Fuzzy free disposal hull models under possibility and credibility measures

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Abstract: The free disposal hull (FDH) models are used as an alternative to data envelopment analysis (DEA) models for performance measurement and efficiency assessment. The conventional FDH models are used to evaluate the performance of a set of firms or decision-making units (DMUs) using deterministic input and output data. However, the input and output data in the real-life performance evaluation problems are often imprecise and ambiguous. The impreciseness and ambiguity associated with the input and output data in FDH can be represented with fuzzy variables. In this paper, the concept of chance-constrained programming is used to develop FDH models with various returns to scale assumptions, including variable returns to scale (VRS), variable non-increasing returns to scale (NIRS), variable non-decreasing returns to scale (NDRS), and constant returns to scale (CRS), for efficient DMUs with fuzzy data. We propose two fuzzy FDH models with respect to possibility and expected value (credibility approach) constraints. Finally, a numerical example is presented to demonstrate the efficacy of the proposed procedures and algorithms.

Keywords: data envelopment analysis; DEA; free disposal hull; FDH; credibility; possibility; fuzzy set.

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1 Introduction

Data envelopment analysis (DEA), introduced by Charnes et al. (1978), is a powerful mathematical method that utilises linear programming (LP) to determine the relative efficiencies of a set of functionally similar decision-making units (DMUs). A DMU is considered efficient when no other DMU can produce more outputs using an equal or lesser amount of inputs. The DEA generalises the usual efficiency measurement from a single-input single-output ratio to a multiple-input multiple-output ratio by using a ratio of the weighted sum of outputs to the weighted sum of inputs. A score of one is assigned to the frontier (efficient) units. The frontier units in DEA are those with maximum output levels for given input levels or with minimum input levels for given output levels.

The free disposal hull (FDH) model, introduced by Deprins et al. (1984), is designed as an alternative to DEA, where only the strong (free) disposability of inputs and outputs is assumed. It is used to establish a best practice group amongst a set of observed units and to identify the units that are inefficient when compared to the best practice group. FDH relaxes the convexity assumption in the variable returns to scale (VRS) DEA models. The original FDH model proposed by Deprins et al. (1984) has been further explored by Tulkens (1993), Tulkens and Vanden Eeckaut (1995a, 1995b), and Kerstens and Vanden Eeckaut (1999).

FDH models are traditionally based on a VRS assumption and are computed via enumeration algorithms (Tulkens, 1993). Kerstens and Vanden Eeckaut (1999) introduced returns to scale in the FDH technology but they did not develop any linear programmes for computing the technical inefficiency. Agrell and Tind (2001) derived a linear programme for the FDH model but without returns to scale assumptions and with a radial output distance function. Finally, Leleu (2006) provided a LP framework to compute the technical inefficiency for FDH technology under various returns to scale assumptions.

The conventional DEA and FDH evaluation methods are based on well-defined, precise and deterministic data for the production set. However, this assumption may not be true in the real-world situations since the precise measurement of data is not possible or expensive in many applications. A few researchers have proposed various methods for dealing with imprecise and ambiguous data in FDH. Jahanshahloo et al. (2004) considered the efficiency analysis in the FDH model where the amount of inputs and outputs were located within the bounded intervals. They then converted the non-linear model into a LP model. Farnoosh et al. (2011) studied FDH models with stochastic data. However, many real-life problems use linguistic data such as good, fair or poor that cannot be mapped to interval data. Fuzzy set theory can be used to deal with the imprecise inputs and outputs in fuzzy DEA and FDH problems.

Fuzzy set theory was introduced by Zadeh (1965) as a means of representing and manipulating data that was not precise, but rather fuzzy. It was specifically designed to mathematically represent uncertainty and vagueness and to provide formalised tools for dealing with the imprecision intrinsic to many problems.

The fuzzy input and output variations in DEA have been studied by many researchers such as Sengupta (1992a, 1992b), Triantis and Girod (1998), Guo and Tanaka (2001), Lertworasirikul et al. (2003), León et al. (2003), Kao and Liu (2000a, 2000b, 2003, 2005), Pei-Huang (2006), Liu (2008), Liu and Chuang (2009), Zhou et al. (2012), Wang and Chin (2011), Majid Zerafat Angiz et al. (2012), and Pendharkar (2012).

Sengupta (1992a, 1992b) was the first to introduce a fuzzy mathematical programming approach in which fuzziness was incorporated into the DEA model by defining tolerance levels on both the objective function and constraint violations. Guo and Tanaka (2001) presented a fuzzy CCR model by converting fuzzy constraints such as fuzzy equalities and fuzzy inequalities into crisp constraints by predefining a possibility level and using the comparison rule for fuzzy numbers Kao and Liu (2000a, 2000b, 2003, 2005) transformed fuzzy input and fuzzy output into intervals by using α-level sets and built a family of crisp DEA models for the intervals.

Using an α-cut method proposed by Sakawa (1993), Lertworasirikul et al. (2003) proposed the possibility and necessity methods for solving a fuzzy DEA-CCR model. They introduced a possibility approach in which the constraints were treated as fuzzy events and transformed fuzzy DEA models into possibility DEA models by using possibility measures of the fuzzy events (fuzzy constraints). The possibility theory is based on two dual fuzzy measures – possibility and necessity measures (Dubois and Prade, 1988; Klir, 1999; Zadeh, 1978).

Liu (2008) and Liu and Chuang (2009) developed a fuzzy DEA/assurance region model for the selection of flexible manufacturing systems and the assessment of university libraries, respectively. Zhou et al. (2012) proposed a generalised fuzzy DEA model with assurance regions, whose lower and upper bounds at given levels could be obtained. Wang and Chin (2011) proposed a 'fuzzy expected value approach' for DEA in which fuzzy inputs and fuzzy outputs were first weighted respectively, and their expected values then used to measure the optimistic and pessimistic efficiencies of DMUs in fuzzy environments.

Majid Zerafat Angiz et al. (2012) introduced an alternative LP model that included some uncertainty information from the intervals within the α-cut approach and proposed the concept of 'local a-level' to develop a multi-objective LP to measure the efficiency of DMUs under uncertainty. Pei-Huang (2006) used multiple criteria ranking by fuzzy DEA. León et al. (2003) developed some fuzzy versions of the classical DEA models (in particular, the BCC model) by using ranking methods based on the comparison of α-cuts.

Pendharkar (2012) developed a fuzzy classification system using DEA and illustrated its application using a simple graduate admissions decision-making problem. Triantis and Girod (1998) used the approach proposed by Carlsson and Korhonen (1986) to formulate the fuzzy BCC and FDH models which were radial measures of efficiency.

Although research on FDH with fuzzy data has been limited, there are a number of recent studies dealing with fuzzy data in the FDH literature. To the best of our knowledge, so far, there is very little research in fuzzy FDH. In this paper, the concept of chance-constrained programming is used to develop FDH models with various returns to scale assumptions, including VRS, variable non-increasing returns to scale (NIRS), variable non-decreasing returns to scale (NDRS), and constant returns to scale (CRS), for efficient DMUs with fuzzy data. We propose two fuzzy DEA models with respect to the possibility and expected value (credibility) approaches.

The remainder of the paper is organised as follows. In the next section, we present some preliminaries and definitions for fuzzy sets. In Section 3, we present the basic FDH model. In Section 4, we present the fuzzy FDH model. In Section 5, we present the result of a numerical example to demonstrate the efficacy of the procedures and algorithms. Section 6 presents our conclusion and future research directions.

2 Background on fuzzy set theory

In this section, some basic concepts on fuzzy sets are given as follows:

Definition 1 (Dubois and Prade, 1980): Let U be a universe set involving a classical set of objects. A fuzzy set $\tilde{A}$ of U is defined by a membership function $\mu_{\tilde{A}}(x) \to [0, 1]$, where $\mu_{\tilde{A}}(x)$, $\forall x \in U$, indicates the degree of membership of $\tilde{A}$ to U.

Definition 2 (Dubois and Prade, 1980): The α-cut of a fuzzy set $\tilde{A}$, $\tilde{A}_\alpha$, is the crisp set of elements belonging to $\tilde{A}$ at least to the degree α defined as $\tilde{A}_\alpha = \{x \in U \mid \mu_{\tilde{A}}(x) \geq \alpha\}$.

Definition 3 (Zimmermann, 1996): A fuzzy subset $\tilde{A}$ of real number R is convex if and only if $\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min(\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)), \forall x, y \in R, \forall \lambda \in [0, 1]$. Alternatively, a fuzzy set is convex if all α-cuts are convex.

Definition 4 (Dubois and Prade, 1980): A fuzzy number of generalised left and right type is denoted by $\tilde{A} = (\alpha, m_1, m_2, \beta)_{LR}$ where α and β are the (non-negative) left and right spreads, respectively, and m_1 and m_2 are the mean values of $\tilde{A}$. The membership function of $\tilde{A}$ can be expressed as

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\dfrac{m_1 - x}{\alpha}\right), & x \leq m_1, \\ 1, & m_1 \leq x \leq m_2 \\ R\left(\dfrac{x - m_2}{\beta}\right), & x \geq m_2. \end{cases} \tag{1}$$

where L and R are the left and right functions, respectively. In particular, suppose that:

$$L(x) = R(x) = \begin{cases} 1 - x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$\tilde{A} = (\alpha, m_1, m_2, \beta)_{LR} = (\alpha, m_1, m_2, \beta) = (m_1 - \alpha, m_1, m_2, m_2 + \beta)$ is then called a trapezoidal fuzzy number. Also, if $m_1 = m_2 = m$, $\tilde{A} = (\alpha, m, \beta)_{LR} = (\alpha, m, \beta)$ $= (m - \alpha, m, m + \beta)$ is called a triangular fuzzy number.

Definition 5 (fuzzy arithmetic) (Dubois and Prade, 1980): Let $\tilde{A} = (\alpha, m_1, m_2, \beta)_{LR}$ and $\tilde{B} = (\bar{\alpha}, \bar{m}_1, \bar{m}_2, \bar{\beta})_{LR}$ be two positive fuzzy numbers. Then, the fuzzy arithmetic of $\tilde{A}$ and $\tilde{B}$ can be defined as follows:

- *Addition:*

$$(\alpha, m_1, m_2, \beta)_{LR} + (\bar{\alpha}, \bar{m}_1, \bar{m}_2, \bar{\beta})_{LR} = (\alpha + \bar{\alpha}, m_1 + \bar{m}_1, m_2 + \bar{m}_2, \beta + \bar{\beta})_{LR}$$

- *Subtraction:*

$$(\alpha, m_1, m_2, \beta)_{LR} - (\bar{\alpha}, \bar{m}_1, \bar{m}_2, \bar{\beta})_{LR} = (\alpha + \bar{\beta}, m_1 - \bar{m}_2, m_2 - \bar{m}_1, \beta + \bar{\alpha})_{LR}$$

Definition 6 (Zadeh, 1978; Zimmermann, 1996): Let $(\Theta, P(\Theta), Pos)$ be a possibility space where Θ is a non-empty set involving all possible potentially events, where $P(\Theta)$ is the power set of Θ. For each $A \subseteq P(\Theta)$, there is a non-negative number $Pos(A)$, the so-called *possibility measure*, with the following properties:

1 $Pos\{\varnothing\} = 1, \quad Pos\{\Theta\} = 1$

2 $A \subseteq B$ implies $Pos(A) \leq Pos(B)$ for any $A, B \in P(\Theta)$

3 $Pos\{\bigcup_k A_k\} = Sup_k Pos\{A_k\}$.

Definition 7(Zimmermann, 1996; Dubois and Prade, 1978, 1988): The *necessity measure* of A, denoted by $Nec(A)$, is defined on $(\Theta, P(\Theta), Pos)$ as $Nec\{A\} = 1 - Pos\{A^c\}$ where A^c is the complement set of A. For any sets A and B, the properties of the necessity measure are presented as follows:

1 $Nec\{\varnothing\} = 0, Nec\{\Theta\} = 1$

2 $Pos(A) \geq Nec(A)$

3 $A \subseteq B$ implies $Nec(A) \leq Nec(B)$

4 $Pos(A) < 1 \Rightarrow Nec(A) = 0$

5 $Nec(A) > 0 \Rightarrow Pos(A) = 1$.

Definition 8 (Liu and Liu, 2002): Let ξ be a fuzzy variable on the possibility space $(U, P(U), Pos)$. The possibility, necessity and credibility of a fuzzy event $\{\xi \geq r\}$ are represented by:

$$Pos\{\xi \geq r\} = \underset{t \geq r}{Sup}\, \mu_\xi(t),$$

$$Nes\{\xi \geq r\} = 1 - \underset{t < r}{Sup}\, \mu_\xi(t),$$

$$Cr(\xi \geq r) = \frac{1}{2}\left[Pos\{\xi \geq r\} + Nec\{\xi \geq r\}\right]$$

where μ_ξ: $\Re \rightarrow [0, 1]$ is the membership function of ξ and r is a real number. The credibility measure is formed on the basis of the possibility and necessity measures and the simplest case is taken as their average. The credibility measure is the self-dual set function $Cr(\xi \geq r) = 1 - Cr(\xi < r)$ (see, Liu and Liu, 2002).

Definition 9 (extension principle) (Zimmermann, 1996): Assume that X is a Cartesian product of universe $X = X_1 \times \ldots \times X_r$ and $\tilde{A}_1, \ldots, \tilde{A}_r$ are r fuzzy sets in $X_1, \ldots, X_r$, respectively. The function f is a mapping from X to a universe Y, $y = f(x_1, \ldots, x_r)$. The extension principle enables us to introduce a fuzzy set $\tilde{B}$ in Y as follows:

$$\tilde{B} = \left\{(y, \mu_{\tilde{B}}(y) \mid y = f\left(x_1, \ldots, x_r\right), \left(x_1, \ldots, x_r\right) \in X\right\}$$

where

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{(x_1, \ldots, x_r) \in f^{-1}(y)} \min\left\{\mu_{\tilde{A}_1}(x_1), \ldots, \mu_{\tilde{A}_r}(x_r)\right\}, & f^{-1}(y) \neq \theta \\ 0, & o.w. \end{cases}$$

where f^{-1} is the inverse of f. Dubois and Prade (1980) modified the extension principle by the algebraic sum and product instead of sup and min, respectively.

3 FDH models

The conventional FDH technology is defined under the VRS assumption, that is, VRS-FDH technology is represented by Tulkens (1993) as follows:

$$T_{FDH}^{VRS} = \left\{ \begin{array}{l} (x, y): \quad \sum_{j=1}^{n} \lambda_j x_{ij} \le xi \quad \forall j = 1, \ldots, n, \\ \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_r \quad \forall m, \quad \sum_{j=1}^{n} \lambda_j = 1, \quad \lambda_j \in \{0, 1\} \end{array} \right\} \tag{2}$$

Tulkens (1993) obtained the input orientation of FDH analysis for DMU*p* by solving the following linear integer programming for various returns to scale:

$$\begin{aligned} & \min \theta \\ & s.t. \\ & \sum_{j=1}^{n} x_{ij} \lambda_j \le x_{ip} \theta, \quad i = 1, \ldots, m, \\ & \sum_{j=1}^{n} y_{rj} \lambda_j \ge y_{rp}, \quad r = 1, \ldots, s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j \in \{0, 1\}, \quad j = 1, \ldots, n. \end{aligned} \tag{3}$$

Agrell and Tind (2001) showed that there exists an equivalent LP problem to the traditional MILP problem to compute the FDH efficiency measure under VRS. The following linear programme is derived from Agrell and Tind (2001).

$$\begin{aligned} & \min \sum_{j=1}^{n} \theta_j \\ & \text{subject to:} \\ & x_{ij} \lambda_j \le x_{ip} \theta_j, \quad i = 1, \ldots m; \ j = 1, \ldots, n, \\ & y_{rj} \lambda_j \ge \lambda_j y_{rp}, \quad r = 1, \ldots, s; \ j = 1, \ldots, n, \\ & \sum_{j=1}^{J} \lambda_j = 1, \\ & \lambda_j \ge 0, \quad j = 1, \ldots, n. \end{aligned} \tag{4}$$

Leleu (2006) introduced NIRS, NDRS and CRS specifications by reference to VRS technology as:

$$T_{FDH}^{\Gamma} = \left\{ (x', y'): \quad (x', y') = (1 + \delta)(x, y), \quad (x, y) \in T_{FDH}^{VRS}, \quad \delta \in \Gamma \right\} \tag{5}$$

where $(1 + \delta)$ is a scaling parameter that is utilised to introduce various returns to scale assumptions. In (5), $\Gamma \in \{VRS, NIRS, NDRS, CRS\}$ with $VRS = \{\delta: \delta = 0\}$, $NIRS = \{\delta: -1 \le \delta \le 0\}$, $NDRS = \{\delta: \delta \ge 0\}$ and $CRS = \{\delta: \delta \ge -1\}$. The integration of returns to scale assumptions into FDH permits the development of four technologies. Leleu (2006) extended the estimation of the FDH using the following LP model with NIRS, NDRS and CRS technologies.

$$
\begin{aligned}
&\min \sum_{j=1}^{n} \theta_j \\
&\text{subject to:} \\
&x_{ij}\left(\lambda_j + \omega_j\right) \le x_{ip}\theta_j \quad i = 1,\ldots,m;\ j = 1,\ldots,n, \\
&y_{rj}\left(\lambda_j + \omega_j\right) \ge \lambda_j y_{rp} \quad r = 1,\ldots,s;\ j = 1,\ldots,n, \\
&\sum_{j=1}^{J} \lambda_j = 1, \\
&\lambda_j \ge 0, \quad j = 1,\ldots,n.
\end{aligned} \tag{6}
$$

where

$$
\begin{aligned}
&\omega_j \in \Gamma_j, \quad \Gamma_j \in \{VRS, NIRS, NDRS, CRS\} \\
&VRS = \left\{\omega_j : \omega_j = 0\right\}, NIRS = \left\{\omega_j : \omega_j \le 0\right\} \\
&NDRS = \left\{\omega_j : \omega_j \ge 0\right\}, CRS = \left\{\omega_j : \omega_j \ unconstrained\right\}
\end{aligned} \tag{7}
$$

In (7), the variable ω_j is the scaling factor for each DMU. This form is appealing because it has the dual formulation and the shadow profit interpretation, as noted by Leleu (2006). The dual formulation of the FDH model with various returns to scale assumptions as presented by Leleu (2006) enhances the economic interpretation of the FDH technology in terms of shadow prices. The dual of the above model as presented by Leleu (2006) is as follows:

$$
\begin{aligned}
&\max \pi \\
&s.t. \\
&\sum_{i=1}^{m} v_{ij} x_{ip} = 1, \quad j = 1,\ldots,n, \\
&\sum_{r=1}^{s} u_{rj}\left(y_{rj} - y_{rp}\right) - \sum_{i=1}^{m} v_{ij} x_{ij} + \pi \le 0, \quad j = 1,\ldots,n, \\
&u_{rj}, v_{ij} \ge 0, \quad r = 1,\ldots,s; i = 1,\ldots,m;\ j = 1,\ldots,n.
\end{aligned} \tag{8}
$$

and

$$
\begin{aligned}
&\sum_{r=1}^{s} u_{rj} y_{rj} - \sum_{i=1}^{m} v_{ij} x_{ij} \ge 0, \quad j = 1,\ldots,n; \text{ under NIRS}, \\
&\sum_{r=1}^{s} u_{rj} y_{rj} - \sum_{i=1}^{m} v_{ij} x_{ij} \le 0, \quad j = 1,\ldots,n; \text{ under NDRS}, \\
&\sum_{r=1}^{s} u_{rj} y_{rj} - \sum_{i=1}^{m} v_{ij} x_{ij} = 0, \quad j = 1,\ldots,n; \text{ under CRS}, \\
&\sum_{r=1}^{s} u_{rj} y_{rj} - \sum_{i=1}^{m} v_{ij} x_{ij}, \quad j = 1,\ldots,n; \text{ unconstrained under VRS}.
\end{aligned} \tag{9}
$$

where u_{rj} ($r = 1,\dots,s; j = 1,\dots,n$) and v_{ij} ($i = 1,\dots,m; j = 1,\dots,n$) are the weights assigned to the r^{th} output and the i^{th} input from the j^{th} DMU, respectively. Notice that the objective function of models (4), (6) and (8) represents the best relative efficiency and the DMUs with $\theta_p^* = 1$ ($\pi^* = 1$), are called the technically input efficient, and those units with $\theta_p^* \neq 1$ ($\pi^* \neq 1$) are called technically input (output)-inefficient.

4 Fuzzy FDH model

In this section, we develop an imprecise FDH-based formulation for dealing with the fuzzy parameters on a possibility space (Θ, $P(\Theta)$, *Pos*) through efficiency measurement. Let us consider n DMUs, indexed by $j = 1,\dots,n$, where each of the DMUs consumes m different fuzzy inputs, indexed by $\tilde{x}_{ij}$ ($i = 1,\dots,m$), to secure s different fuzzy outputs indexed by $\tilde{y}_{rj}$ ($r = 1,\dots,s$).

The following fuzzy FDH model results from consideration of the fuzzy inputs and outputs for DMU_p:

$$\begin{aligned}
&\max \pi \\
&s.t. \\
&\sum_{i=1}^{m} v_{ij}\tilde{x}_{ip} = 1, \quad j = 1,\dots,n, \\
&\sum_{r=1}^{s} u_{rj}\left(\tilde{y}_{rj} - \tilde{y}_{rp}\right) - \sum_{i=1}^{m} v_{ij}\tilde{x}_{ij} + \pi \le 0, \quad j = 1,\dots,n, \\
&u_{rj}, v_{ij} \ge 0, \quad r = 1,\dots,s; i = 1,\dots,m; j = 1,\dots,n.
\end{aligned} \tag{10}$$

And the following holds for various returns to scale assumptions:

$$\begin{aligned}
&\sum_{r=1}^{s} u_{rj}\tilde{y}_{rj} - \sum_{i=1}^{m} v_{ij}\tilde{x}_{ij} \ge 0, \quad j = 1,\dots,n; \text{ under NIRS,} \\
&\sum_{r=1}^{s} u_{rj}\tilde{y}_{rj} - \sum_{i=1}^{m} v_{ij}\tilde{x}_{ij} \le 0, \quad j = 1,\dots,n; \text{ under NDRS,} \\
&\sum_{r=1}^{s} u_{rj}\tilde{y}_{rj} - \sum_{i=1}^{m} v_{ij}\tilde{x}_{ij} = 0, \quad j = 1,\dots,n; \text{ under CRS.}
\end{aligned}$$

We construct the following generic FDH model, called the possibility constrained programming model as follows:

$$\begin{aligned}
&\max \pi \\
&s.t. \\
&Pos\left(\sum_{i=1}^{m} v_{ij}\tilde{x}_{ip} = 1\right) \ge \delta, \quad j = 1,\dots,n, \\
&Pos\left(\sum_{r=1}^{s} u_{rj}\left(\tilde{y}_{rj} - \tilde{y}_{rp}\right) - \sum_{i=1}^{m} v_{ij}\tilde{x}_{ij} + \pi \le 0\right) \ge \delta, \quad j = 1,\dots,n, \\
&u_{rj}, v_{ij} \ge 0, \quad r = 1,\dots,s; i = 1,\dots,m; j = 1,\dots,n.
\end{aligned} \tag{11}$$

And for various returns to scale assumptions we have the following:

$$Pos\left(\sum_{r=1}^{s} u_{rj}\tilde{y}_{rj} - \sum_{i=1}^{m} v_{ij}\tilde{x}_{ij} \geq 0\right) \geq \delta, \quad j = 1,\ldots,n; \text{ under NIRS,}$$

$$Pos\left(\sum_{r=1}^{s} u_{rj}\tilde{y}_{rj} - \sum_{i=1}^{m} v_{ij}\tilde{x}_{ij} \leq 0\right) \geq \delta, \quad j = 1,\ldots,n; \text{ under NDRS,}$$

$$Pos\left(\sum_{r=1}^{s} u_{rj}\tilde{y}_{rj} - \sum_{i=1}^{m} v_{ij}\tilde{x}_{ij} = 0\right) \geq \delta, \quad j = 1,\ldots,n; \text{ under CRS.}$$

where δ is a pre-specified acceptable level of possibility, and varies between [0, 1]. These parameters, assumed known a priori, are also called the threshold (or aspiration) levels. Let us assume that the inputs and outputs, $\tilde{x}_{ij} = (x_{ij}^{\alpha}, x_{ij}^{m_1}, x_{ij}^{m_2}, x_{ij}^{\beta})_{LR}$, and $\tilde{y}_{rj} = (y_{rj}^{\alpha}, y_{rj}^{m_1}, y_{rj}^{m_2}, y_{rj}^{\beta})_{LR}$ are characterised by the left and right trapezoidal fuzzy numbers. Notice that in model (10) the extension principle (see Definition 9) enables us to generalise the membership functions of $\sum_{r=1}^{s} u_{rj}\left(\tilde{y}_{rj} - \tilde{y}_{rp}\right)$ and $\sum_{i=1}^{m} v_{ij}\tilde{x}_{ij}$ as follows:

$$\mu_{\sum_{i=1}^{m} v_{ij}\tilde{x}_{ij}}(t) = \begin{cases} L\left(\dfrac{\sum_{i=1}^{m} v_{ij}x_{ij}^{m_1} - t}{\sum_{i=1}^{m} v_{ij}x_{ij}^{\alpha}}\right) & t \leq \sum_{i=1}^{m} v_{ij}x_{ij}^{m_1}, \\ R\left(\dfrac{t - \sum_{i=1}^{m} v_{ij}x_{ij}^{m_2}}{\sum_{i=1}^{m} v_{ij}x_{ij}^{\beta}}\right) & t \geq \sum_{i=1}^{m} v_{ij}x_{ij}^{m_2}. \end{cases} \tag{12}$$

and

$$\mu_{\sum_{r=1}^{s} u_{rj}\tilde{y}_{rj}}(t) = \begin{cases} L\left(\dfrac{\sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_1} - y_{rp}^{m_2}\right) - t}{\sum_{r=1}^{s} u_{rj}\left(y_{rj}^{\alpha} + y_{rp}^{\beta}\right)}\right) & t \leq \sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_1} - y_{rp}^{m_2}\right), \\ R\left(\dfrac{t - \sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_2} - y_{rp}^{m_1}\right)}{\sum_{r=1}^{s} u_{rj}\left(y_{rj}^{\beta} + y_{rp}^{\alpha}\right)}\right) & t \geq \sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_2} - y_{rp}^{m_1}\right). \end{cases} \tag{13}$$

Therefore, $\sum_{i=1}^{m} v_i\tilde{x}_{ij}$ and $\sum_{r=1}^{s} u_{rj}(\tilde{y}_{rj} - \tilde{y}_{rp})$ can be denoted as:

$$\left(\sum_{i=1}^{m} v_i x_{ij}^{\alpha}, \sum_{i=1}^{m} v_i x_{ij}^{m_1}, \sum_{i=1}^{m} v_i x_{ij}^{m_2}, \sum_{i=1}^{m} v_i x_{ij}^{\beta}\right)_{LR}$$

and

$$\left(\sum_{r=1}^{s} u_{rj}\left(y_{rj}^{\alpha}+y_{rp}^{\beta}\right), \sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_1}-y_{rp}^{m_2}\right), \sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_2}-y_{rp}^{m_1}\right), \sum_{r=1}^{s} u_{rj}\left(y_{rj}^{\beta}+y_{rp}^{\alpha}\right)\right)_{LR},$$

respectively. In order to solve the possibility constrained programming model (11), we convert the constraints in this model into their respective crisp equivalents. We use Theorem 1 to solve PCCP model (11).

4.1 The fuzzy FDH model with possibility measure

Theorem 1 (Sakawa, 1993): Let λ_1 and λ_2 be two independent fuzzy numbers with continuous membership functions. For a given confidence level $\alpha \in [0, 1]$,

$Pos\{\lambda_1 \geq \lambda_2\} \geq \alpha$ if and only if $\lambda_{1,\alpha}^{R} \geq \lambda_{2,\alpha}^{L}$,

where $\lambda_{1,\alpha}^{L}, \lambda_{1,\alpha}^{R}$ and $\lambda_{2,\alpha}^{L}, \lambda_{2,\alpha}^{R}$ are the left and the right side extreme points of the α-level sets $[\lambda_{1,\alpha}^{L}, \lambda_{1,\alpha}^{R}]$ and $[\lambda_{2,\alpha}^{L}, \lambda_{2,\alpha}^{R}]$ of λ_1 and λ_2, respectively, and $Pos\{\lambda_1 \geq \lambda_2\}$ means that the degree of possibility λ_1 is greater than or equal to λ_2.

Consider the last set of constraints in model (11) for the deterministic equivalent. Using the first part of Theorem 1, these constraints can be written as follows:

$$Pos\left(\sum_{r=1}^{s} u_{rj}\left(\tilde{y}_{rj}-\tilde{y}_{rp}\right)-\sum_{i=1}^{m} v_{ij}\tilde{x}_{ij}+\pi \leq 0\right) \geq \delta$$

$$\Rightarrow \left(\sum_{r=1}^{s} u_{rj}\left(\tilde{y}_{rj}-\tilde{y}_{rp}\right)\right)_{\delta}^{L} - \left(\sum_{i=1}^{m} v_{ij}\tilde{x}_{ij}\right)_{\delta}^{R} + \pi \leq 0$$

$$\Rightarrow \sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_1}-y_{rp}^{m_2}-R^{-1}(\delta)\left(y_{rj}^{\alpha}+y_{rp}^{\beta}\right)\right)$$

$$-\sum_{i=1}^{m} v_{ij}\left(\overline{x}_{ij}^{m_2}+R^{-1}(\delta)x_{ij}^{\beta}\right)+\pi \leq 0.$$

Similarly, Theorem 1 can be applied to the remaining constraints and ultimately model (11) is transformed into model (14).

$$\begin{aligned}
&\max \pi \\
&s.t. \\
&\sum_{i=1}^{m} v_{ij}\left(x_{ip}^{m_2}+R^{-1}(\delta)x_{ip}^{\beta}\right) \geq 1, \quad j=1,\ldots,n, \\
&\sum_{i=1}^{m} v_{ij}\left(x_{ip}^{m_1}-L^{-1}(\delta)x_{ip}^{\alpha}\right) \leq 1, \quad j=1,\ldots,n,
\end{aligned} \tag{14}$$

$$\sum_{r=1}^{s} u_{rj}\left(\left(y_{rj}^{m_1}-y_{rp}^{m_2}\right)-L^{-1}(\delta)\left(y_{rj}^{\alpha}+y_{rp}^{\beta}\right)\right)$$
$$-\sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_2}+R^{-1}(\delta)x_{ij}^{\beta}\right)+\pi \leq 0, \quad j=1,\ldots,n,$$
$$u_{rj}, v_{ij} \geq 0, \quad r=1,\ldots,s; i=1,\ldots,m; j=1,\ldots,n.$$

and

$$\sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_1}-L^{-1}(\delta)y_{rj}^{\alpha}\right)-\sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_2}+R^{-1}(\delta)x_{ij}^{\beta}\right) \leq 0, \quad j=1,\ldots,n, \text{ under NDRS}$$
$$\sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_2}+R^{-1}(\delta)y_{rj}^{\beta}\right)-\sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_1}-L^{-1}(\delta)x_{ij}^{\alpha}\right) \geq 0, \quad j=1,\ldots,n, \text{ under NIRS}$$
$$\begin{cases}\sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_1}-L^{-1}(\delta)y_{rj}^{\alpha}\right)-\sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_2}+R^{-1}(\delta)x_{ij}^{\beta}\right) \leq 0; \quad j=1,\ldots,n, \\ \sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_2}+R^{-1}(\delta)y_{rj}^{\beta}\right)-\sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_1}-L^{-1}(\delta)x_{ij}^{\alpha}\right) \geq 0; \quad j=1,\ldots,n.\end{cases} \text{ under CRS}$$

We present the following definition to define the efficiency of a DMU:

Definition 10: A DMU is called possibilisticly δ-efficient if the objective function of model (14), π^*, is greater than or equal to one at the possibility level δ; otherwise, it is called possibilisticly δ-inefficient.

Theorem2: The h-possibility efficiency score is a non-increasing function of the possibility level h.

Proof: Since $L(x)$ and $R(x)$ are continuous non-increasing functions, then we have as follows: $R^{-1}(\delta_2) \geq R^{-1}(\delta_1)$, $L^{-1}(\delta_2) \geq L^{-1}(\delta_1)$

Then we must have as follows:

$$\sum_{i=1}^{m} v_{ij}\left(x_{ip}^{m_2}+R^{-1}\left(\delta_2\right)x_{ip}^{\beta}\right) \geq \sum_{i=1}^{m} v_{ij}\left(x_{ip}^{m_2}+R^{-1}\left(\delta_1\right)x_{ip}^{\beta}\right), \quad j=1,\ldots,n$$
$$\sum_{i=1}^{m} v_{ij}\left(x_{ip}^{m_1}-R^{-1}\left(\delta_2\right)x_{ip}^{\alpha}\right) \leq \sum_{i=1}^{m} v_{ij}\left(x_{ip}^{m_1}-R^{-1}\left(\delta_1\right)x_{ip}^{\alpha}\right), \quad j=1,\ldots,n$$
$$\sum_{r=1}^{s} u_{rj}\left(\left(y_{rj}^{m_1}-y_{rp}^{m_2}\right)-R^{-1}\left(\delta_2\right)\left(y_{rj}^{\alpha}+y_{rp}^{\beta}\right)\right)-\sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_2}+R^{-1}\left(\delta_2\right)x_{ij}^{\beta}\right)+\pi$$
$$\leq \sum_{r=1}^{s} u_{rj}\left(\left(y_{rj}^{m_1}-y_{rp}^{m_2}\right)-R^{-1}\left(\delta_1\right)\left(y_{rj}^{\alpha}+y_{rp}^{\beta}\right)\right)-\sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_2}+R^{-1}\left(\delta_1\right)x_{ij}^{\beta}\right)+\pi \leq 0,$$
$$j=1,\ldots,n$$

Also, similar to the above constraints, the following can be verified for CRS, NIRS and NDRS technologies:

$$\sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_1} - R^{-1}(\delta) y_{rj}^{\alpha}\right) - \sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_2} + R^{-1}(\delta) x_{ij}^{\beta}\right) \le 0, \quad j = 1, \ldots, n, \quad \text{under NDRS}$$

$$\sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_1} + R^{-1}(\delta) y_{rj}^{\alpha}\right) - \sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_2} - R^{-1}(\delta) x_{ij}^{\beta}\right) \ge 0, \quad j = 1, \ldots, n, \quad \text{under NIRS}$$

$$\begin{cases} \sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_1} - R^{-1}(\delta) y_{rj}^{\alpha}\right) - \sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_2} + R^{-1}(\delta) x_{ij}^{\beta}\right) \le 0; \quad j = 1, \ldots, n, \\ \sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_1} + R^{-1}(\delta) y_{rj}^{\alpha}\right) - \sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_2} - R^{-1}(\delta) x_{ij}^{\beta}\right) \ge 0; \quad j = 1, \ldots, n. \end{cases} \quad \text{under CRS.}$$

Then, from the above we have as follows:

$$\sum_{i=1}^{m} v_{ij}\left(x_{ip}^{m_2} + R^{-1}(\delta_2) x_{ip}^{\beta}\right) \ge 1, \quad j = 1, \ldots, n,$$

$$\sum_{i=1}^{m} v_{ij}\left(x_{ip}^{m_1} - L^{-1}(\delta_2) x_{ip}^{\alpha}\right) \le 1, \quad j = 1, \ldots, n,$$

$$\sum_{r=1}^{s} u_{rj}\left(\left(y_{rj}^{m_1} - y_{rp}^{m_2}\right) - R^{-1}(\delta_2)\left(y_{rj}^{\alpha} + y_{rp}^{\beta}\right)\right)$$

$$-\sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_2} + R^{-1}(\delta_2) x_{ij}^{\beta}\right) + \pi \le 0, \quad j = 1, \ldots, n$$

and

$$\sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_1} - R^{-1}(\delta_2) y_{rj}^{\alpha}\right) - \sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_2} + R^{-1}(\delta_2) x_{ij}^{\beta}\right) \le 0, \quad j = 1, \ldots, n, \text{ under NDRS}$$

$$\sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_1} + R^{-1}(\delta_2) y_{rj}^{\alpha}\right) - \sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_2} - R^{-1}(\delta_2) x_{ij}^{\beta}\right) \ge 0, \quad j = 1, \ldots, n, \text{ under NIRS}$$

$$\begin{cases} \sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_1} - R^{-1}(\delta_2) y_{rj}^{\alpha}\right) - \sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_2} + R^{-1}(\delta_2) x_{ij}^{\beta}\right) \le 0; \; j = 1, \ldots, n, \\ \sum_{r=1}^{s} u_{rj}\left(y_{rj}^{m_1} + R^{-1}(\delta_2) y_{rj}^{\alpha}\right) - \sum_{i=1}^{m} v_{ij}\left(x_{ij}^{m_2} - R^{-1}(\delta_2) x_{ij}^{\beta}\right) \ge 0; \; j = 1, \ldots, n, \end{cases} \quad \text{under CRS}$$

Let (u_{rj}^*, v_{ij}^*) be optimal solutions of model (14) for possibility level δ_1. Based upon the above discussion, (u_{rj}^*, v_{ij}^*) is a feasible solution of (14) for any possibility level δ_2 such that $\delta_1 \ge \delta_2$. Consequently we must have $\pi_{\delta_1}^* \ge \pi_{\delta_2}^*$. □

Proposition 1: If DMU_p is possibilisticδ-efficient under possibility level $\delta = 1$, then for all threshold levels δ such that $\delta < 1$ the DMU_p is efficient.

Proof: This can obviously be obtained from Theorem 2. □

Proposition 2: If DMU_p under threshold level $\delta = 0$ is inefficient, then for all possibility levels δ such that $\delta > 0$ the DMU_p is inefficient.

Proof: This can obviously be obtained from Theorem 2. □

4.2 The fuzzy FDH model with credibility measure

In this section, we introduce the credibility approach to solve the fuzzy FDH model and deduce its equivalent crisp model. In this case, the expected value operator is used to convert the chance constraints into deterministic constraints. Similar to the expected value operator for a random variable in probability theory, the expected value operator for a fuzzy variable using the credibility measure is defined by Liu (2002) as follows:

Definition 11 (Liu, 2002): Let ξ be a fuzzy variable. Then the expected value of ξ is defined by $E(\xi) = \int_0^{+\infty} Cr\{\xi \geq r\}dr - \int_{-\infty}^{0} Cr\{\xi \leq r\}dr$ provided that at least one of the two integrals is finite.

Theorem 3: Let $\tilde{\lambda} = (\alpha, m_1, m_2, \beta)_{LR}$ be a left and right fuzzy variable where m_1 and m_2 are *the* mean values and α and β are the (non-negative) left and right spreads. Then the expected value of fuzzy variable $\tilde{\lambda}$ is as follows:

$$E[\tilde{\lambda}] = \frac{1}{2}\left[(m_1 + m_2) - \alpha\int_0^1 L(t)dt + \beta\int_0^1 R(t)dt\right]$$

Proof: Based on the definition of the credibility measure we have the following relationship:

$$Cr\{\tilde{\lambda} \geq r\} = \frac{1}{2}\left(Pos\{\tilde{\lambda} \geq r\} + Nec\{\tilde{\lambda} \geq r\}\right)$$

$$Cr(\tilde{\lambda} \geq r) = \begin{cases} 1, & r \leq m_1 - \alpha \\ 1 - \frac{1}{2}L\left(\frac{m_1 - r}{\alpha}\right), & m_1 - \alpha \leq r \leq m_1 \\ \frac{1}{2}, & m_1 \leq r \leq m_2 \\ \frac{1}{2}R\left(\frac{r - m_2}{\beta}\right), & m_2 \leq r \leq m_2 + \beta \\ 0, & r > m_2 + \beta \end{cases} \quad (15)$$

$$Cr(\tilde{\lambda} \le r) = \begin{cases} 0, & r \le m_1 - \alpha \\ \frac{1}{2} L\left(\frac{m_1 - r}{\alpha}\right), & m_1 - \alpha \le r \le m_1 \\ \frac{1}{2}, & m_1 \le r \le m_2 \\ 1 - \frac{1}{2} R\left(\frac{r - m_2}{\beta}\right), & m_2 \le r \le m_2 + \beta \\ 1, & r > m_2 + \beta \end{cases} \tag{16}$$

Obviously, it follows from (16) that $\int_{-\infty}^{0} Cr\{\tilde{\lambda} \le r\} dr = 0$ and accordingly:

$$\begin{aligned} E(\tilde{\lambda}) &= \int_0^{+\infty} Cr\{\tilde{\lambda} \ge r\} dr - \int_{-\infty}^{0} Cr\{\tilde{\lambda} \le r\} dr \\ &= \int_0^{m_1 - \alpha} dr + \int_{m_1 - \alpha}^{m_1} \left(1 - \frac{1}{2} L\left(\frac{m_1 - r}{\alpha}\right)\right) dr + \int_{m_1}^{m_2} \frac{dr}{2} + \int_{m_2}^{m_2 + \beta} \frac{1}{2} R\left(\frac{r - m_2}{\beta}\right) dr \end{aligned}$$

By making the following variables change, $\frac{m_1 - r}{\alpha} = t, \frac{r - m_2}{\beta} = t$ we can obtain:

$$E(\tilde{\lambda}) = \frac{m_1 + m_2}{2} - \frac{\alpha}{2} \int_0^1 L(t) dt + \frac{\beta}{2} \int_0^1 R(t) dt.$$

Proposition 3: In particular, let $\tilde{A} = (a, b, c, d)$ and $\tilde{B} = (a, b, c)$ be trapezoidal and triangular fuzzy variables, respectively, then:

$$E[\tilde{A}] = \frac{1}{4}(a + b + c + d) \text{ and } E[\tilde{B}] = \frac{1}{4}(a + 2b + c).$$

Proof: Let $\tilde{A} = (a, b, c, d)$. For a trapezoidal fuzzy number we have $R(x) = L(x) = 1 - x$, and $b - a = \alpha$, $d - c = \beta$. By using Theorem 3, we have following:

$$\begin{aligned} E(\tilde{A}) &= \frac{m_1 + m_2}{2} - \frac{\alpha}{2} \int_0^1 L(t) dt + \frac{\beta}{2} \int_0^1 R(t) dt \\ &= \frac{m_1 + m_2}{2} - \frac{\alpha}{2} \int_0^1 (1 - t) dt + \frac{\beta}{2} \int_0^1 (1 - t) dt = \frac{m_1 + m_2}{2} + \frac{\beta - \alpha}{4} \\ &= \frac{1}{4}\left(2(m_1 + m_2) + \beta - \alpha\right) = \frac{1}{4}(a + b + c + d). \end{aligned}$$

The credibility approach uses the ‘expected value’ of fuzzy variables to transform the fuzzy FDH model into the credibility programming FDH model. Similar to the expected value approach in stochastic programming where random variables are replaced by their expected values, in the credibility programming FDH (CP-FDH) model fuzzy variables are replaced by ‘expected value’, which are derived by using credibility measures. In this way, the fuzzy FDH (FFDH) model is transformed into the following credibility programming-FDH model:

$$
\begin{aligned}
&\max \pi \\
&s.t. \\
&E\left[\sum_{i=1}^{m} v_{ij}\tilde{x}_{ip}\right] = E[\tilde{1}], \quad j = 1,\ldots,n, \\
&E\left[\sum_{r=1}^{s} u_{rj}\left(\tilde{y}_{rj} - \tilde{y}_{rp}\right) - \sum_{i=1}^{m} v_{ij}\tilde{x}_{ij} + \pi\right] \le 0, \quad j = 1,\ldots,n, \\
&u_{rj}, v_{ij} \ge 0, \quad r = 1,\ldots,s; i = 1,\ldots,m; j = 1,\ldots,n.
\end{aligned} \tag{17}
$$

and

$$
\begin{aligned}
&E\left[\sum_{r=1}^{s} u_{rj}\tilde{y}_{rj} - \sum_{i=1}^{m} v_{ij}\tilde{x}_{ij}\right] \ge 0, \quad j = 1,\ldots,n; \quad \text{under NIRS,} \\
&E\left[\sum_{r=1}^{s} u_{rj}\tilde{y}_{rj} - \sum_{i=1}^{m} v_{ij}\tilde{x}_{ij} \le 0\right], \quad j = 1,\ldots,n; \quad \text{under NDRS,} \\
&E\left[\sum_{r=1}^{s} u_{rj}\tilde{y}_{rj} - \sum_{i=1}^{m} v_{ij}\tilde{x}_{ij}\right] = 0, \quad j = 1,\ldots,n; \quad \text{under CRS.}
\end{aligned}
$$

Now, we substitute the following expected value of fuzzy variables to obtain a deterministic FDH model with various returns to scale assumptions:

$$
E\left[\sum_{r=1}^{s} u_{rj}\left(\tilde{y}_{rj} - \tilde{y}_{rp}\right)\right] = \frac{1}{2}\sum_{r=1}^{s} u_r \left[\begin{array}{l} \left(y_{rj}^{m_2} + y_{rj}^{m_1} - y_{rp}^{m_2} - y_{rp}^{m_1}\right) \\ -\left(y_{rj}^{\alpha} + y_{rp}^{\beta}\right)\int_0^1 L(t)dt + \left(y_{rj}^{\beta} + y_{rp}^{\alpha}\right)\int_0^1 R(t)dt \end{array}\right],
$$

and

$$
E\left[\sum_{i=1}^{m} v_{ij}\tilde{x}_{ij}\right] = \frac{1}{2}\sum_{i=1}^{m} v_{ij}\left[\left(x_{ip}^{m_2} + x_{ip}^{m_1}\right) - x_{ip}^{\alpha}\int_0^1 L(t)dt + x_{ip}^{\beta}\int_0^1 R(t)dt\right].
$$

By applying the following expected value of the Fuzzy variables, we can construct the following deterministic LP model to evaluate the efficiency of each DMU in an uncertain environment:

$$
\begin{aligned}
&\max \pi \\
&s.t. \\
&\sum_{i=1}^{m} v_{ij}\left[\left(x_{ip}^{m_2} + x_{ip}^{m_1}\right) - x_{ip}^{\alpha}\int_0^1 L(t)dt + x_{ip}^{\beta}\int_0^1 R(t)dt\right] = 2, \quad j = 1,\ldots,n, \\
&\sum_{r=1}^{s} u_{rj}\left[\left(y_{rj}^{m_2} + y_{rj}^{m_1} - y_{rp}^{m_2} - y_{rp}^{m_1}\right) - \left(y_{rj}^{\alpha} + y_{rp}^{\beta}\right)\int_0^1 L(t)dt + \left(y_{rj}^{\beta} + y_{rp}^{\alpha}\right)\int_0^1 R(t)dt\right] \\
&-\sum_{i=1}^{m} v_{ij}\left[\left(x_{ij}^{m_2} + x_{ij}^{m_1}\right) - x_{ij}^{\alpha}\int_0^1 L(t)dt + x_{ij}^{\beta}\int_0^1 R(t)dt\right] + 2\pi \le 0, \quad j = 1,\ldots,n, \\
&u_{rj}, v_{ij} \ge 0, \quad r = 1,\ldots,s; i = 1,\ldots,m; j = 1,\ldots,n.
\end{aligned} \tag{18}
$$

For NIRS, NDRS, and CRS we have respectively:

$$\sum_{r=1}^{s} u_{rj}\left[\left(y_{rj}^{m_2}+y_{rj}^{m_1}\right)-y_{rj}^{\alpha}\int_0^1 L(t)dt+y_{rj}^{\beta}\int_0^1 R(t)dt\right]$$
$$-\sum_{i=1}^{m} v_{ij}\left[\left(x_{ij}^{m_2}+x_{ij}^{m_1}\right)-x_{ij}^{\alpha}\int_0^1 L(t)dt+x_{ij}^{\beta}\int_0^1 R(t)dt\right]\geq 0, \quad j=1,\ldots,n, \quad \text{under NIRS}$$
$$\sum_{r=1}^{s} u_{rj}\left[\left(y_{rj}^{m_2}+y_{rj}^{m_1}\right)-y_{rj}^{\alpha}\int_0^1 L(t)dt+y_{rj}^{\beta}\int_0^1 R(t)dt\right]$$
$$-\sum_{i=1}^{m} v_{ij}\left[\left(x_{ij}^{m_2}+x_{ij}^{m_1}\right)-x_{ij}^{\alpha}\int_0^1 L(t)dt+x_{ij}^{\beta}\int_0^1 R(t)dt\right]\leq 0, \quad j=1,\ldots,n, \quad \text{under NDRS}$$
$$\sum_{r=1}^{s} u_{rj}\left[\left(y_{rj}^{m_2}+y_{rj}^{m_1}\right)-y_{rj}^{\alpha}\int_0^1 L(t)dt+y_{rj}^{\beta}\int_0^1 R(t)dt\right]$$
$$-\sum_{i=1}^{m} v_{ij}\left[\left(x_{ij}^{m_2}+x_{ij}^{m_1}\right)-x_{ij}^{\alpha}\int_0^1 L(t)dt+x_{ij}^{\beta}\int_0^1 R(t)dt\right]=0, \quad j=1,\ldots,n, \quad \text{under CRS.}$$

We should note that in the above model, the DMU_p under assessment is said to be efficient if the corresponding optimal solution is equal to unity; otherwise, DMU_p is inefficient.

5 Numerical example

In this section, we use a hypothetical example to examine the applicability of the proposed models. Consider five DMUs with two fuzzy triangular inputs and two fuzzy triangular outputs as reported in Table 1. This data is denoted by (α, m, β) where m is the centre value and α and β are the left and right tails, respectively.

Table 1 Fuzzy inputs and fuzzy outputs

DMU	*Input 1*	*Input 2*	*Output 1*	*Output 2*
1	(0.2, 4, 0.5)	(0.2, 5.1, 0.2)	(0.2, 2.6, 0.2)	(0.3, 4.1, 0.3)
2	(0.1, 5.9, 0.1)	(0.1, 5.5, 0.1)	(0.1, 2.2, 0.1)	(0.2, 3.5, 0.2)
3	(0.2, 4.9, 0.5)	(0.1, 2.6, 0.4)	(0.2, 3.2, 0.5)	(0.5, 5.1, 0.8)
4	(0.4, 8.1, 0.7)	(0.1, 5.3, 0.1)	(0.1, 4.9, 0.4)	(0.2, 5.7, 0.2)
5	(0.3, 6.5, 0.6)	(0.2, 4.1, 0.5)	(0.4, 6.1, 0.7)	(0.6, 7.4, 0.9)

Six different possibility (threshold) levels of $\delta = 0$, $\delta = 0.2$, $\delta = 0.5$, $\delta = 0.7$, $\delta = 0.9$ and $\delta = 1$ are considered to compare the results from the FDH models with various returns to scale assumptions, including CRS, NIRS, NDRS, and VRS. The computational results of the deterministic equivalent of the efficiency model (11), for $\delta = 0$, $\delta = 0.2$, $\delta = 0.5$, $\delta = 0.7$, $\delta = 0.9$ and $\delta = 1$ are presented in Tables 2, 3, 4 and 5 for the CRS, NIRS, NDRS, and VRS cases, respectively.

Table 2 FDH model results under the CRS assumption

DMU	*δ = 0*	*δ = 0.2*	*δ = 0.5*	*δ = 0.7*	*δ = 0.9*	*δ = 1.0*	*Credibility approach (CRS)*
1	1.3791	1.2961	1.1525	1.0527	0.9515	0.9003	0.9012
2	0.8074	0.7631	0.6784	0.6169	0.5535	0.5211	0.5330
3	1.5391	1.4151	1.2452	1.1418	1.0455	1.0000	1.0000
4	0.8747	0.8295	0.7611	0.7149	0.6682	0.6446	0.6480
5	1.4397	1.3402	1.2024	1.1175	1.0379	1.0000	1.0000

Table 3 FDH model results under the NIRS assumption

DMU	*δ = 0*	*δ = 0.2*	*δ = 0.5*	*δ = 0.7*	*δ = 0.9*	*δ = 1.0*	*Credibility approach (NIRS)*
1	1.3791	1.2997	1.1525	1.0527	0.9515	0.9003	0.9012
2	0.8074	0.7631	0.6784	0.6169	0.5535	0.5211	0.5330
3	1.6708	1.5585	1.3862	1.2684	1.1480	1.0000	1.0000
4	0.8747	0.8295	0.7611	0.7149	0.6682	0.6446	0.6480
5	4.2653	4.1554	3.9914	3.8824	2.9610	1.0000	1.0000

Table 4 FDH model results under the NDRS assumption

DMU	*δ = 0*	*δ = 0.2*	*δ = 0.5*	*δ = 0.7*	*δ = 0.9*	*δ = 1.0*	*Credibility approach (NDRS)*
1	1.3816	1.2961	1.1769	1.1031	1.0334	1.0000	1.0000
2	0.931	0.9107	0.8803	0.8603	0.8404	0.8305	0.8432
3	1.5391	1.4151	1.2452	1.1418	1.0455	1.0000	1.0000
4	0.9221	0.8972	0.8608	0.8371	0.8139	0.8025	0.8043
5	1.4397	1.3402	1.2024	1.1175	1.0379	1.0000	1.0000

Table 5 FDH model results under the VRS assumption

DMU	*δ = 0*	*δ = 0.2*	*δ = 0.5*	*δ = 0.7*	*δ = 0.9*	*δ = 1.0*	*Credibility approach (VRS)*
1	1.4211	1.3802	1.3205	1.2817	1.2437	1.0000	1.0000
2	0.9310	0.9107	0.8803	0.8603	0.8404	0.8305	0.8432
3	1.8400	1.7857	1.7059	1.6537	1.6023	1.0000	1.0000
4	0.9221	0.8972	0.8608	0.8371	0.8139	0.8025	0.8043
5	1.4194	1.3834	1.3307	1.2964	1.2628	1.0000	1.0000

As shown in these tables, the DMUs have a higher efficiency score under $\delta = 0$ compared with other probability levels. We have confirmed Proposition 2.

In addition, under all given possibility levels, DMU 3 and DMU 5 perform better than DMUs 1, 2, and 4 according to the fuzzy possibility FDH models. Furthermore, DMU 2 and DMU 4 are inefficient under all given possibility levels. This result shows that the efficiency is non-increasing (non-decreasing) in possibility level δ and therefore we have confirmed Theorem 2.

We apply the fuzzy expected value model (17) to calculate the efficiency measure of the DMUs as reported in the last column of Table 2, Table 3, Table 4, and Table 5, for the CRS, NIRS, NDRS and VRS cases, respectively. According to Table 1, DMU 5 in the CRS and NIRS cases, and DMUs 1, 3 and 5 in the NDRS and VRS cases are identified as the efficient units with a unity score.

In summary, the proposed credibility approach (fuzzy expected value) model is a rather straightforward formulation for measuring the efficiency of a group of DMUs as well as providing adequate discriminatory power in the presence of the fuzzy inputs and fuzzy outputs.

6 Conclusions and future research directions

Fuzzy DEA is a tool for comparing the performance of a set of activities or organisations in uncertain environments. The conventional FDH is deterministic and assumes that the inputs and the outputs are measured precisely. However, the observed values of the input and output data in real-world problems can potentially be fuzzy in nature.

In this paper, the concept of chance-constrained programming was used to develop FDH models with various returns to scale assumptions, including VRS, NIRS, NDRS, and CRS, for DMUs with fuzzy data. We proposed two fuzzy FDH models with respect to the possibility and the expected value (credibility approach) constraints. Finally, we used a numerical example to show the feasibility and the richness of the obtained solutions, since only real applications can reveal the true value of the framework. We plan to apply the proposed models to a real-life elaborate case study in the near future.

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