A fuzzy linear programming model with fuzzy parameters and decision variables

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Abstract: Linear programming (LP) is an optimisation technique most widely used for optimal allocation of limited resources amongst competing activities. Precise data are fundamentally indispensable in standard LP problems. However, the observed values of the data in real-world problems are often imprecise or vague. Fuzzy set theory has been extensively used to represent ambiguous, uncertain or imprecise data in LP by formalising the inaccuracies inherent in human decision-making. We propose a new method for solving fuzzy LP (FLP) problems in which the right-hand side parameters and the decision variables are represented by fuzzy numbers. A new fuzzy ranking model and a new supplementary variable are utilised in the proposed FLP method to obtain the fuzzy and crisp optimal solutions by solving one LP model. Moreover, we introduce an alternative model with deterministic variables and parameters derived from the proposed FLP model. Interestingly, the result of the alternative model is identical to the crisp solution of the proposed FLP model. We use a numerical example from the FLP literature for comparison purposes and to demonstrate the applicability of the proposed method and exhibit the efficacy of the procedure.

Keywords: fuzzy linear programming; trapezoidal fuzzy numbers; duality; complementary slackness theory.

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1 Introduction

Linear programming (LP) is a quantitative tool for optimal allocation of limited resources amongst competing activities. It is perhaps the most popular operations research technique with applications in several functional areas of business such as production, finance, marketing, distribution, advertising and so forth (Chen and Ko, 2009, 2010; Hassanzadeh Amin et al., 2011; Peidro et al., 2010; Rong and Lahdelma, 2008). The conventional LP deals with crisp parameters. However, managerial decision-making is subject to professional judgments usually based on imprecise, vague, uncertain or incomplete information. Fuzzy set theory has been used to handle such imprecision by generalising the notion of membership in a set. Essentially, each element in a fuzzy set is associated with a point-value selected from the unit interval [0, 1]. Fuzzy set theory has been extensively employed in LP. The main objective in fuzzy LP (FLP) is to find the best solution possible with imprecise, vague, uncertain or incomplete information. There are many sources of imprecision in FLP. The sources of imprecision in FLP vary. For example, sometimes constraint satisfaction limits are vague and other times coefficient variables are not known precisely. The fundamental challenge in FLP is to construct an optimisation model that can produce the optimal solution with subjective professional judgments.

The problem of ranking fuzzy numbers plays an important role in decision-making. Numerous researchers have proposed different methods for ranking fuzzy numbers since the inception of fuzzy sets theory by Zadeh (1965). Wang and Kerre (2001a, 2001b) classified the ranking methods into three main categories. The first category is composed of ranking methods based on fuzzy mean and spread (e.g., Adamo, 1980; Liou and Wang, 1992), the second category consists of ranking methods based on fuzzy scoring (e.g., Bortolan and Degani, 1985; Kim and Park, 1990) and the third category is composed of methods based on preference relations (e.g., Dubois and Prade, 1983; Yuan, 1991).

We propose a new method for solving FLP problems in which the right-hand side parameters and the decision variables are represented by fuzzy numbers. A new fuzzy ranking model and a new supplementary variable are utilised in the proposed FLP method to obtain the fuzzy and crisp optimal solutions by solving one LP model.

Moreover, we introduce an alternative model with deterministic variables and parameters derived from the proposed FLP model. The proposed method:

a is computationally simple

b captures the ambiguity and impreciseness in DMs' judgments.

This paper is organised as follows: The next section presents a brief review of the existing literature followed by a description of the FLP problem. We then introduce the mathematical details of the proposed method and after that we discuss the concept of duality in FLP. Finally, we present the results from a numerical example and finish the paper with our conclusions and future research directions.

2 Literature review

The theory of fuzzy mathematical programming was first proposed by Tanaka et al. (1974) based on the fuzzy decision framework of Bellman and Zadeh (1970). Zimmermann (1978) introduced the first formulation of FLP to address the impreciseness and vagueness of the parameters in LP problems with fuzzy constraints and objective functions. Zimmermann (1978) constructed a crisp model of the problem and obtained its crisp results using an existing algorithm. He then used Bellman and Zadeh's (1970) interpretation that a fuzzy decision is a union of goals and constraints and fuzzified the problem by considering subjective constants of admissible deviations for the goal and the constraints. Finally, he defined an equivalent crisp problem using an auxiliary variable that represented the maximisation of the minimisation of the deviations on the constraints. There are generally five FLP classifications in the literature:

- Zimmermann (1987) has classified FLP problems into two categories: symmetrical and non-symmetrical models. In a symmetrical fuzzy decision there is no difference between the weight of the objectives and constraints while in the asymmetrical fuzzy decision, the objectives and constraints are not equally important and have different weights (Amid et al., 2006).
- Leung (1988) has classified FLP problems into four categories:
  1. a precise objective and fuzzy constraints
  2. a fuzzy objective and precise constraints
  3. a fuzzy objective and fuzzy constraints
  4. robust programming.
- Luhandjula (1989) has classified FLP problems into three categories:
  1. flexible programming
  2. mathematical programming with fuzzy parameters
  3. fuzzy stochastic programming.
- Inuiguchi et al. (1990) have classified FLP problems into six categories:
  1. flexible programming
  2. possibilistic programming
  3. possibilistic LP using fuzzy max

4 robust programming
5 possibilistic programming with fuzzy preference relations
6 possibilistic LP with fuzzy goals.

- Kumar et al. (2011) have divided FLP problems into two categories:
  1 FLP problems with inequality constraints
  2 FLP problems with equality constraints.

Some authors (Buckley and Feuring, 2000; Hashemi et al., 2006; Allahviranloo et al., 2008) have proposed different methods for solving FLP problems with inequality constraints where initially the FLP problem is converted into crisp LP problem and then the obtained crisp LP problem is solved to find the fuzzy optimal solution of the FLP problems. Other authors (Dehghan et al., 2006; Hosseinzadeh Lotfi et al., 2009) have proposed methods for solving FLP problems with equality constraints. However, the solutions to FLP problems with equality constraints are generally approximate (Hosseinzadeh Lotfi et al., 2009).

In the past decade, researchers have discussed various properties of FLP problems and proposed an assortment of models. Zhang et al. (2003) proposed a FLP with fuzzy numbers for the coefficients of objective functions. They introduced a number of optimal solutions for the FLP problems and developed a number of theorems for converting the FLP problems into multi-objective optimisation problems. Stanciulescu et al. (2003) proposed a FLP model with fuzzy coefficients for the objectives and the constraints. He used fuzzy decision variables with a joint membership function instead of crisp decision variables and linked the decision variables together to sum them up to a constant. He considered lower-bounded fuzzy decision variables that set up the lower bounds of the decision variables. He then generalised the method to lower–upper-bounded fuzzy decision variables that set up also the upper bounds of the decision variables. Katagiri et al. (2004) considered a multi-objective 0–1 programming problem with fuzzy random variables as coefficients of objective functions and proposed a decision-making model for maximising the expected degrees of possibility that the objective function values attained the fuzzy goals.

Ganesan and Veeramani (2006) proposed a FLP model with symmetric trapezoidal fuzzy numbers. They proved fuzzy analogues of some important LP theorems and obtained some interesting results which in turn led to the solution for FLP problems without converting them into crisp LP problems. Ebrahimnejad (2011a) showed that the method proposed by Ganesan and Veermani (2006) stops in a finite number of iterations and proposed a revised version of their method that was more efficient and robust in practice. He also proved the absence of degeneracy and showed that if an FLP problem has a fuzzy feasible solution, it also has a fuzzy basic feasible solution and if an FLP problem has an optimal fuzzy solution, it also has an optimal fuzzy basic solution. To solve a multi-objective programming problem with fuzzy coefficients, Wu (2008a) transformed the problem into a vector optimisation problem by applying the embedding theorem and using a suitable linear defuzzification function.

Lodwick and Jamison (2007) developed the theory underlying fuzzy, possibilistic and mixed fuzzy/possibilistic optimisation and demonstrated the appropriate use of distinct solution methods associated with each type of optimisation dependent on the semantics of the problem. Mahdavi-Amiri and Nasseri (2007) developed some methods for solving FLP problems by introducing and solving certain auxiliary problems. They applied a linear ranking function to order trapezoidal fuzzy numbers and deduced some duality results by establishing the dual problem of the LP problem with trapezoidal fuzzy variables. Rommelfanger (2007) showed that both the probability distributions and fuzzy sets should be used in parallel or in combination, to model imprecise data dependent on the real situation. van Hop (2007) presented a model to measure attainment values of fuzzy numbers/fuzzy stochastic variables and used these new measures to convert the FLP problem or the fuzzy stochastic LP problem into the corresponding deterministic LP problem.

Mahdavi-Amiri and Nasseri (2006) proposed a FLP model where a linear ranking function was used to rank order trapezoidal fuzzy numbers. They established the dual problem of the LP problem with trapezoidal fuzzy variables and deduced some duality results to solve the FLP problem directly with the primal simplex tableau. Ebrahimnejad (2010) introduced a new primal-dual algorithm for solving FLP problems by using the duality results proposed by Mahdavi-Amiri and Nasseri (2007). Ebrahimnejad (2011b) has also generalised the concept of sensitivity analysis in FLP problems by applying fuzzy simplex algorithms and using the general linear ranking functions on fuzzy numbers.

Ghodousian and Khorram (2008) studied the new linear objective function optimisation with respect to the fuzzy relational inequalities defined by max-min composition in which fuzzy inequality replaces ordinary inequality in the constraints. They showed that their method attains the optimal points that are better solutions than those resulting from the resolution of the similar problems with ordinary inequality constraints. Tan et al. (2008) developed a FLP extension of the general life cycle model using a concise and consistent linear model that makes identification of the optimal solution straightforward. Wu (2008b) derived the optimality conditions for FLP problems by proposing two solution concepts based on similar solution concept, called the non-dominated solution, in the multi-objective programming problem.

Hosseinzadeh Lotfi et al. (2009) considered full FLP problems where all variable and parameters were triangular fuzzy numbers. They showed that there is no method in the literature for finding the fuzzy optimal solution of full FLP problems and introduced a new method for solving full FLP problems with equality constraints. They used the concept of the symmetric triangular fuzzy numbers and proposed an approach to defuzzify a general fuzzy quantity. They first approximated the fuzzy triangular numbers to its nearest symmetric triangular numbers with the assumption that all decision variables were symmetric triangular. They then converted every FLP model into two crisp complex LP models and used a special ranking for fuzzy numbers to transform their full FLP model into a multi-objective LP where all variables and parameters were crisp. Kumar et al. (2011) further studied the full FLP problems with equality introduced by Hosseinzadeh Lotfi et al. (2009) and proposed a new method for finding the fuzzy optimal solution in these problems.

Gupta and Mehlawat (2009) studied a pair of fuzzy primal-dual LP problems and calculated duality results using an aspiration level approach. Their approach is particularly important for FLP where the primal and dual objective values may not be bounded. Peidro et al. (2010) used fuzzy sets and developed a FLP to model the supply chain uncertainties. Chen and Ko (2010), Inuiguchi and Ramík (2000) and Peidro et al. (2010) have developed a number of FLP models to solve problems ranging from supply chain management to product development. In summary, the FLP models in the literature could be classified into the following seven groups:

- *Group 1*: the FLP problems in this group involve fuzzy numbers for the decision variables and the right-hand-side of the constraints (e.g., Mahdavi-Amiri and Nasseri, 2007).
- *Group 2*: the FLP problems in this group involve fuzzy numbers for the coefficients of the decision variables in the objective function (e.g., Wu 2008b).
- *Group 3*: the FLP problems in this group involve fuzzy numbers for the coefficients of the decision variables in the constraints and the right-hand-side of the constraints (e.g., Xinwang, 2001).
- *Group 4*: the FLP problems in this group involve fuzzy numbers for the decision variables, the coefficients of the decision variables in the objective function and the right-hand-side of the constraints (e.g., Ganesan and Veeramani, 2006).
- *Group 5*: the FLP problems in this group involve fuzzy numbers for the coefficients of the decision variables in the objective function, the coefficients of the decision variables in the constraints and the right-hand-side of the constraints (e.g., Mahdavi-Amiri and Nasseri, 2006; Hatami-Marbini et al., 2013; Hatami-Marbini and Tavana, 2011).
- *Group 6*: the FLP problems in this group involve fuzzy numbers for the coefficients of the decision variables in the decision variables, the coefficients of the decision variables in the constraints and the right-hand-side of the constraints (e.g., Saati et al., 2012).
- *Group 7*: the FLP problems in this group, so-called fully FLP (FFLP) problems, involve fuzzy numbers in the decision variables, the coefficients of the decision variables in the objective function, the coefficients of the decision variables in the constraints and the right-hand-side of the constraints (e.g., Hosseinzadeh Lotfi et al., 2009).

Although the FFLP group is the general case of the FLP, it may not be suitable for all FLP problems with different assumptions and sources of fuzziness. The FLP model proposed in this study belongs to *Group 1* in which the right-hand side parameters and the decision variables are represented by fuzzy numbers. The proposed method is computationally simple and provides a better solution compared with the existing FLP methods in the literature (e.g., Mahdavi-Amiri and Nasseri, 2007).

3 Fuzzy sets theory

This section introduces some basic definitions for fuzzy sets (Dubois and Prade, 1978, 1980; Kaufmann and Gupta, 1991; Klir and Yuan, 1995; Zadeh, 1965; Zimmermann, 1996).

Definition 3.1. Let U be a universe set. A fuzzy set $\tilde{E}$ in U is defined by a set of ordered pairs $\tilde{E} = \{(x, \mu_{\tilde{E}}(x)|x \in U\}$ where $\mu_{\tilde{E}}(x), \ \forall x \in U,$ indicates the degree of membership of $\tilde{E}$ to U.

Definition 3.2. A fuzzy subset $\tilde{A}$ of real number U is convex if and only if

$$\mu_{\tilde{E}}\left(\lambda x+(1-\lambda)y\right) \geq \left(\mu_{\tilde{E}}(x) \wedge \mu_{\tilde{E}}(y)\right), \forall x, y \in U, \forall \lambda \in [0, 1],$$

where '$\wedge$' denotes the minimum operator.

Definition 3.3. The α-level of fuzzy set Ẽ, $\tilde{E}_\alpha$, is the crisp set $\tilde{E}_\alpha = \{x \mid \mu_{\tilde{E}}(x) \geq \alpha\}$. The support of Ẽ is the crisp set $Sup(\tilde{E}) = \{x \mid \mu_{\tilde{E}}(x) > 0\}$. Ẽ is normal if and only if $Sup_{x \in U}\mu_{\tilde{E}}(x) = 1$, where U is the universal set.

Definition 3.4. Ẽ is a fuzzy number if Ẽ is a normal and convex fuzzy subset of U.

Definition 3.5. A fuzzy number $\tilde{E} = (a^{m1}, a^{m2}, a^l, a^u)$, is called a generalised trapezoidal fuzzy number with membership function $\mu_{\tilde{E}}$ and the following properties:

a $\mu_{\tilde{E}}$ is a continuous mapping from R to the closed interval [0, 1]

b $\mu_{\tilde{E}}(x) = 0$ for all $x \in (-\infty, a^l]$,

c $\mu_{\tilde{E}}$ is strictly increasing on $[a^l, a^{m1}]$

d $\mu_{\tilde{E}}(x) = 1$ for all $x \in [a^{m1}, a^{m2}]$

e $\mu_{\tilde{E}}$ is strictly decreasing on $[a^{m2}, a^u]$

f $\mu_{\tilde{E}}(x) = 0$ for all $x \in [a^u, +\infty)$.

The membership function $\mu_{\tilde{E}}$ of Ẽ can be defined as follows:

$$\mu_{\tilde{E}}(x) = \begin{cases} f_a(x), & a^l \leq x \leq a^{m1}, \\ 1, & a^{m1} \leq x \leq a^{m2}, \\ g_a(x), & a^{m2} \leq x \leq a^u, \\ 0, & \text{Otherwise.} \end{cases} \tag{1}$$

where $f_a: [a^l, a^{m1}] \rightarrow [0, 1]$ and $g_a: [a^{m2}, a^u] \rightarrow [0, 1]$.

The inverse functions of f_a and g_a, denoted as f_a^{-1} and g_a^{-1}, exist. Since $f_a: [a^l, a^{m1}] \rightarrow [0, 1]$ is continuous and strictly increasing, $f_a^{-1}: [0, 1] \rightarrow [a^l, a^{m1}]$ is also continuous and strictly decreasing. Similarly, since $g_a: [a^{m2}, a^u] \rightarrow [0, 1]$ is continuous and strictly decreasing, $g_a^{-1}: [0, 1] \rightarrow [a^{m2}, a^u]$ is also continuous and strictly increasing. That is, both $\int_0^1 f_a^{-1}$ and $\int_0^1 g_a^{-1}$ exist (Liou and Wang, 1992).

Moreover, the parametric form of a fuzzy number Ẽ can be denoted by $(\underline{a}(r), \overline{a}(r))$, $0 \leq r \leq 1$, which satisfies the following requirements:

1 $\underline{a}(r)$ is a bounded increasing left continuous function

2 $\overline{a}(r)$ is a bounded decreasing right continuous function

3 $\underline{a}(r) \leq \overline{a}(r)$, where $0 \leq r \leq 1$.

Particularly, we are working with a special type of the trapezoidal fuzzy number with a membership function $\mu_{\tilde{E}}$ expressed by:

$$\mu_{\tilde{E}}(x)=\begin{cases}\dfrac{x-a^l}{a^{m1}-a^l}, & a^l\le x\le a^{m1},\\ 1, & a^{m1}\le x\le a^{m2},\\ \dfrac{a^u-x}{a^u-a^{m2}}, & a^{m2}\le x\le a^u,\\ 0, & \text{Otherwise.}\end{cases} \tag{2}$$

The trapezoidal fuzzy number $\tilde{E} = (a^{m1}, a^{m2}, a^l, a^u)$ is reduced to a real number A if $a^l = a^{m1} = a^{m2} = a^u$. Conversely, a real number A can be written as a trapezoidal fuzzy number $\tilde{A} = (a, a, a, a)$. If $\tilde{E} = (a^m, a^l, a^u)$ then, is called a triangular fuzzy number. A triangular fuzzy number has the following membership function:

$$\mu_{\tilde{E}}(x)=\begin{cases}\dfrac{x-a^l}{a^m-a^l}, & a^l\le x\le a^m,\\ 1, & x=a^m,\\ \dfrac{a^u-x}{a^u-a^m}, & a^m\le x\le a^u,\\ 0, & \text{Otherwise.}\end{cases} \tag{3}$$

For the purpose of simplicity and without loss of generality, we assume that all fuzzy numbers used throughout the paper are trapezoidal fuzzy numbers. Among the various types of fuzzy numbers, trapezoidal fuzzy numbers are used most often for characterising information in practical applications (Klir and Yuan, 1995; Yeh and Deng, 2004). The common use of trapezoidal fuzzy numbers is mainly attributed to their simplicity in both concept and computation.

Definition 3.6. The minimum t-norm is usually applied in FLP to evaluate a linear combination of fuzzy quantities. Therefore, for a given set of trapezoidal fuzzy numbers $\tilde{e}_j = (a_j^{m1}, a_j^{m2}, a_j^l, a_j^u)$, $j = 1,2,\ldots,n$ and λ_j, $\tilde{E} = \sum_{j=1}^{n}\lambda_j\tilde{e}_j$ as a trapezoidal fuzzy number is defined as follows:

If $\lambda_j \ge 0$, then $\tilde{E} = \sum_{j=1}^{n}\lambda_j\tilde{e}_j = \left(\sum_{j=1}^{n}\lambda_j a_j^{m1}, \sum_{j=1}^{n}\lambda_j a_j^{m2}, \sum_{j=1}^{n}\lambda_j a_j^{l}, \sum_{j=1}^{n}\lambda_j a_j^{u}\right)$

If $\lambda_j \ge 0$, then $\tilde{E} = \sum_{j=1}^{n}\lambda_j\tilde{e}_j = \left(\sum_{j=1}^{n}\lambda_j a_j^{m2}, \sum_{j=1}^{n}\lambda_j a_j^{m1}, \sum_{j=1}^{n}\lambda_j a_j^{u}, \sum_{j=1}^{n}\lambda_j a_j^{l}\right)$

where $\sum_{j=1}^{n}\lambda_j\tilde{e}_j$ denotes the combination $\lambda_1\tilde{e}_1 \oplus \lambda_2\tilde{e}_2 \oplus \ldots \oplus \lambda_n\tilde{e}_n$.

4 The FLP problem

In this section we introduce a FLP model where decision variables and resources (right hand side values) are fuzzy quantities. A LP problem may be defined as the problem of maximising or minimising (in the proposed model) a linear function subject to linear constraints:

$$\begin{aligned}&\text{Minimise } Z = cx\\&\text{Subject to:}\\&\qquad A.x \geq b,\ x \geq 0.\end{aligned} \tag{4}$$

where $x \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and A is an $(m \times n)$ real matrix. Contrary to the classical LP problem, here, x and b will be the fuzzy numbers and they are denoted by symbols with the tilde above. Let $\mu_{\tilde{b}} : R \to [0,1]$, $\mu_{\tilde{x}} : R \to [0,1]$ be membership functions of the fuzzy numbers, $\tilde{b}$ and $\tilde{x}$, respectively. To define a FLP problem, we will use the following proposition:

Proposition 4.1. Let $\tilde{b}, \tilde{x} \in F(R)$ where $F(R)$ presents the set of all fuzzy subsets. Then the fuzzy set $c\tilde{x}$ and $A.\tilde{x}$, based on the *extension principle*, is a fuzzy number. The FLP problem can be expressed as follows:

$$\begin{aligned}&\text{Minimise } \tilde{Z} = cx\\&\text{Subject to:}\\&\qquad A.\tilde{x} \geq \tilde{b},\ \tilde{x} \geq \tilde{0}.\end{aligned} \tag{5}$$

where $\tilde{b} = (\tilde{b}_1, \tilde{b}_2, \ldots \tilde{b}_m)$ and $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)^t$ represent, respectively, fuzzy parameters for the right-hand side and fuzzy decision variables involved in the objective function and constraints. $c = (c_1, c_2, \ldots, c_n)$ and $A = [a_{ij}]_{m \times n}$ are crisp parameters. Note here that we propose a new definition for non-negative fuzzy variables (i.e., see Definition 5.3).

Definition 4.1. (Feasible solution): The set of values of the fuzzy variables $\tilde{x}$ which satisfies all constraints of model (2) is called a feasible solution.

Definition 4.2. (Optimal solution): $\tilde{x}^*$ is the optimal solution for model (5), if for all feasible solutions $\tilde{x}$, we have $c\tilde{x}^* \leq c\tilde{x}$.

Let us further discuss the fuzzy basic feasible solution and the optimal solution. Consider the FLP problem (5). Let *rank* $(A) = m$ and define partition A as $[B\ N]$ where B, $m \times m$, is non-singular. Let y be the solution to $By = a_j$ where a_j is the j^{th} column of the coefficient matrix. Thus, $\tilde{x}_B = (\tilde{x}_{B_1}, \ldots, \tilde{x}_{B_m})^t = B^{-1}\tilde{b}$ and $\tilde{x}_N = \tilde{0}$ is a solution of $A\tilde{x} = \tilde{b}$. $\tilde{x} = (\tilde{x}_B^t, \tilde{0})$ is called a *fuzzy basic feasible solution* (*FBFS*) corresponding to the basic B when $\tilde{x}_B \geq \tilde{0}$. The fuzzy objective value is $\tilde{z} = c_B \tilde{x}_B$ where $c_B = (c_{B_1}, \ldots c_{B_m})$. In addition, for the basic variables we have

$$z_j - c_j = c_B B^{-1} a_j - c_j = c_B e_j - c_j = c_{B_i} - c_j = c_j - c_j = \tilde{0}$$

Note that $B^{-1}a_j = e_j$ where $e_j = (0,\dots, 1,\dots, 0)^t$.

If $\tilde{x}_B > \tilde{0}$; then $\tilde{x}$ is called a *non-degenerate* FBFS, and if one component of $\tilde{x}$ is zero, then $\tilde{x}$ is called a *degenerate* FBFS. A FBFS is optimal if and only if $z_j = c_B B^{-1} a_j \le c_j$ for all j. In other words, the FLP problem can be rewritten as follows:

$$\begin{aligned} &\text{Minimise} \quad \tilde{Z} = c_B\tilde{x}_B + c_N\tilde{x}_N \\ &\text{Subject to:} \\ &\qquad B\tilde{x}_B + N\tilde{x}_N \ge \tilde{b}, \\ &\qquad \tilde{x}_B \ge \tilde{0},\ \tilde{x}_N \ge \tilde{0}. \end{aligned} \tag{6}$$

If $\tilde{x}^* = (\tilde{x}_B^t, x_N^t) = (B^{-1}\tilde{b}, \tilde{0})$ is a FBFS, then $z^* = c_B\tilde{x}_B = c_B B^{-1}\tilde{b}$. Now, the objective function of (6) can be written as:

$$\begin{aligned} \tilde{z} = c\tilde{x} = c_B\tilde{x}_B + c_N\tilde{x}_N = c_B B^{-1}\tilde{b} - (c_B B^{-1}N - c_N)\,\tilde{x}_N = c_B B^{-1}\tilde{b} - \sum_{j\in N}(c_B B^{-1}a_j - c_j)\tilde{x}_j = \\ = c_B B^{-1}\tilde{b} - \sum_{j\in N}(z_j - c_j)\tilde{x}_j = \tilde{z}^* - \sum_{j\in N}(z_j - c_j)\tilde{x}_j \end{aligned}$$

where N is the index set of non-basic variables. For each feasible case of $\tilde{x}$, z_j is smaller than or equal to c_j, therefore, $(z_j - c_j)\tilde{x}_j \le \tilde{0}$ and $\sum_{j\in N}(z_j - c_j)\tilde{x}_j \le \tilde{0}$ or $\tilde{z}^* \le \tilde{z}$. That is to say, $\tilde{x}^*$ is an optimal solution.

5 The proposed method

In this section we present the mathematical details of the proposed method. We first propose a new solution method for the FLP model (2) where the variables and resources are assumed to be fuzzy numbers. We can then obtain the fuzzy optimal solution and the crisp optimal solution using one LP. Our solving method is developed without the use of fuzzy arithmetic.

Definition 5.1. $\tilde{A} = (a^{m_1}, a^{m_2}, a^l, a^u)$ is a non-negative trapezoidal fuzzy number if $a^l + a^u \ge 0$ and $a^{m_1} + a^{m_2} \ge 0$. Obviously, $\tilde{A}$ is a non-positive trapezoidal fuzzy number if $a^l + a^u \le 0$ and $a^{m_1} + a^{m_2} \le 0$. In relation to this definition we cannot examine the negativity and positivity of a fuzzy number in the remaining cases.

According to a given definition, a trapezoidal fuzzy number might be either positive or negative. For instance, a common alternative definition for positivity of a trapezoidal fuzzy number $(a^{m_1}, a^{m_2}, a^l, a^u)$ when $a^l > 0$. However, this definition is not consistent with the concept of fuzzy logic and it derives from crisp logic while the Definition 5.1 is deduced with respect to the fuzzy concept. Let (2, 1, –3, 8) be a trapezoidal fuzzy number. It is a non-negative fuzzy number based on Definition 5.1 while it would be negative because of $a^l < 0$. In this paper, we use the Definition 5.1 to determine the non-negative trapezoidal fuzzy numbers.

Definition 5.2. Let $\tilde{A} = (a^{m_1}, a^{m_2}, a^l, a^u)$ be a non-negative trapezoidal fuzzy number. Therefore, $\tilde{A}$ corresponds to the following relations:

$$
\begin{aligned}
&a^u \geq a^{m_2}, \\
&a^{m_2} \geq a^{m_1}, \\
&a^{m_1} \geq a^l, \\
&a^l + a^u \geq 0, \\
&a^{m_1} + a^{m_2} \geq 0.
\end{aligned}
$$

Definition 5.3. A trapezoidal fuzzy variable $\tilde{x} = (x^{m_1}, x^{m_2}, x^l, x^u)$ is a non-negative trapezoidal fuzzy variable if we impose the following conditions:

$$
\begin{aligned}
&x^u \geq x^{m_2}, \\
&x^{m_2} \geq x^{m_1}, \\
&x^{m_1} \geq x^l, \\
&x^l + x^u \geq 0, \\
&x^{m_1} + x^{m_2} \geq 0.
\end{aligned} \tag{7}
$$

Accordingly, note that here we have always $x^{m_2} \geq 0$ and $x^u \geq 0$.

Definition 5.4. Let $A\tilde{x} \geq \tilde{b}$ be the constraint of model (2). This constraint corresponds to the following four constraints:

$$
\begin{aligned}
&Ax^{m_1} \geq b^{m_1}, \\
&Ax^{m_2} \geq b^{m_2}, \\
&Ax^l \geq b^l, \\
&Ax^u \geq b^u.
\end{aligned}
$$

Note that Definition 5.1 is given for identification of the positive fuzzy numbers while Definition 5.4 enables us to solve the linear inequalities with fuzzy variables and fuzzy right hand side variables so that the solution is able to satisfy the four conditions. Using this approach we can then obtain the fuzzy optimal solution and the crisp optimal solution by solving one LP.

Definition 5.5. The proposed method is intended to obtain both the optimal solution of the FLP model with fuzzy variables and the crisp optimal solution. Hence, we define a new variable, namely *x*, which is smaller than or equal to x^{m_2} and bigger than or equal to x^{m_1} (i.e., $x^{m_1} \leq x \leq x^{m_2}$). Note that by the use of this definition and the proposed method, we can calculate the crisp solution on top of the fuzzy solutions.

We use the Definitions 5.3 and 5.4 to obtain the FLP model (5) to the following LP model:

$$\begin{aligned}&\text{Minimise} \quad Z = cx\\&\text{Subject to:}\\&\qquad Ax^i \geq b^i, \quad i = m_1, m_2, l, u, \quad (1)\\&\qquad \left.\begin{aligned}&x^{m_1} \leq x \leq x^{m_2},\\&x^u - x^{m_2} \geq 0,\\&x^{m_2} - x^{m_1} \geq 0,\\&x^{m_1} - x^l \geq 0,\\&x^l + x^u \geq 0,\\&x^{m_1} + x^{m_2} \geq 0.\end{aligned}\right\} \quad (2)\end{aligned} \tag{8}$$

where $x^{m_1}, x^{m_2}, x^l, x^u$ and x are the decision variables which their optimal values are determined by the proposed model. Note that (2) in model (8) that is the set of constraints can be shown as

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^{m_1} \\ x^{m_2} \\ x^l \\ x^u \\ x \end{bmatrix} \geq 0$$

In consequence, $\tilde{x} = (x^{m_1}, x^{m_2}, x^l, x^u)$ is the fuzzy optimal solution and x is the crisp optimal solution for FLP model (5). Thus, the fuzzy and crisp objective function values can be computed, respectively, as $Z = cx$ and $\tilde{Z} = c\tilde{x} = (cx^{m_1}, cx^{m_2}, cx^l, cx^u)$ where c is positive.

By applying the following theorem, model (5) can be reduced to a LP problem as shown by Maleki et al. (2000) and Maleki (2002).

It is interesting to mention that the optimal solution of the following programme with positive A is equal to the crisp optimal solution of (8).

$$\begin{aligned}&\text{Minimise} \quad Z = cx\\&\text{Subject to:}\\&\qquad Ax \geq b^{m_1}, \quad x \geq 0.\end{aligned} \tag{9}$$

Note that in the maximisation of the FLP problems, the right hand side value of (9) must be b^{m_2}. The proposed model (8) is equivalent to the general LP model (9) with more constraints. In other words, model (8) involves $4m + 7n$ constraints and $5n$ decision variables while model (9) has m constraints and n decision variables.

6 Duality in FLP

In this section we examine the well-known concept of duality in LP for FLP problems based on fuzzy relations. The dual of the proposed model (8) is as follows:

$$\begin{aligned}&\text{Maximise}\quad D = b^{m_1} y^1 + b^{m_1} y^2 + b^l y^3 + b^u y^4\\&\text{Subject to:}\\&\qquad y^5 - y^6 = c,\\&\qquad A^T y^1 - y^5 - y^8 + y^9 + y^{11} = 0,\\&\qquad A^T y^2 + y^6 - y^7 + y^8 + y^{11} = 0,\\&\qquad A^T y^3 - y^9 + y^{10} = 0,\\&\qquad A^T y^4 + y^7 + y^{10} = 0,\\&\qquad y^i \geq 0, \quad i = 1,\ldots,11.\end{aligned} \tag{10}$$

In comparison with the primal model (8), the dual of the proposed model (10) is computationally efficient because it involves $5n$ constrains. Considering model (9) as a general form of the primal proposed model, model (10) as its dual form is equivalent to the following LP model:

$$\begin{aligned}&\text{Maximise}\quad D = b^t y\\&\text{Subject to:}\\&\qquad A^t y = c, \quad y \geq 0\end{aligned} \tag{11}$$

where $y \in \mathbb{R}^m, c \in \mathbb{R}^n, b \in \mathbb{R}^m$ and A is an $(n \times m)$ real matrix.

The following theorems for duality hold between (9) and (11).

Theorem 6.1 (Weak duality theorem): Let $\hat{x}$ be a feasible solution of the primal problem and $\hat{y}$ be a feasible solution of the dual problem. Then, $c\hat{x} \geq b^t \hat{y}$.

Proof. Let $\hat{x}$ and $\hat{y}$ be the feasible solutions. Then, $A\hat{x} \geq b$ and $A^t \hat{y} = c$. If we multiply $A\hat{x} \geq b$ and $A^t \hat{y} = c$, respectively, by $\hat{y} \geq 0$ and $\hat{x} \geq 0$ on both sides of the relationships; then we get $c\hat{x} = \hat{y}A\hat{x} \geq b^t \hat{y}$ or $c\hat{x} \geq b^t \hat{y}$.

Corollary 6.1. Let x^* and y^* be the feasible solutions of the primal and dual problems, respectively, such that $cx^* = b^t y^*$. Then x^* and y^* are optimal solutions to models (9) and (11), respectively.

Theorem 6.2. (Duality theorem): Let $\hat{x}$ be an optimal solution of the primal model. Then, there exists $\hat{y}$ which is optimal to the dual model and conversely. Further, $c\hat{x} = b^T \hat{y}$.

Theorem 6.3. (Existence theorem): If the primal model is unbounded; then, the dual model is infeasible and if the primal model is infeasible and the dual model is feasible; then, the dual model is unbounded. Furthermore, it is possible that both the primal and dual models are infeasible.

Theorem 6.4. (Complementary slackness theorem): If in any optimal solution of the primal problem, the slack variable $x_{n+i}^* > 0$; then, in every optimal solution of the dual problem, $y_i^* = 0$. Conversely, if $y_i^* > 0$ in any optimal solution of the dual model; then, in every optimal solution of (LP) $x_{n+i}^* = 0$; i.e., for a pair of optimal solutions of primal and dual, $x_{n+i}^* y_i^* = 0$.

7 Numerical example

In this section, we use the numeric example proposed by Mahdavi-Amiri and Nasseri (2007) to demonstrate the advantage of the proposed model as well as comparing our results with Mahdavi-Amiri and Nasseri's (2007) results.

Mahdavi-Amiri and Nasseri (2007) used a linear ranking function to solve a LP model with trapezoidal fuzzy parameters and trapezoidal fuzzy decision variables by means of fuzzy simplex algorithm. Maleki et al. (2000) and Mahdavi-Amiri and Nasseri (2006, 2007) used the following ranking function to represent each fuzzy number, $A = (a^{m_1}, a^{m_2}, a^l, a^u)$, into the real line $F(\mathbb{R}) \rightarrow \mathbb{R}$:

$$\Re(\tilde{A}) = \frac{a^{m_1} + a^{m_2} + a^l + a^u}{2},$$

and consequently we have

$$\begin{aligned}
&\tilde{A} \geq \tilde{B} \quad \text{if} \quad \Re(\tilde{A}) \geq \Re(\tilde{B}), \\
&\tilde{A} > \tilde{B} \quad \text{if} \quad \Re(\tilde{A}) > \Re(\tilde{B}), \\
&\tilde{A} = \tilde{B} \quad \text{if} \quad \Re(\tilde{A}) = \Re(\tilde{B}), \\
&\tilde{A} \leq \tilde{B} \quad \text{if} \quad \tilde{B} \geq \tilde{A}.
\end{aligned}$$

Moreover, we have:

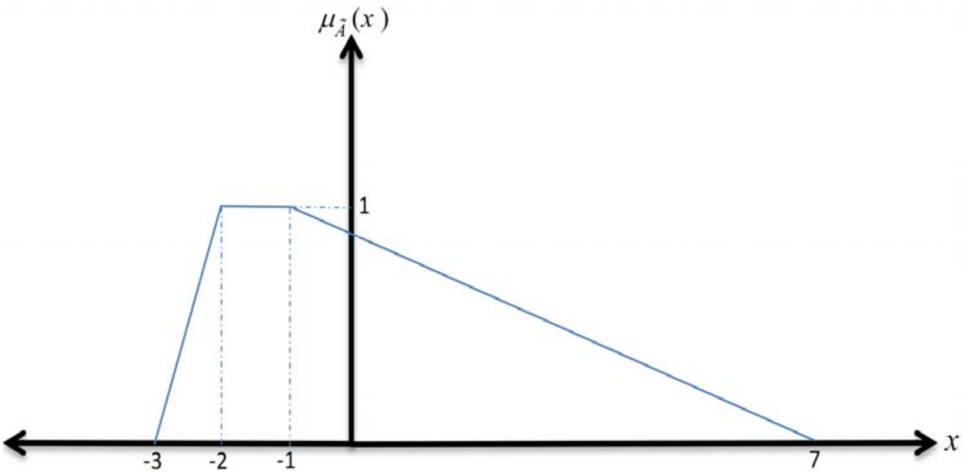
$$\tilde{A} \geq \tilde{0} \quad \text{if} \quad \frac{a^{m_1} + a^{m_2} + a^l + a^u}{2} \geq \frac{0^{m_1} + 0^{m_2} + 0^l + 0^u}{2}$$

or

$$\tilde{A} \geq \tilde{0} \quad \text{if} \quad (A^{m_1} + A^{m_2} + A^l + A^u) \geq 0.$$

This ranking function cannot provide a correct definition for non-negativity in trapezoidal fuzzy number. For example, consider $\tilde{A} = (-2, -1, -3, 7)$ that is shown in Figure 1.

Figure 1 A fuzzy number (see online version for colours)



Corresponding to the above definition of the ranking function, $R(\tilde{A}) = (-2-1-3+7)/2$ $= 0.5 \geq 0$ and $\tilde{A}$ is the non-negative fuzzy number. However, the question arises here is that how $\tilde{A}$ can be a non-negative when the main portion of $\tilde{A}$ places in the negative axis (see Figure 1). That is why the unexpected results reveal from the model proposed by Mahdavi-Amiri and Nasseri (2007).

Consider the FLP problem with two trapezoidal fuzzy variables and trapezoidal fuzzy resources for two corresponding constrains as follows:

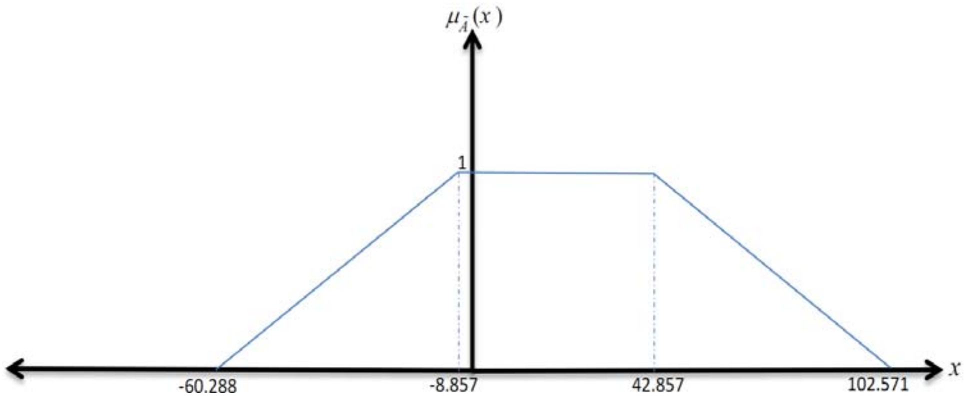
$$\begin{aligned} \min \quad & z = 6\tilde{x}_1 + 10\tilde{x}_2 \\ s.t. \quad & 2\tilde{x}_1 + 5\tilde{x}_2 \geq (5, 8, 3, 13) \\ & 3\tilde{x}_1 + 4\tilde{x}_2 \geq (6, 0, 4, 16) \\ & \tilde{x}_1, \tilde{x}_2 \geq \tilde{0} \end{aligned}$$

The optimal solutions and the objective function as trapezoidal fuzzy numbers for Mahdavi-Amiri and Nasseri (2007) are: as follows:

$$\begin{aligned} \tilde{x}_1 &= (-0.286, 4.286, -4.571, 9.715), \\ \tilde{x}_2 &= (-0.714, 1.714, -3.286, 7.428), \\ \tilde{z} &= (-8.857, 42.857, -60.288, 102.571). \end{aligned}$$

Figure 2 shows the corresponding fuzzy numbers for the optimal objective function value.

Figure 2 The optimal number of the objective functions for Mahdavi-Amiri and Nasseri's models (see online version for colours)



Next, we apply model (8) proposed in this study to the FLP problem to create the following LP formulation:

$$\begin{aligned}
\min \quad & z = 6x_1 + 10x_2 \\
s.t. \quad & 2x_1^{m_1} + 5x_2^{m_1} \ge 5, \quad 2x_1^{m_2} + 5x_2^{m_2} \ge 8, \quad 2x_1^{l} + 5x_2^{l} \ge 3, \quad 2x_1^{u} + 5x_2^{u} \ge 13, \\
& 3x_1^{m_1} + 4x_2^{m_1} \ge 6, \quad 3x_1^{m_2} + 4x_2^{m_2} \ge 10, \quad 3x_1^{l} + 4x_2^{l} \ge 4, \quad 3x_1^{u} + 4x_2^{u} \ge 16, \\
& x_1^{m_1} \le x_1 \le x_1^{m_2}, \quad x_2^{m_1} \le x_2 \le x_2^{m_2}, \\
& x_1^{u} - x_1^{m_2} \ge 0, \quad x_1^{m_2} - x_1^{m_1} \ge 0, \quad x_1^{m_2} - x_1^{l} \ge 0, \quad x_1^{u} + x_1^{l} \ge 0, \\
& x_1^{m_1} + x_1^{m_2} \ge 0, \quad x_2^{u} - x_2^{m_2} \ge 0, \quad x_2^{m_2} - x_2^{m_1} \ge 0, \quad x_2^{m_1} - x_2^{l} \ge 0, \\
& x_2^{u} + x_2^{l} \ge 0, \quad x_2^{m_1} + x_2^{m_2} \ge 0.
\end{aligned}$$

Solving the above LP model will result in the following optimal solutions:

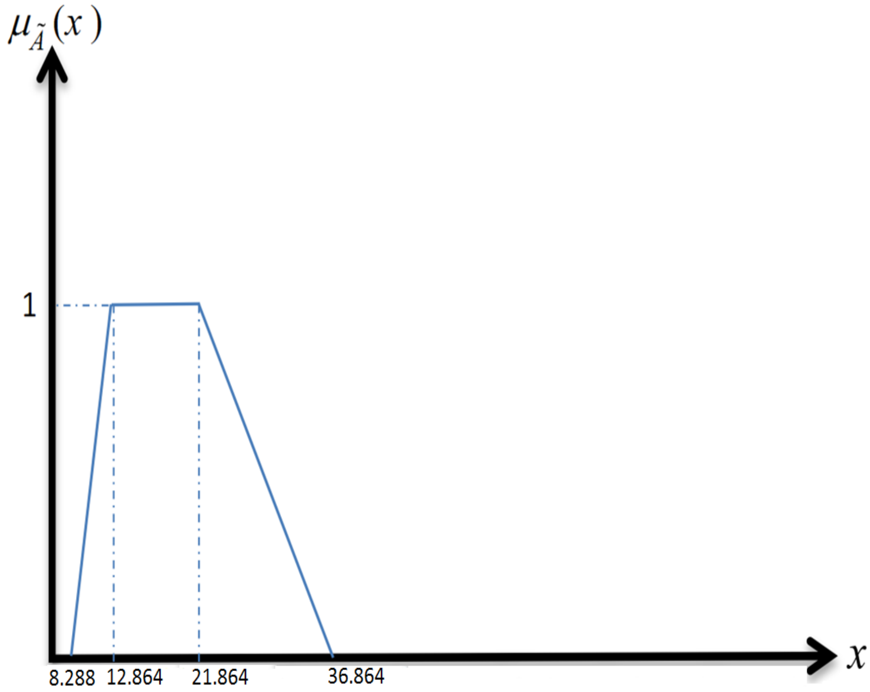
$$\begin{aligned}
&\tilde{x}_1 = (x_1^{m1}, x_1^{m2}, x_1^{l}, x_1^{u}) = (1.429, 2.929, 1.143, 5.429), \\
&\tilde{x}_2 = (x_2^{m1}, x_2^{m2}, x_2^{l}, x_2^{u}) = (0.429, 0.429, 0.143, 0.429), \\
&x_1 = 1.429, \\
&x_2 = 0.429.
\end{aligned}$$

and by putting these optimal solutions composed of the crisp measure (x_1, x_2) and the fuzzy measure $(\tilde{x}_1, \tilde{x}_2)$ into the objective function, we can achieve the deterministic and fuzzy objective function values at the same time that are reported as follows:

$$\begin{aligned}
&z = 6x_1 + 10x_2 = 12.857 \\
&\tilde{z} = 6\tilde{x}_1 + 10\tilde{x}_2 = (12.864, 21.864, 8.288, 36.864).
\end{aligned}$$

We should note that the proposed model enables us to obtain the crisp and fuzzy optimal solutions (consequently crisp and fuzzy optimal objective function values) by solving LP problem simultaneously. The fuzzy optimal value of the objective function is depicted in Figure 3.

Figure 3 Fuzzy optimal number for the objective function in model (8) (see online version for colours)



If we take model (9) into consideration for this example, the following model can be formulated as follows:

$$\begin{aligned} \min \quad & z = 6x_1 + 10x_2 \\ s.t. \quad & 2x_1 + 5x_2 \geq 5, \\ & 3x_1 + 4x_2 \geq 6, \\ & x_1, x_2 \geq 0. \end{aligned}$$

After solving the above LP model, the optimal solutions and the optimal value for the objective function are $x_1 = 1.429$, $x_2 = 0.429$ and $z = 12.857$. The interesting finding is that that the optimal solutions of model (9) and the crisp optimal solutions of model (8) are the same.

When the results of the proposed model and Mahdavi-Amiri and Nasseri's (2007) model are compared, it becomes clear that the fuzzy objective function of the former is quite better with respect to two reasons. First, the coefficients of the objective function are positive, six for $\tilde{x}_1$ and ten for $\tilde{x}_2$, and the fuzzy decision variable are non-negative. Therefore, the optimal value of the objective function must be non-negative while a portion of the respective fuzzy number in Mahdavi-Amiri and Nasseri's (2007) model are placed in the negative area (see Figure 2). However, the fuzzy optimal value of the objective function proposed in this study lies in the positive territory as shown in Figure 3. This finding confirms that our model produces a more rational solution compared with the model proposed by Mahdavi-Amiri and Nasseri (2007). Second, the model in this example is the minimisation problem, thereby; it aims at decreasing the value of the objective function as much as possible subject to the constraints. As it is shown in Figures 1 and 2, the fuzzy objective function of the proposed model in this study is much smaller than the Mahdavi-Amiri and Nasseri's (2007) model. The distance between the lower and upper bounds of the fuzzy objective function for Mahdavi-Amiri and Nasseri's (2007) model is 162.859 while the associated distance in our model is 28.576.

8 Conclusions and future research directions

Numerous FLP models with different levels of sophistication have been proposed over the past decade. However, many of these models have limited to real-world applications because of their methodological complexities and applicability. In contrast, the method proposed in this study is straight-forward and applicable to a wide range of real-world problems such as supply chain management, performance evaluation by means of data envelopment analysis, marketing management, failure mode and effect analysis and product development (Baykasoğlu and Göçken, 2008; Chen and Ko, 2010; Inuiguchi and Ramík, 2000; Peidro et al., 2010).

We proposed a new method for solving FLP problems in which the right-hand side parameters and the decision variables are represented by fuzzy numbers. We utilised a new fuzzy ranking model and a new supplementary variable in the proposed FLP method to obtain the fuzzy and crisp optimal solutions by solving one LP model. Moreover, we introduced an alternative model with deterministic variables and parameters derived from the proposed FLP model. We showed that the result of the alternative model is identical to the crisp solution of the proposed FLP model. We also used the numerical example of Mahdavi-Amiri and Nasseri (2007) for comparison purposes and to demonstrated the applicability of the proposed method and exhibited the efficacy of the procedure.

Future research will concentrate on the comparison of results obtained with those that might be obtained with other methods. In addition, we plan to extend the FLP approach proposed here to deal with fuzzy nonlinear optimisation problems with multiple objectives where the vagueness or impreciseness appears in all the components of the optimisation problem. Such an extension also implies the study of new practical experiments. Finally, we plan to focus on the use of co-evolutionary algorithms to solve fuzzy optimisation problems. This approach would permit the search for solutions covering optimality, diversity and interpretability. We hope that the concepts introduced here will provide inspiration for future research.

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References

Adamo, J.M. (1980) 'Fuzzy decision trees', *Fuzzy Sets and Systems*, Vol. 4, No. 3, pp.207–219.

Allahviranloo, T., Hosseinzadeh Lotfi, F., Kiasary, M.K., Kiani, N.A. and Alizadeh, L. (2008) 'Solving full fuzzy linear programming problem by the ranking function', *Applied Mathematical Sciences*, Vol. 2, No. 1, pp.19–32.

Amid, A., Ghodsypour, S.H. and O'Brien, C. (2006) 'Fuzzy multiobjective linear model for supplier selection in a supply chain', *International Journal of Production Economics*, Vol. 104, No. 2, pp.394–407.

Baykasoğlu, A. and Göçken, T. (2008) 'A review and classification of fuzzy mathematical programs', *Journal of Intelligent and Fuzzy Systems: Applications in Engineering and Technology*, Vol. 19, No. 3, pp.205–229.

Bellman, R.E. and Zadeh, L.A. (1970) 'Decision making in a fuzzy environment', *Management Science*, Vol. 17, No. 4, pp.141–164.

Bortolan, G. and Degani, R. (1985) 'A review of some methods for ranking fuzzy subsets', *Fuzzy Sets and Systems*, Vol. 15, No. 1, pp.1–19.

Buckley, J.J. and Feuring, T. (2000) 'Evolutionary algorithm solution to fuzzy problems: fuzzy linear programming', *Fuzzy Sets and Systems*, Vol. 109, No. 1, pp.35–53.

Chen, L.H. and Ko, W.C. (2009) 'Fuzzy linear programming models for new product design using QFD with FMEA', *Applied Mathematical Modelling*, Vol. 33, No. 2, pp.633–647.

Chen, L.H. and Ko, W.C. (2010) 'Fuzzy linear programming models for NPD using a four-phase QFD activity process based on the means-end chain concept', *European Journal of Operational Research*, Vol. 201, No. 2, pp.619–632.

Dehghan, M., Hashemi, B. and Ghatee, M. (2006) 'Computational methods for solving fully fuzzylinear systems', *Applied Mathematics and Computations*, Vol. 179, No. 1, pp.328–343.

Dubois, D. and Prade, H. (1978) 'Operations on fuzzy numbers', *International Journal of Systems Science*, Vol. 9, No. 6, pp.613–626.

Dubois, D. and Prade, H. (1980) *Operations on fuzzy numbers. Fuzzy Sets and System: Theory and Applications*, Academic Press, New York.

Dubois, D. and Prade, H. (1983) 'Ranking fuzzy numbers in the setting of possibility theory', *Information Sciences*, Vol. 30, No. 3, pp.183–224.

Ebrahimnejad, A. (2011a) 'Some new results in linear programs with trapezoidal fuzzy numbers: finite convergence of the Ganesan and Veeramani's method and a fuzzy revised simplex method', *Applied Mathematical Modelling*, Vol. 35, No. 9, pp.4526–4540.

Ebrahimnejad, A. (2011b) 'Sensitivity analysis in fuzzy number linear programming problems', *Mathematical and Computer Modelling*, Vol. 53, Nos. 9–10, pp.1878–1888.

Ebrahimnejad, A., Nasseri, S.H., Hosseinzadeh Lotfi, F. and Soltanifar, M. (2010) 'A primal-dual method for linear programming problems with fuzzy variables', *European Journal of Industrial Engineering*, Vol. 4, No. 2, pp.189–209.

Ganesan, K. and Veeramani, P. (2006) 'Fuzzy linear programs with trapezoidal fuzzy numbers', *Annals of Operations Research*, Vol. 143, No. 1, pp.305–315.

Ghodousian, A. and Khorram, E. (2008) 'Fuzzy linear optimization in the presence of the fuzzy relation inequality constraints with max–min composition', *Information Sciences*, Vol. 178, No. 2, pp.501–519.

Gupta, P and, Mehlawat, M.K. (2009) 'Bector-Chandra type duality in fuzzy linear programming with exponential membership functions', *Fuzzy Sets and Systems*, Vol. 160, No. 22, pp.3290–3308.

Hashemi, S.M., Modarres, M., Nasrabadi, E. and Nasrabadi, M.M. (2006) 'Fully fuzzified linear programming, solution and duality', *Journal of Intelligent and Fuzzy Systems*, Vol. 17, No. 3, pp.253–261.

Hassanzadeh Amin, S., Razmi, J. and Zhang, G. (2011) 'Supplier selection and order allocation based on fuzzy SWOT analysis and fuzzy linear programming', *Expert Systems with Applications*, Vol. 38, No. 1, pp.334–342.

Hatami-Marbini, A. and Tavana, M. (2011) 'An extension of the linear programming method with fuzzy parameters', *International Journal of Mathematics in Operational Research*, Vol. 3, No. 1, pp.44–55.

Hatami-Marbini, A., Agrell, P., Tavana, M. and Emrouznejad, A. (2013) 'A stepwise fuzzy linear programming model with possibility and necessity relation', *Journal of Intelligent & Fuzzy Systems*, Vol. 25, No. 1, pp.81–93.

Hosseinzadeh Lotfi, F., Allahviranloo, T., Alimardani Jondabeh, M. and Alizadeh, L. (2009) 'Solving a full fuzzy linear programming using lexicography method and fuzzy approximate solution', *Applied Mathematical Modelling*, Vol. 33, No. 7, pp.3151–3156.

Inuiguchi, M. and Ramík, J. (2000) 'Possibilistic linear programming: a brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem', *Fuzzy Sets and Systems*, Vol. 111, No. 1, pp.3–28.

Inuiguchi, M., Ichihashi, H. and Tanaka, H. (1990) 'Fuzzy programming: a survey of recent developments', in Slowinski, R. and Teghem, J. (Eds.): *Stochastic Versus Fuzzy Approaches to Multiobjective Mathematical Programming Under Uncertainty*, pp.45–68, Kluwer Academic Publishers, Dordrecht.

Katagiri, H., Sakawa, M., Kato, K. and Nishizaki, I. (2004) 'A fuzzy random multiobjective 0–1 programming based on the expectation optimization model using possibility and necessity measures', *Mathematical and Computer Modelling*, Vol. 40, Nos. 3–4, pp.411–421.

Kaufmann, A. and Gupta, M.M. (1991) *Introduction to Fuzzy Arithmetic: Theory and Application*, 3rd ed., Van Nostrand Reinhold, USA.

Kim, K. and Park, K.S. (1990) 'Ranking fuzzy numbers with index of optimism', *Fuzzy Sets and Systems*, Vol. 35, No. 2, pp.143–150.

Klir, G.J. and Yuan, B. (1995) *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. Prentice Hall, New Jersey.

Kumar, A., Kaur, J. and Singh, P. (2011) 'A new method for solving fully fuzzy linear programming problems', *Applied Mathematical Modelling*, Vol. 35, No. 2, pp.817–823.

Leung, Y. (1988) *Spatial Analysis and Planning under Imprecision*, North-Holland, Amsterdam.

Liou, T.S. and Wang, M.J. (1992) 'Ranking fuzzy numbers with integral value', *Fuzzy Sets and Systems*, Vol. 50, No. 3, pp.247–255.

Lodwick, W.A. and Jamison, K.D. (2007) 'Theoretical and semantic distinctions of fuzzy, possibilistic, and mixed fuzzy/possibilistic optimization', *Fuzzy Sets and Systems*, Vol. 158, No. 17, pp.1861–1872.

Luhandjula, M.K. (1989) 'Fuzzy optimization: an appraisal', *Fuzzy Sets and Systems*, Vol. 30, No. 3, pp.257–282.

Mahdavi-Amiri, N. and Nasseri, S.H. (2006) 'Duality in fuzzy number linear programming by use of a certain linear ranking function', *Applied Mathematical and Computation*, Vol. 180, No. 1, pp.206–216.

Mahdavi-Amiri, N. and Nasseri, S.H. (2007) 'Duality results and a dual simplex method for linear programming problems with trapezoidal fuzzy variables', *Fuzzy Sets and Systems*, Vol. 158, No. 17, pp.1961–1978.

Maleki, H.R. (2002) 'Ranking functions and their applications to fuzzy linear problems', *Far East Journal of Mathematical Sciences*, Vol. 4, No. 3, pp.283–301.

Maleki, H.R., Tata, M. and Mashinchi, M. (2000) 'Linear programming with fuzzy variables', *Fuzzy Sets and Systems*, Vol. 109, No. 1, pp.21–33.

Peidro, D., Mula, J., Jiménez, M. and Botella, M. (2010) 'A fuzzy linear programming based approach for tactical supply chain planning in an uncertainty environment', *European Journal of Operational Research*, Vol. 205, No. 1, pp.65–80.

Rommelfanger, H. (2007) 'A general concept for solving linear multicriteria programming problems with crisp, fuzzy or stochastic values', *Fuzzy Sets and Systems*, Vol. 158, No. 17, pp.1892–1904.

Rong, A. and Lahdelma, R. (2008) 'Fuzzy chance constrained linear programming model for optimizing the scrap charge in steel production', *European Journal of Operational Research*, Vol. 186, No. 3, pp.953–964.

Saati, S., Hatami-Marbini, A., Tavana, M. and Hajiahkondi, E. (2012) 'A two-fold linear programming model with fuzzy data', *International Journal of Fuzzy System Applications*, Vol. 2, No. 3, pp.1–12.

Stanciulescu, C., Fortemps, Ph., Installé, M. and Wertz, V. (2003) 'Multiobjective fuzzy linear programming problems with fuzzy decision variables', *European Journal of Operational Research*, Vol. 149, No. 3, pp.654–675.

Tan, R.R., Culaba, A.B. and Aviso, K.B. (2008) 'A fuzzy linear programming extension of the general matrix-based life cycle model', *Journal of Cleaner Production*, Vol. 16, No. 13, pp.1358–1367.

Tanaka, H., Okuda, T. and Asai, K. (1974) 'On fuzzy mathematical programming', *Journal of Cybernetics*, Vol. 3, No. 4, pp.37–46.

van Hop, N. (2007) 'Solving linear programming problems under fuzziness and randomness environment using attainment values', *Information Sciences*, Vol. 177, No. 14, pp.2971–2984.

Wang. X. and Kerre, E.E. (2001a) 'Reasonable properties for the ordering of fuzzy quantities (I)', *Fuzzy Sets and Systems*, Vol. 118, No. 3, pp.375–385.

Wang, X. and Kerre, E.E. (2001b) 'Reasonable properties for the ordering of fuzzy quantities (II)', *Fuzzy Sets and Systems*, Vol. 118, No. 3, pp.387–405.

Wu, H.C. (2008a) 'Using the technique of scalarization to solve the multiobjective programming problems with fuzzy coefficients', *Mathematical and Computer Modelling*, Vol. 48, Nos. 1–2, pp.232–248.

Wu, H.C. (2008b) 'Optimality conditions for linear programming problems with fuzzy coefficients', *Computers and Mathematics with Applications*, Vol. 55, No. 12, pp.2807–2822.

Xinwang, L. (2001) 'Measuring the satisfaction of constraints in fuzzy linear programming', *Fuzzy Sets and Systems*, Vol. 122, No. 2, pp.263–275.

Yeh, C.H. and Deng, H. (2004) 'A practical approach to fuzzy utilities comparison in fuzzy multi-criteria analysis', *International Journal of Approximate Reasoning*, Vol. 35, No. 2, pp.179–194.

Yuan, Y. (1991) 'Criteria for evaluating fuzzy ranking methods', *Fuzzy Sets and Systems*, Vol. 43, No. 2, pp.139–157.

Zadeh, L.A. (1965) 'Fuzzy sets', *Information and Control*, Vol. 8, No. 3, pp.338–353.

Zhang, G., Wu, Y.H., Remias, M. and Lu, J. (2003) 'Formulation of fuzzy linear programming problems as four-objective constrained optimization problems', *Applied Mathematics and Computation*, Vol. 139, Nos. 2–3, pp.383–399.

Zimmerman, H.J. (1978) 'Fuzzy programming and linear programming with several objective functions', *Fuzzy Sets and Systems*, Vol. 1, No. 1, pp.45–55.

Zimmermann, H.J. (1987) *Fuzzy Sets, Decision Making and Expert Systems*, Kluwer Academic Publishers, Boston.

Zimmermann, H.J., (1996) *Fuzzy Set Theory and its Applications*, 2nd ed., Kluwer Academic Publishers, Dordrecht, Netherlands.