy basic solution. Mahdavi-Amiri and Nasseri [36] proposed a FLP model where a linear ranking function was used to order trapezoidal fuzzy numbers. They established the dual problem of the LP problem with trapezoidal fuzzy variables and deduced some duality results to solve the FLP problem directly with the primal simplex tableau. Mahdavi-Amiri and Nasseri [37] developed some methods for solving FLP problems by introducing and solving certain auxiliary problems. They applied linear ranking function to order trapezoidal fuzzy numbers and deduced some duality results by establishing the dual problem of the LP problem with trapezoidal fuzzy variables. Ebrahimnejad [20] introduced a new primal-dual algorithm for solving FLP problems by using the duality results proposed by Mahdavi-Amiri and Nasseri [37]. Ebrahimnejad [20] also generalized the concept of sensitivity analysis in FLP problems by applying a fuzzy simplex algorithm and by using general linear ranking functions on fuzzy numbers.

Lodwick and Jamison [34] developed the theory underlying fuzzy, possibilistic, and mixed fuzzy/possibilistic optimization and demonstrated the appropriate use of distinct solution methods associated with each type of optimization dependent on the semantics of the problem. Rommelfanger [41] showed that both the probability distributions and fuzzy sets should be used in parallel or in combination, to model imprecise data dependent on the real situation. Van Hop [46] presented a model to measure attainment values of fuzzy numbers/fuzzy stochastic variables and used these new measures to convert the FLP problem or the fuzzy stochastic linear programming problem into the corresponding deterministic linear programming problem. Ghodousian and Khorram [22] studied a new linear objective function optimization with respect to the fuzzy relational inequalities defined by max-min composition in which fuzzy inequality replaces ordinary inequality in the constraints. They showed that their method attains the optimal points that are better solutions than those resulting from the resolution of the similar problems with ordinary inequality constraints.

Wu [51] derived the optimality conditions for FLP problems by proposing two solution concepts based on similar solution concept, called the nondominated solution, in the multiobjective programming problem. To solve a multi objective programming problem with fuzzy coefficients, Wu [52] transformed the problem into a vector optimization problem by applying the embedding theorem and using a suitable linear defuzzification function. Gupta and Mehlawat [23] studied a pair of fuzzy primal-dual linear programming problems and calculated duality results using an aspiration level approach. Their approach is particularly important for FLP where the primal and dual objective values may not be bounded. Peidro et al. [39] used fuzzy sets and developed a FLP to model the supply chain uncertainties.

Hosseinzadeh Lotfi et al. [25] considered full FLP problems where all parameters and variable were triangular fuzzy numbers. They pointed out that there is no method in the literature for finding the fuzzy optimal solution of the generalized FLP problem and proposed a new method to find the fuzzy optimal solution of full FLP problems with equality constraints. They used the concept of the symmetric triangular fuzzy numbers and introduced an approach to defuzzify a general fuzzy quantity. They first approximate the fuzzy triangular numbers to its nearest symmetric triangular numbers and convert every FLP model into two crisp complex LP models. They then use a special ranking for fuzzy numbers to transform their full FLP model into a multiobjective linear programming where all variables and parameters are crisp. In summary, there are generally five FLP classifications in the literature:

– Zimmerman [57] classifieds FLP problems into two categories: symmetrical and nonsymmetrical models. In a symmetrical fuzzy decision there is no difference between the weight of the objectives and constraints while in the asymmetrical fuzzy decision, the objectives and constraints are not equally important and have different weights [2].
– Leung [32] classifies FLP problems into four categories: (1) a precise objective and fuzzy constraints; (2) a fuzzy objective and precise constraints; (3) a fuzzy objective and fuzzy constraints; and (4) robust programming.
– Luhandjula [35] distinguishes three categories of FLP: (1) flexible programming; (2) mathematical programming with fuzzy parameters; and (3) fuzzy stochastic programming.
– Inuiguchi et al. [27] extend the classification to six categories: (1) flexible programming; (2) possibilistic programming; (3) possibilistic linear programming using fuzzy max; (4) robust programming; (5) possibilistic programming with fuzzy preference relations; and (6) possibilistic linear programming with fuzzy goals.
– Kumar et al. [30] proposed two categories: FLP problems with inequality constraints and FLP problems with equality constraints. Some authors (e.g., [1, 11, 24]) have proposed different methods for solving FLP problems with inequality constraints where initially the FLP problem is converted into crisp LP problem and then the obtained crisp LP problem is solved to find the fuzzy optimal solution of the FLP problems. Other authors (e.g., [15, 26]) have proposed methods for solving FLP problems with equality constraints. However, the solutions to FLP problems with equality constraints are generally approximate [26].

Practically, the source of imprecisions in FLP problems differs in different circumstances. This has been the driving force in the development of different FLP problems in the literature. We have identified six different classes of FLP models in the literature according to their sources of fuzziness (see Table 1):

– Class (i)–The FLP problems in this class consider fuzzy numbers for the decision variables and the right-hand-side of the constraints (e.g., [37]).
– Class (ii)–The FLP problems in this class consider fuzzy numbers for the coefficients of the decision variables in the objective function (e.g., [42, 51]).
– Class (iii)–The FLP problems in this class consider fuzzy numbers for the coefficients of the decision variables in the constraints and the right-hand-side of the constraints (e.g., [53]).
– Class (iv)–The FLP problems in this class consider fuzzy numbers for the decision variables, the coefficients of the decision variables in the objective function and the right-hand-side of the constraints (e.g., [21]).
– Class (v)–The FLP problems in this class consider fuzzy numbers for the coefficients of the decision variables in the objective function, the coefficients of the decision variables in the constraints and the

Table 1
The classification of the FLP models in the literature

Class	Decision variables	Coefficients of the decision variables in the objective function	Coefficients of the decision variables in the constraints	The right-hand-side of the constraints	Reference
i	Fuzzy	Precise	Precise	Fuzzy	[37]
ii	Precise	Fuzzy	Precise	Precise	[42, 51]
iii	Precise	Precise	Fuzzy	Fuzzy	[53]
iv	Fuzzy	Fuzzy	Precise	Fuzzy	[21]
v	Precise	Fuzzy	Fuzzy	Fuzzy	[36, 40, 51], Proposed method
vi	Fuzzy	Fuzzy	Fuzzy	Fuzzy	[25, 30]

right-hand-side of the constraints (e.g., [36, 40, 51]).

– Class (vi)–The FLP problems in this class, known as Fully FLP (FFLP) problems, consider fuzzy numbers in the decision variables, the coefficients of the decision variables in the objective function, the coefficients of the decision variables in the constraints and the right-hand-side of the constraints (e.g., [25, 30]).

Each class of FLP problems is developed for special circumstances. While a FFLP model is the general case of FLP and subsumes different FLP models, it may not be suitable for all FLP problems with different assumptions and sources of fuzziness. The FLP model proposed in this study belongs to class (v) where fuzzy numbers are considered for the coefficients of the decision variables in the objective function, the coefficients of the decision variables in the constraints and the right-hand-side of the constraints.

3. Preliminary definitions

In this section, we review some basic definitions of fuzzy sets [29, 58].

Definition 1. Let U be a universe set. A fuzzy set $\tilde{A}$ of U is defined by a membership function $\mu_{\tilde{A}}(x) \to [0, 1]$, where $\mu_{\tilde{A}}(x)$, $\forall x \in U$, indicates the degree of membership of $\tilde{A}$ to U.

Definition 2. A fuzzy subset $\tilde{A}$ of real number R is convex iff

$$\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq (\mu_{\tilde{A}}(x) \wedge \mu_{\tilde{A}}(y)), \quad \forall x, y \in R, \forall \lambda \in [0,\ 1]$$

where "$\wedge$" denotes the minimum operator.

Definition 3. The α-level of fuzzy set $\tilde{A}$, denoted by $\tilde{A}_\alpha$, is the crisp set $\tilde{A}_\alpha = \{x \,|\mu_{\tilde{A}}(x) \geq \alpha\}$. The support of $\tilde{A}$ is the crisp set $Sup(\tilde{A}) = \{x \,|\mu_{\tilde{A}}(x) > 0\}$. $\tilde{A}$ is normal iff $Sup_{x\in U}\mu_{\tilde{A}}(x) = 1$, where U is the universal set.

Definition 4. $\tilde{A}$ is a fuzzy number iff $\tilde{A}$ is a normal and convex fuzzy subset of R.

Definition 5. A fuzzy number $\tilde{A} = (a^l, a^u, \alpha, \beta)$ is called a trapezoidal fuzzy number where a^l and a^u are the central points and $\alpha > 0$ and $\beta > 0$ present, respectively, the right and left fuzziness. The membership function of $\tilde{A}$ can be defined as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{1}{\alpha}(x - a^l + \alpha), & a^l - \alpha \leq x \leq a^l, \\ 1, & a^l \leq x \leq a^u, \\ \frac{1}{\beta}(a^u - x + \beta), & a^u \leq x \leq a^u + \beta, \\ 0, & O.W. \end{cases} \quad (1)$$

Note that the trapezoidal fuzzy number is a triangular fuzzy number if $a^l = a^u = a$, denoted by a triple (a, α, β).

Definition 6. In fuzzy linear programming, the min T-norm is usually applied to assess a linear combination of fuzzy quantities. Thus, for a given set of trapezoidal fuzzy numbers $\tilde{A}_j = (a^l_j, a^u_j, \alpha_j, \beta_j)$ $j = 1, 2, ..., n$, and $\lambda_j \geq 0$, $\sum_{j=1}^{n} \lambda_j \tilde{A}_j$ is explained as follows:

$$\sum_{j=1}^{n} \lambda_j \tilde{A}_j = \left(\sum_{j=1}^{n} \lambda_j a^l_j, \sum_{j=1}^{n} \lambda_j a^u_j, \sum_{j=1}^{n} \lambda_j \alpha_j, \sum_{j=1}^{n} \lambda_j \beta_j \right) \quad (2)$$

Ranking fuzzy numbers plays a pivotal role in decision-making, data analysis, artificial intelligence and FLP. Many studies have investigated various ranking methods [14, 47–49]. Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers, Wu [50] used the following possibility and necessity indices to order the fuzzy numbers:

$$Pos(\tilde{a} \preceq \tilde{b}) = (\tilde{a} \preceq^{Pos} \tilde{b}) = \sup_{u \le v} \min\{u_{\tilde{a}}(u),\ u_{\tilde{b}}(v)\},$$

$$\begin{aligned} Nes(\tilde{a} \prec \tilde{b}) &= (\tilde{a} \prec^{Nes} \tilde{b}) \\ &= \inf_{u \ge v} \max\{1 - u_{\tilde{a}}(u),\ 1 - u_{\tilde{b}}(v)\} \\ &= 1 - \sup_{u \ge v} \min\{u_{\tilde{a}}(u),\ u_{\tilde{b}}(v)\}, \end{aligned} \tag{3}$$

With respect to the level of satisfaction $\alpha \in [0, 1]$, we adopted the definition of Li and Gong [33] that states the above relationships as:

$$Pos(\tilde{a} \preceq \tilde{b}) \ge \alpha \text{ iff } \tilde{a}_{\alpha}^{L} \le \tilde{b}_{\alpha}^{U}$$

$$Nes(\tilde{a} \prec \tilde{b}) \ge \alpha \text{ iff } \tilde{a}_{1-\alpha}^{U} \le \tilde{b}_{1-\alpha}^{L}$$

4. The FLP problem

Fuzzy sets theory proposed by Zadeh [54, 55] and developed by Dubois and Prade [16–18] has been extensively used to solve optimization problems with imprecise, vague, or incomplete data (see [4–7, 9, 10, 28, 31]). Let us consider a generic LP model (4):

$$\begin{aligned} &\text{Maximize} \quad Z(x) = cx \\ &\text{Subject to:} \\ &\qquad Ax \le b, \\ &\qquad x \ge 0. \end{aligned} \tag{4}$$

where $c = (c_1, c_2, ..., c_n)$, $A = [a_{ij}]_{m\times n}$ and $b = (b_1, b_2, ..., b_m)^t$ represent crisp parameters. Imprecisions in the parameters of model (4) can be modeled by fuzzy sets, leading to the following formulation:

$$\begin{aligned} &\text{Maximize} \quad Z(x) = \tilde{c}x \\ &\text{Subject to:} \\ &\qquad \tilde{A}x \le \tilde{b}, \\ &\qquad x \ge 0. \end{aligned} \tag{5}$$

where $\tilde{c} = (\tilde{c}_1, \tilde{c}_2, ..., \tilde{c}_n)$, $\tilde{A} = [\tilde{a}_{ij}]_{m\times n}$ and $\tilde{b} = (\tilde{b}_1, \tilde{b}_2, ..., \tilde{b}_m)^t$ represent the fuzzy parameters involved in the objective function and the constraints, and, $x = (x_1, x_2, ..., x_m)$ represent the crisp decision variables. FLP models in the literature generally either incorporate the imprecisions related to the objective function or the imprecisions related to the technical coefficients. A few studies have incorporated simultaneously the uncertainties and the imprecisions related to the objective function and the technical coefficients. In this paper, we develop a new FLP model with fuzzy parameters in both the objective function and the technical coefficients. The proposed model is inspired by the work of Li and Gong [33] on possibility and necessity relation and the work of Rommelfanger et al. [42] on solving FLP problems with fuzzy parameters in the objective function.

5. The proposed model

In the proposed method, we first use the possibility and necessity relations for fuzzy constraints without considering the fuzzy objective function. We then extend our method to a fuzzy objective function. In this paper, we use trapezoidal fuzzy numbers for the parameters in the constraints and the objective function to increase the simplicity of operations and problem formulations. In the first step, based on possibility and necessity concepts, model (5) can be transformed into models (6) and (7) as follows:

$$\begin{aligned} &\text{Maximize} \quad Z^p(x^p) = \tilde{c}x^p \\ &\text{Subject to:} \\ &\qquad \tilde{A}x^p \prec^{pos} \tilde{b}, \\ &\qquad x^p \ge 0. \end{aligned} \tag{6}$$

$$\begin{aligned} &\text{Maximize} \quad Z^n(x^n) = \tilde{c}x^n \\ &\text{Subject to:} \\ &\qquad \tilde{A}x^n \prec^{nes} \tilde{b}, \\ &\qquad x^n \ge 0. \end{aligned} \tag{7}$$

where $\tilde{c} = (c^l, c^u, \nu, \mu)$, $\tilde{A} = (a^l, a^u, \alpha, \beta)$ and $\tilde{b} = (b^l, b^u, \gamma, \theta)$ are trapezoidal fuzzy numbers. To simplify:

$$\begin{aligned} &\text{Maximize} \quad Z^p(x^p) = \tilde{c}x^p \\ &\text{Subject to:} \\ &\qquad b^u - a^l x^p \ge (h-1)(\alpha x^p + \theta), \\ &\qquad x^p \ge 0. \end{aligned} \tag{8}$$

$$\begin{aligned} &\text{Maximize} \quad Z(x^n) = \tilde{c}x^n \\ &\text{Subject to:} \\ &\qquad b^l - a^u x^n \ge (1-h)(\beta x^n + \gamma), \\ &\qquad x^n \ge 0. \end{aligned} \tag{9}$$

where h is the degree of satisfaction defined on the domain [0, 1].

We then deal with the fuzzy objective function. When an h-cut is applied to the fuzzy objective function, the interval model (10) for a given $h \in [0, 1]$ can be obtained as follows:

$$\begin{aligned} &\text{Maximize} \quad Z_h(x) = x[\underline{c}_h, \bar{c}_h] \\ &\text{Subject to:} \\ &\qquad Const. \end{aligned} \tag{10}$$

We assume that the interpretation of h in the above model is similar to the interpretation of h in models (8) and (9). We thus use the same $h \in [0, 1]$ in both the fuzzy constraint and the fuzzy objective function. In this paper we use a compromise solution to determine all efficient solutions to (10). We assume that the preferences of the decision maker are quasi-monotonously increasing over the objective function values. To find the upper and lower bounds for the objective function value, we make use of the upper and lower limits, respectively, for each h-cut.

$$\begin{aligned} &\text{Maximize} \quad Z_h(x) = \begin{pmatrix} x\underline{c}_h \\ x\bar{c}_h \end{pmatrix} \\ &\text{Subject to:} \\ &\qquad Const. \end{aligned} \tag{11}$$

We then split model (11) into the following two LP models to obtain the optimal solutions:

$$\begin{aligned} &\text{Maximize} \quad Z_h^u(x) = x\bar{c}_h \\ &\text{Subject to:} \\ &\qquad Const. \end{aligned} \tag{12}$$

$$\begin{aligned} &\text{Maximize} \quad Z_h^l(x) = x\underline{c}_h \\ &\text{Subject to:} \\ &\qquad Const. \end{aligned} \tag{13}$$

$\bar{x}^*$ and $\underline{x}^*$ are the optimal solutions to models (12) and (13), respectively, for a given $h \in [0,\ 1]$ and $Z_h^u(\bar{x}^*) \geq Z_h^l(\underline{x}^*)$. Accordingly, $[\bar{Z}_h^{l*}, \bar{Z}_h^{u*}]$ where $\bar{Z}_h^l = Z_h^l(\bar{x}^*)$ and $\bar{Z}_h^u = Z_h^u(\underline{x}^*)$ represent the bounded objective values for each h-cut., along the principles in Zimmermann [57] It is presumed that a decision maker is only eager to accept a solution with following properties:

$$Z_h^l(\underline{x}^*) \geq \bar{Z}_h^l \text{ and } Z_h^u(\bar{x}^*) \geq \bar{Z}_h^u$$

The decision maker then identifies his/her objective with membership functions $f_{Z_\alpha^l}(x)$ and $f_{Z_h^u}(x)$ for each h-cut which express his/her satisfaction with the obtained objective values. In this paper, we use the following linear membership functions for representing the possibility and necessity point of views,

$$f_{Z_h^u}(x) = \begin{cases} \frac{Z_h^u(x) - \bar{Z}_h^u}{Z_h^{u*} - \bar{Z}_h^u}, & \bar{Z}_h^u \leq Z_h^u(x) \leq Z_h^{u*} \\ 0, & O.W. \end{cases}$$

$$f_{Z_h^l}(x) = \begin{cases} \frac{Z_h^l(x) - \bar{Z}_h^l}{Z_h^{l*} - \bar{Z}_h^l}, & \bar{Z}_h^l \leq Z_h^l(x) \leq Z_h^{l*} \\ 0, & O.W. \end{cases} \tag{14}$$

The above-defined function cannot be used when $\bar{Z}_h^u = Z_h^{u*}$. That is why it is important to note that if more information is available, it can be incorporated into the model by using other membership functions describing the decision maker's satisfaction. Next, the following fuzzy optimization system is formulated to take into consideration all information concerning all α:

$$\begin{aligned} &\text{Maximize} \quad \begin{pmatrix} f_{Z_\alpha^l}(x) \\ f_{Z_\alpha^u}(x) \end{pmatrix} \\ &\text{Subject to:} \\ &\qquad Const. \end{aligned} \tag{15}$$

By using the Zadeh [55]'s minimum operator as the preference function and the linear membership functions, a compromise solution can be calculated using the approach proposed by Rommelfanger et al. [42] and Negoita and Sularia [38]:

$$\begin{aligned} &\text{Maximize} \quad \omega \\ &\text{Subject to:} \\ &\qquad (Z_h^{l^*} - \bar{Z}_h^l)\omega - Z_h^l(x) \leq -\bar{Z}_h^l, \\ &\qquad (Z_h^{u^*} - \bar{Z}_h^u)\omega - Z_h^u(x) \leq -\bar{Z}_h^u, \\ &\qquad Const. \end{aligned} \tag{16}$$

Note that when dealing with higher h-cut, there is the risk that both extreme objective functions Z_h^l and Z_h^u will have the same optimal solution. Therefore, the maximum values of h should not be chosen, particularly level $h = 1$ can be avoided. Combining the transformation on the fuzzy constraints proposed in (8) and (9) with model (16) yields:

$$\begin{aligned}
&\text{Maximize} \quad \omega_h^p \\
&\text{Subject to:} \\
&\qquad (Z_h^{l^*} - \bar{Z}_h^l)\omega_h^p - Z_h^l(x^p) \leq -\bar{Z}_h^l, \\
&\qquad (Z_h^{u^*} - \bar{Z}_h^u)\omega_h^p - Z_h^u(x^p) \leq -\bar{Z}_h^u, \\
&\qquad b^u - a^l x^p \geq (h-1)(\alpha x^p + \theta), \\
&\qquad x^p \geq 0. \qquad (17)
\end{aligned}$$

$$\begin{aligned}
&\text{Maximize} \quad \omega_h^n \\
&\text{Subject to:} \\
&\qquad (Z_h^{l^*} - \bar{Z}_h^l)\omega_h^n - Z_h^l(x^n) \leq -\bar{Z}_h^l, \\
&\qquad (Z_h^{u^*} - \bar{Z}_h^u)\omega_h^n - Z_h^u(x^n) \leq -\bar{Z}_h^u, \\
&\qquad b^l - a^u x^n \geq (1-h)(\beta x^n + \gamma), \\
&\qquad x^n \geq 0. \qquad (18)
\end{aligned}$$

In conclusion, we calculate the objective function values of the FLP model (5) from the possibility and necessity viewpoints for each h-cut using models (17) and (18). We find the optimal solution by taking an arithmetic average of these objective function values for all h-cuts as

$$Z_{Ave}^* = \frac{1}{4h}\sum_{h=1}^{5}(Z_h^l(x^{n*}) + Z_h^u(x^{n*}) + Z_h^l(x^{p*}) + Z_h^u(x^{p*}))$$

where h is the number of cuts.

6. Stability of the results

In order to identify the stability of the result in the proposed method, we discuss the bounded objective values for each h-cut using a linear membership function. Without loss of generality, we use the following scaling function for the possibility and necessity relations:

$$f_{Z_h^e}(x^*) = \frac{Z_h^e(x^*) - \bar{Z}_h^e}{Z_h^{e*} - \bar{Z}_h^e}, \quad e = u, l \qquad (19)$$

The scaling metric corresponds to a linear approximation of the satisfaction level of the decision maker about the objective function values for each h-cut.

7. Numerical examples

In this section, we first use the numerical example of Rommelfanger et al. [42] with an additional fuzzy constraint to demonstrate the applicability of our method. We then use the numerical example of Ramík [40] to compare his results with ours and to establish the pros and cons of solving a FLP with the possibility/necessity approach.

7.1. Example 1

Consider the numerical example of Rommelfanger et al. [42] with an additional fuzzy constraint to demonstrate the applicability of the proposed method and exhibit the efficacy of the procedures and algorithms. Let us consider:

$$\begin{aligned}
&\text{Maximize} \quad \tilde{c}_1 x_1 + \tilde{c}_2 x_2 \\
&\text{Subject to:} \\
&\qquad 1x_1 + 4x_2 \leq 100, \quad 1x_1 + 3x_2 \leq 76, \\
&\qquad 3x_1 + 5x_2 \leq 138, \quad 3x_1 + 4x_2 \leq 120, \\
&\qquad 1x_1 + 1x_2 \leq 36, \quad 3x_1 + 2x_2 \leq 103, \\
&\qquad 1x_1 + 2x_2 \leq 53, \quad 7x_1 + 8x_2 \leq 260, \\
&\qquad 2x_1 + 1x_2 \leq 68, \quad \tilde{a}_1 x_1 + \tilde{a}_2 x_2 \leq \tilde{b}, \\
&\qquad x_1, x_2 \geq 0. \qquad (20)
\end{aligned}$$

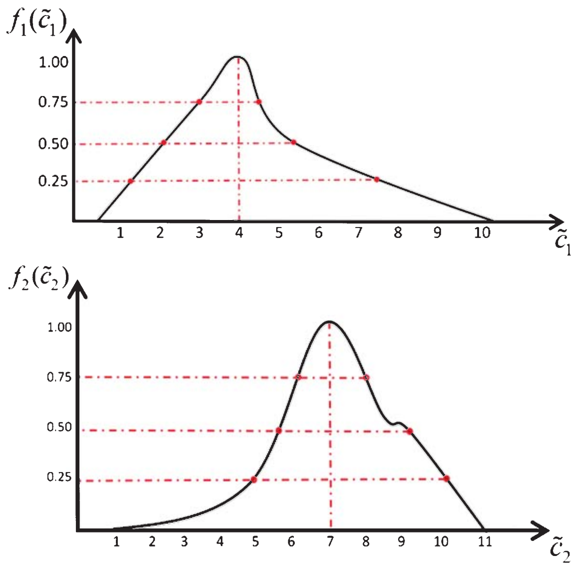

Fig. 1. The membership functions of $\tilde{c}_1$ and $\tilde{c}_2$.

where $\tilde{c}_1$ and $\tilde{c}_2$ are fuzzy numbers and their membership functions are depicted in Fig. 1.

Furthermore, let us define $\tilde{a}_1 = (2, 3, 1, 2)$, $\tilde{a}_2 = (1, 2, 2, 1)$ and $\tilde{b} = (2, 3, 1, 2)$ in the constraint $(\tilde{a}_1 x_1 + \tilde{a}_2 x_2 \le \tilde{b})$ as fuzzy trapezoidal numbers. Note that some coefficient values in model (21) are crisp parameters, as often found in real applications. We take into consideration five cuts involving {0, 0.25, 0.5, 0.75, 1} for analyzing this problem. Table 2 presents the interval value for two fuzzy coefficients of the objective function with respect to the aforementioned cuts.

The proposed possibility and necessity models correspond to the following models:

$$\text{Maximize } [\underline{c}_{1(h)}, \bar{c}_{1(h)}]x_1^p + [\underline{c}_{2(h)}, \bar{c}_{2(h)}]x_2^p$$

Subject to:

$$\begin{aligned} &1x_1^p + 4x_2^p \le 100, \quad 1x_1^p + 3x_2^p \le 76,\\ &3x_1^p + 5x_2^p \le 138, \quad 3x_1^p + 4x_2^p \le 120,\\ &1x_1^p + 1x_2^p \le 36, \quad 3x_1^p + 2x_2^p \le 103,\\ &1x_1^p + 2x_2^p \le 53, \quad 7x_1^p + 8x_2^p \le 260,\\ &2x_1^p + 1x_2^p \le 68,\\ &(h+1)x_1^p + 2hx_2^p \le 5 - 2h,\\ &x_1^p, x_2^p \ge 0. \end{aligned} \tag{21}$$

$$\text{Maximize } [\underline{c}_{1(h)}, \bar{c}_{1(h)}]x_1^n + [\underline{c}_{2(h)}, \bar{c}_{2(h)}]x_2^n$$

Subject to:

$$\begin{aligned} &1x_1^n + 4x_2^n \le 100, \quad 1x_1^n + 3x_2^n \le 76,\\ &3x_1^n + 5x_2^n \le 138, \quad 3x_1^n + 4x_2^n \le 120,\\ &1x_1^n + 1x_2^n \le 36, \quad 3x_1^n + 2x_2^n \le 103,\\ &1x_1^n + 2x_2^n \le 53, \quad 7x_1^n + 8x_2^n \le 260,\\ &2x_1^n + 1x_2^n \le 68,\\ &(5-2h)x_1^n + (3-h)x_2^n \le 4 - 2h,\\ &x_1^n, x_2^n \ge 0. \end{aligned} \tag{22}$$

On the basis of models (12) and (13), we solve the above mentioned models. The results form the possibility and necessity viewpoints are reported in Tables 3 and 4, respectively. Note that in Table 2 (or 3), $\bar{x}^{p*}$ (or $\bar{x}^{n*}$) and $\underline{x}^{p*}$ (or $\underline{x}^{n*}$) represent the upper and lower optimal solutions of model (21) (or 22) for various h-cuts. We then compute the minimum objective values, $\bar{Z}_h^l$ and $\bar{Z}_h^u$, proposed by Zimmermann [57] for each h-cut presented in Table 5 and it shows that $Z_h^l(\underline{x}^*) \ge \bar{Z}_h^l$ and $Z_h^u(\bar{x}^*) \ge \bar{Z}_h^u$. We use the membership function (14) proposed by Rommelfanger et al. [42] for all h-cuts in the possibility and necessity cases. The membership functions $f_{Z_\alpha^l}(x_h)$ and $f_{Z_h^u}(x_h)$ approximate the preference function of the decision maker. For example when $h = 0.25$, $f_{Z_{0.25}^l}(x^p)$ and $f_{Z_{0.25}^u}(x^p)$ are expressed as follows for the *possibility* cases:

Table 2
The interval values of $\tilde{c}_1$ and $\tilde{c}_2$ for different h-cuts

h-cut	$[\underline{c}_{1(h)}, \bar{c}_{1(h)}]$	$[\underline{c}_{2(h)}, \bar{c}_{2(h)}]$
0	[1.4, 11]	[0.5, 10]
0.25	[5, 10]	[1, 7]
0.5	[6, 9]	[2, 5]
0.75	[6.5, 7.5]	[3.2, 4.5]
1	7	4

Table 3
The results from the possibility viewpoint

h-cut	Upper bound			Lower bound		
	$\bar{x}_1^{p*}$	$\bar{x}_2^{p*}$	$Z_h^{u*} = Z_h^u(\bar{x}^{p*})$	$\underline{x}_1^{p*}$	$\underline{x}_2^{p*}$	$Z_h^{l*} = Z_h^l(\underline{x}^{p*})$
0	5	23.67	310.33	5	23.67	35.63
0.25	0	9	90	0	9	45
0.5	0	4	36	0	4	24
0.75	0	2.33	17.5	0	2.33	15.67
1	0	1.5	10.5	0	1.5	10.5

Table 4
The results from the necessity viewpoint

h-cut	Upper bound			Lower bound		
	$\bar{x}_1^*$	$\bar{x}_2^*$	$Z_h^{u*} = Z_h^u(\bar{x}_h^*)$	$\underline{x}_1^*$	$\underline{x}_2^*$	$Z_h^{l*} = Z_h^l(\underline{x}_h^*)$
0	0	1.33	14.67	0	1.33	1.87
0.25	0	1.25	12.73	0	1.27	6.36
0.5	0	1.2	10.8	0	1.2	7.2
0.75	0	1.11	8.33	0	1.11	7.22
1	0	1	7	0	1	7

Table 5
The minimum objective values

h-cut	Possibility		Necessity	
	$\bar{Z}_h^u = Z_h^u(\underline{x}_h^*)$	$\bar{Z}_h^l = Z_h^l(\bar{x}_h^*)$	$\bar{Z}_h^u = Z_h^u(\underline{x}_h^*)$	$\bar{Z}_h^l = Z_h^l(\bar{x}_h^*)$
0	291.7	18.835	13.3	0.665
0.25	63	9	8.89	1.25
0.5	20	8	6	2.4
0.75	10.485	7.456	4.995	3.552
1	6	6	4	4

Table 6
The results from two different viewpoints

h-cut	Possibility			Necessity		
	x_1^*	x_2^*	w	x_1^*	x_2^*	w
0	5	23.67	2.243	0.8	0	1.481
0.25	3.6	0	0.75	0.778	0	0.457
0.5	2.667	0	1.5	0.75	0	1.437
0.75	2	0	2.49	0.714	0	2.234
1	1.5	0	4.714	0.667	0	2.889

$$f_{Z^l_{0.25}}(x^p) = \begin{cases} \frac{Z^l_{0.25}(x^p)-9}{45-9}, & 9 \le Z^l_{0.25}(x^p) \le 45 \\ 0, & O.W. \end{cases}$$

$$f_{Z^u_{0.25}}(x^p) = \begin{cases} \frac{Z^u_{0.25}(x^p)-63}{90-63}, & 63 \le Z^u_{0.25}(x^p) \le 90 \\ 0, & O.W. \end{cases}$$

Similarly, $f_{Z^l_{0.25}}(x^n)$ and $f_{Z^u_{0.25}}(x^n)$ for $h = 0.25$ are expressed as follows for the *necessity* cases:

$$f_{Z^u_h}(x^n) = \begin{cases} \frac{Z^u_h(x^n)-8.89}{12.73-8.89}, & 8.89 \le Z^u_h(x^n) \le 12.73 \\ 0, & O.W. \end{cases}$$

$$f_{Z^l_h}(x^n) = \begin{cases} \frac{Z^l_h(x^n)-1.25}{6.36-1.25}, & 1.25 \le Z^l_h(x^n) \le 6.36 \\ 0, & O.W. \end{cases}$$

Ultimately models (17) and (18) are evolved for this example and Table 6 shows the optimal solution for various h-cuts. For instance, in the possibility case, the following proposed model under $h = 0.25$ is formulated:

Maximize $\omega^p_{0.25}$

Subject to:

$$\begin{aligned} &(45-9)\omega^p_{0.25} - Z^l_{0.25}(x^p) \le 9, \\ &(90-63)\omega^p_{0.25} - Z^u_{0.25}(x^p) \le 63, \\ &1x_1^p + 4x_2^p \le 100, \quad 1x_1^p + 3x_2^p \le 76, \\ &3x_1^p + 5x_2^p \le 138, \quad 3x_1^p + 4x_2^p \le 120, \\ &1x_1^p + 1x_2^p \le 36, \quad 3x_1^p + 2x_2^p \le 103, \\ &1x_1^p + 2x_2^p \le 53, \quad 7x_1^p + 8x_2^p \le 260, \\ &2x_1^p + 1x_2^p \le 68, \\ &(0.25+1)x_1^p + 0.5x_2^p \le 5 - 0.5, \\ &x_1^p, x_2^p \ge 0. \end{aligned}$$

where $Z^l_h(x^p)$ and $Z^u_h(x^p)$ are $(5x_1^p + x_2^p)$ and $(10x_1^p + 7x_2^p)$, respectively. Analogously, the proposed model for the necessity case with $h = 0.25$ is

Table 7
The satisfaction level

h-cut	Possibility		Necessity	
	$f_{Z^l_h}(x^*)$	$f_{Z^u_h}(x^*)$	$f_{Z^l_h}(x^*)$	$f_{Z^u_h}(x^*)$
0	0	0	0.378	−3.285
0.25	0.25	1	0.517	−0.289
0.5	0.5	0.25	0.437	0.156
0.75	0.675	0.644	0.297	0.102
1	1	1	0.216	0.216

Table 8
The attained objective function values

h-cut	Possibility		Necessity	
	$Z^l_h(x^*)$	$Z^u_h(x^*)$	$Z^l_h(x^*)$	$Z^u_h(x^*)$
0	18.835	291.7	1.12	8.8
0.25	18	36	3.89	7.78
0.5	16.002	24.003	4.5	6.75
0.75	13	15	4.641	5.335
1	10.5	10.5	4.648	4.648

Maximize $\omega^n_{0.25}$

Subject to:

$$\begin{aligned} &(6.36-1.25)\omega^n_{0.25} - Z^l_{0.25}(x^n) \le 1.25, \\ &(12.73-8.89)\omega^n_{0.25} - Z^u_{0.25}(x^n) \le 8.89, \\ &1x_1^n + 4x_2^n \le 100, \quad 1x_1^n + 3x_2^n \le 76, \\ &3x_1^n + 5x_2^n \le 138, \quad 3x_1^n + 4x_2^n \le 120, \\ &1x_1^n + 1x_2^n \le 36, \quad 3x_1^n + 2x_2^n \le 103, \\ &1x_1^n + 2x_2^n \le 53, \quad 7x_1^n + 8x_2^n \le 260, \\ &2x_1^n + 1x_2^n \le 68, \\ &(5-0.5)x_1^n + (3-0.25)x_2^n \le 4 - 0.5, \\ &x_1^n, x_2^n \ge 0. \end{aligned}$$

where $Z^l_h(x^n)$ and $Z^u_h(x^n)$ are $(5x_1^n + x_2^n)$ and $(10x_1^n + 7x_2^n)$, respectively. The average objective value is 23.325.

Here, we implement the stability analysis of the attained results. Table 7 presents the corresponding results for h-cuts 0, 0.25, 0.5, 0.75 and 1. The computed values for $Z^e_h(x^*)$ are reported in Table 8. Notice that in the necessity aspect $f_{Z^u_h}(x^*)$ for $h = 0$ and 0.25 lead to the non-feasible measures. By using different h-cuts in the possibility and necessity viewpoints, we nearly satisfy all aspects of a calculated solution. The decision maker is able to select a given membership level such as 0.5.

7.2. Example 2

We use the following problem proposed by Ramík [40] to compare his results with ours and to establish the pros and cons of solving a FLP with possibility/necessity approach.

$$\text{Maximize}\quad \tilde{c}_1 x_1 + \tilde{c}_1 x_1$$

$$\text{Subject to:}$$

$$\tilde{a}_{11}x_1 + \tilde{a}_{12}x_2 \leq \tilde{b}_1,$$

$$\tilde{a}_{21}x_1 + \tilde{a}_{22}x_2 \leq \tilde{b}_2,$$

$$x_1, x_2 \geq 0.$$

where

$$\tilde{c}_1 = (4, 4, 1, 1), \quad \tilde{c}_1 = (4, 4, 2, 2)$$

$$\tilde{a}_{11} = (3, 3, 2, 2), \quad \tilde{a}_{12} = (1, 1, 0, 0)$$

$$\tilde{a}_{21} = (3, 3, 2, 2), \quad \tilde{a}_{22} = (3, 3, 0, 0)$$

$$\tilde{b}_1 = (11, 11, 3, 3), \quad \tilde{b}_2 = (12, 12, 1, 3)$$

Ramík [40] converted the above fuzzy program to the following linear models based on the fuzzy relations:

$$\text{Maximize}\quad (3+h)x_1 + (2+2h)x_2$$

$$\text{Subject to:}$$

$$(5-2h)x_1 + x_2 \leq 8 + 3h,$$

$$(3-h)x_1 + 3x_2 \leq 11 + h,$$

$$x_1, x_2 \geq 0. \tag{23}$$

$$\text{Minimize}\quad (8+3h)y_1 + (11+h)y_2$$

$$\text{Subject to:}$$

$$(5-2h)y_1 + (3-h)y_2 \geq 3 + h,$$

$$y_1 + 3y_2 \geq 2 + 2h,$$

$$y_1, y_2 \geq 0. \tag{24}$$

Considering $\alpha = 0.7$ as an appropriate level of satisfaction, the optimal solution of model (23) is $x_1^* = 4.15$ and $x_2^* = 1.95$ (with an optimal objective function value of 26.80). Similarly, the optimal solution of model (24) is $y_1^* = 0.92$ and $y_2^* = 1.23$ (with an optimal objective function value of 26.80).

We then use our model to solve the above problem and compare the results obtained in the two models. The possibility and necessity models for this problem are formulated as follows:

Table 9
The interval values of $\tilde{c}_1$ and $\tilde{c}_2$ for different h-cuts

$[\underline{c}_{2(h)}, \bar{c}_{2(h)}]$	$[\underline{c}_{1(h)}, \bar{c}_{1(h)}]$	h-cut
[2, 6]	[3, 5]	0
[2.5, 5.5]	[3.25, 4.75]	0.25
[3, 5]	[3.5, 4.5]	0.5
[3.5, 4.5]	[3.75, 4.25]	0.75

Table 10
The results of model (25)

h-cut	Upper bound			Lower bound		
	$\bar{x}_1^{p*}$	$\bar{x}_2^{p*}$	$Z_h^{u*} = Z_h^u(\bar{x}^{p*})$	$\underline{x}_1^{p*}$	$\underline{x}_2^{p*}$	$Z_h^{l*} = Z_h^l(\underline{x}^{p*})$
0	14	0	70	14	0	42
0.25	9.5	0	47.125	9.5	0	30.875
0.5	6.75	0	30.375	6.75	0	23.625
0.75	5.1	0	21.675	5.1	0	19.125
1	4	0	16	4	0	16

$$\text{Maximize}\quad [\underline{c}_{1(h)}, \bar{c}_{1(h)}]x_1^p + [\underline{c}_{2(h)}, \bar{c}_{2(h)}]x_2^p$$

$$\text{Subject to:}$$

$$(1-2h)x_1^p + 3x_2^p \leq 14 - 3h,$$

$$(1+2h)x_1^p + 3x_2^p \leq 15 - 3h,$$

$$x_1^p, x_2^p \geq 0. \tag{25}$$

$$\text{Maximize}\quad [\underline{c}_{1(h)}, \bar{c}_{1(h)}]x_1^n + [\underline{c}_{2(h)}, \bar{c}_{2(h)}]x_2^n$$

$$\text{Subject to:}$$

$$(5-2h)x_1^n + 3x_2^n \leq 8 + 3h,$$

$$(5-2h)x_1^n + 3x_2^n \leq 11 + h,$$

$$w_1^n, w_2^n \geq 0. \tag{26}$$

where $[\underline{c}_{1(h)}, \bar{c}_{1(h)}]$ and $[\underline{c}_{2(h)}, \bar{c}_{2(h)}]$ are respectively, the lower and upper bounds of the variables w_1 and w_2 by means of four h-cuts (i.e., 0, 0.25, 0.5 and 0.75) presented in Table 9 as follows:

The optimal solutions of models (25) and (26) are reported in Tables 10 and 11.

We use the following membership function (14) for all h-cuts to determine a compromise solution in this example.

$$f_{Z_h^u}(x) = \begin{cases} \frac{Z_h^u(x) - Z_h^{l*}}{Z_h^{u*} - Z_h^{l*}}, & Z_h^{l*} \leq Z_h^u(x) \leq Z_h^{u*} \\ 0, & O.W. \end{cases}$$

Table 11
The results of model (26)

h-cut	Upper bound			Lower bound		
	$\bar{x}_1^*$	$\bar{x}_2^*$	$Z_h^{u*} = Z_h^u(\bar{x}_h^*)$	$\underline{x}_1^*$	$\underline{x}_2^*$	$Z_h^{l*} = Z_h^l(\underline{x}_h^*)$
0	0	2.667	16	0	2.667	5.333
0.25	0	2.917	16.042	0	2.917	7.292
0.5	0	3.167	15.833	0	3.167	9.5
0.75	0	3.417	15.375	0	3.417	11.958
1	0	3.667	14.667	0	3.667	14.667

Table 12
The results from the possibility and necessity viewpoints

h-cut	Possibility			Necessity		
	x_1^*	x_2^*	w	x_1^*	x_2^*	w
0	12	0	0.643	0	2.667	0.816
0.25	9.5	0	0.877	0	2.917	0.999
0.5	6.75	0	1	0	3.167	1
0.75	5.1	0	1	0	3.417	1

As a result, we express the following models from the possibility and necessity viewpoints:

$$\text{Maximize } \omega_h^p$$

$$\text{Subject to:}$$

$$\begin{aligned}
&(Z_h^{u*} - Z_h^{l*})\omega_h^p - Z_h^u(x) \le -Z_h^{l*}\\
&(1-2h)x_1^p + 3x_2^p \le 14 - 3h,\\
&(1+2h)x_1^p + 3x_2^p \le 15 - 3h,\\
&x_1^p, x_2^p \ge 0.
\end{aligned}$$

$$\text{Maximize } \omega_h^n$$

$$\text{Subject to:}$$

$$\begin{aligned}
&(Z_h^{u*} - Z_h^{l*})\omega_h^n - Z_h^u(x) \le -Z_h^{l*}\\
&(5-2h)x_1^n + 3x_2^n \le 8 + 3h,\\
&(5-2h)x_1^n + 3x_2^n \le 11 + h,\\
&x_1^n, x_2^n \ge 0.
\end{aligned}$$

Table 12 shows the optimal results of the above models for each h-cut. Based on the Table 12, the optimal value of objective function is 22.759 that is acceptable in comparison with Ramík [40]'s method.

8. Conclusions and future research directions

In spite of the numerous FLP models proposed over the past decade, very few real-world applications are reported. Two explanations could be the methodological complexities and restrictive assumptions of the models. In contrast, the method proposed in this study addresses a fairly comprehensive class of problems with fuzziness in the coefficients of the decision variables in the objective function, the coefficients of the decision variables in the constraints and the right-hand-side of the constraints. The approach essentially draws on the construction of upper and lower bounds for the objective function value, valid for any monotonous preference function. In spite of its comprehensiveness, the proposed model is straight-forward and applicable to a wide range of real-world problems such as supply chain management, performance evaluation by means of data envelopment analysis, marketing management, failure mode and effect analysis and product development.

Future research will concentrate on the comparison of results obtained with those that might be obtained with other methods. The proposed FLP approach may be readily extended to fuzzy non-linear optimization problems with multiple objectives where the vagueness or impreciseness appears in all the components of the optimization problem. Finally, the work may also contribute to the development of platforms for experimental validation of decision making preferences and decision support systems.

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