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A New Method for Solving Dual DEA Problems with Fuzzy Stochastic Data

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Data envelopment analysis (DEA) is a widely used mathematical programming technique for measuring the relative efficiency of decision-making units which consume multiple inputs to produce multiple outputs. Although precise input and output data are fundamentally used in classical DEA models, real-life problems often involve uncertainties characterized by fuzzy and/or random input and output data. We present a new input-oriented dual DEA model with fuzzy and random input and output data and propose a deterministic equivalent model with linear constraints to solve the model. The main contributions of this paper are fourfold: (1) we extend the concept of a normal distribution for fuzzy stochastic variables and propose a DEA model for problems characterized by fuzzy stochastic variables; (2) we transform the proposed DEA model with fuzzy stochastic variables into a deterministic equivalent linear form; (3) the proposed model which is linear and always feasible can overcome the nonlinearity and infeasibility in the existing fuzzy stochastic DEA models; (4) we present a case study in the banking industry to exhibit the applicability of the proposed method and feasibility of the obtained solutions.

Keywords: Data envelopment analysis; fuzzy random variable; normal distribution; banking industry.

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1. Introduction

Data envelopment analysis (DEA) initially introduced by Charnes *et al.*[1] is a well-known nonparametric methodology for computing the relative efficiency of a set of homogeneous units, named as decision-making units (DMUs). DEA generalizes the single-input single-output ratio model for efficiency measurement into a multiple-input multiple-output model by utilizing the ratio for the weighted sum of outputs to the weighted sum of inputs. It computes scalar efficiency scores with a range of zero to one that determine an efficient level or position for each DMU under evaluation among all the DMUs. A DMU is considered efficient if its efficiency score is equal to one; otherwise it is said to be inefficient.

Much research has been carried out on the performance measurement and solution procedures based on DEA models with crisp data. Keshavarz and Toloo[2] clarified the relationships between DEA and the multi-criteria assignment problem and designed a two-phase DEA approach to find and classify all efficient assignments. Azizi and Wang[3] proposed a pair of bounded DEA models for measuring the overall performances of a set of DMUs which were represented by interval efficiencies. He *et al.*[4] introduced a DEA model to improve the interval efficiencies by decreasing inputs, increasing outputs, or altering both concurrently.[4] Boloori *et al.*[5] investigated the duality problem in DEA network structures and proposed equivalent multiplier and envelopment models for performance measurement. Gang *et al.*[6] proposed a pairwise comparison matrix in multiple criteria decision-making. However, a measure of impreciseness is often needed in most real-life DEA problems in which the observed input and output data are often not known precisely. Two typical methods including probability-theoretic and fuzzy-theoretic approaches are most often used for such DEA models involving uncertainty.

The notions of fuzziness and randomness were introduced in DEA to handle imprecise data. Fuzzy sets can be used to represent ambiguous or imprecise information. Hatami-Marbini *et al.*[7] have categorized the fuzzy DEA methods into five general categories: the tolerance approach,[8,9] the α-level-based approach,[9,10] the fuzzy ranking approach,[11,12] the possibility approach,[13] and the fuzzy arithmetic approach.[14] In the α-level-based approach, the fuzzy DEA model is transformed into a pair of parametric programs for each α-level. Kao and Liu,[10] one of the most cited studies in the $\alpha-$level approach's category, used the method of Ref. 15 for ranking fuzzy numbers to transfer the fuzzy DEA model into a pair of parametric mathematical programs for the given level of α. Saati *et al.*[16] represented a fuzzy DEA problem with a possibilistic programming model and used the α-level-based approach to convert this problem into an interval programming problem. Puri and Yadav[17] applied the suggested methodology by Ref. 16 to solve a fuzzy DEA model with undesirable outputs. Khanjani Shiraz *et al.*[18] proposed fuzzy free disposal hull models under possibility and credibility measures. Momeni *et al.*[19] used fuzzy DEA models to represent the imprecise data in supply chain performance measurement problems. Tavana and Khalili-Damaghani[20] proposed a two-stage fuzzy DEA method to decompose the efficiency score of a two-stage DMU into two stages. Payan[21] used the common set of weights to evaluate the performance of DMUs with fuzzy data with a linear program.

Land *et al.*[22] extended the chance-constrained DEA model to evaluate the efficiency of DMUs with deterministic inputs and random outputs. Olesen and Petersen[23] proposed a chance-constrained programming model for efficiency evaluation using a piecewise linear envelopment of confidence regions for the input and output data in DEA. Researchers extend the concept of stochastic efficiency.[24–26] Cooper *et al.*[27] used chance-constrained programming for extending congestion. Huang and Li[28] considered the possibility of random variations in input and output data and proposed stochastic DEA models. The joint chance constraints have observed stochastic been used by several in DEA models. Tsionas and Papadakis[29] proposed Bayesian inference techniques in chance-constrained DEA models. Udhayakumar *et al.*[30] used a genetic algorithm to solve the chance-constrained DEA models involving the concept of satisficing. Udhayakumar *et al.*[30] and Tsolas and Charles[31] used a satisficing DEA model in the banking industry. Farnoosh *et al.*[32] proposed a chance-constrained free disposal hull model with random input and random output. Wu *et al.*[33] considered undesirable outputs with weak disposability and proposed a stochastic DEA model. A review of stochastic DEA models can be found in a recent work by Olesen and Petersen.[34]

Many scholars have used random phenomenon to represent uncertainty in mathematical programming problems. In real-life problems, it is not unusual to have to deal with two or more concurrent uncertainty factors. However, many researchers believe that the classical random and fuzzy variables cannot always be used to clearly represent complicated real-life problems where randomness and fuzziness coexist simultaneously. The concept of a fuzzy stochastic variable can be a useful method for handling these types of uncertainties concurrently. Several approaches have been proposed to study fuzziness and randomness simultaneously in DEA problems. Kwakernaak proposed the concept of a fuzzy random variable which was further enhanced by Refs. 35–38.[39,40] Qin and Liu[40] developed a fuzzy random DEA model and represented the fuzzy random data with well-known possibility and probability distributions. Tavana *et al.*[41] also introduced three different FDEA models consisting of probability-possibility, probability-necessity and probability-credibility constraints in which input and output data entailed fuzziness and randomness at the same time. Tavana *et al.*[42] provided a chance-constrained DEA model with random fuzzy data using Poisson, uniform, and normal distributions. Tavana *et al.*[43] further proposed DEA models with bi-random input–output.

Khanjani *et al.*[44] proposed fuzzy rough DEA models based on the expected value and possibility approaches. Paryab *et al.*[45] proposed DEA models using a bi-fuzzy data-based possibility approach. However, there has been no attempt to study randomness and roughness simultaneously in DEA problems. Nasseri *et al.*[46] proposed a DEA model with undesirable output consisting of probability-possibility, probability-necessity and probability-credibility constraints. To deal with the uncertain environments, especially hybrid environments, the DEA model may disorder its structure when the uncertain parameter of input and output exists. For example, the method proposed by Ref. 41 does not calculate the efficiency scores of DMUs in the range of zero to one for input-oriented DEA models. Another shortcoming of this approach is the nonlinear (quadratic) form of the proposed DEA model. Hence, this study tries to overcome the shortcomings of the existing approaches.

Several researchers have studied efficiency measurement in the banking industry. Ebrahimnejad *et al.*[47] evaluated the performance of 49 branches of Peoples Bank based on a three-stage DEA model. Khoshandam *et al.*[48] introduced a new DEA model for calculating marginal rates with nondiscretionary factors. They applied their model to measure efficiency of 33 bank branches in Iran. Kazemi Matin *et al.*[49] evaluated the 61 banks in the Gulf Cooperation Council countries using a modified semi-oriented radial measure in DEA models with negative data. They considered total assets, capital and deposits as input variables, and loans and equity in each branch as output variables. Charles and Kumar[50] used a satisficing DEA model to measure service quality efficiency. They addressed noise in the data using stochastic simulation and applied their model to 13 major banks operating in Malaysia. Hadi-vencheh *et al.*[51] considered a data set, which consists of 27 banks with three inputs and two outputs in the period 1987–1989 in which managers desire to assess the impact of information technology on bank performance. Toloo and Tichý[52] evaluated 14 banks active in the Czech Republic based on extended multiplier and envelopment forms of DEA models.

The main contributions of this paper are fourfold: (1) we extend the concept of normal distribution for fuzzy stochastic variables and propose a new version of DEA model for problems characterized by fuzzy stochastic variables; (2) we transform the proposed DEA model with fuzzy stochastic variables into a deterministic equivalent linear form; (3) since the proposed model is linear and always feasible, it overcomes the lack of infeasibility and nonlinearity of the existing fuzzy stochastic DEA models; (4) we present a case study in the banking industry to exhibit the feasibility and richness of the obtained solutions.

The remainder of this paper is organized as follows. In Sec. 2, we present some necessary concepts related to fuzzy set theory and probability theory. Section 3 introduces an extended normal distribution and a modified ranking function. Section 4 presents our proposed CCR-DEA model to solve the fuzzy stochastic DEA model. In Sec. 5, we present the results of a case study in the banking industry to evaluate the efficiency of 25 branches. Section 6 presents our conclusions and future research directions.

2. Preliminaries

In this section, we review some necessary concepts related to fuzzy set theory and probability theory, which will be used in the rest of paper (see Refs. 36 and 53–56).

Definition 1. A fuzzy set $\tilde{A}$, defined on universal set X, is given by a set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$ where $\mu_{\tilde{A}}(x)$ gives the membership grade of the element x in the set $\tilde{A}$ and is called the membership function.

Definition 2. A fuzzy set$\tilde{A}$, defined on universal set of real numbers R, is a fuzzy number if its membership function has the following features:

(1) $\tilde{A}$ is convex, i.e., $\forall x, y \in R, \forall \lambda \in [0,1], \mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$,
(2) $\tilde{A}$ is normal, i.e., $\exists \bar{x} \in R; \mu_{\tilde{A}}(\bar{x}) = 1$,
(3) $\mu_{\tilde{A}}$ is piecewise continuous.

Definition 3. A function $L : [0, \infty) \to [0,1]$ (or $R : [0,\infty) \to [0,1]$) is said to be a reference function of a fuzzy number if and only if $L(0) = 1$ (or $R(0) = 1$) and L or R is nonincreasing on $[0, \infty)$.

Definition 4. A fuzzy number $\tilde{A} = (m, \alpha, \beta)_{LR}$ is an $L - R$ fuzzy number if its membership function is given by:

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\dfrac{m-x}{\alpha}\right), & \text{for } x \leq m, \quad \alpha > 0, \\ 1, & \text{for } x = m, \\ R\left(\dfrac{x-m}{\beta}\right), & \text{for } x \geq m, \quad \beta > 0. \end{cases}$$

Remark 1. If $L(x) = R(x) = \max\{0, 1-x\}$ then an $L - R$ fuzzy number $\tilde{A} = (m, \alpha, \beta)_{LR}$ is said to be a triangular fuzzy number (TFN) and is denoted by $\tilde{A} = (m, \alpha, \beta)$.

Definition 5. Let $\tilde{A} = (m, \alpha, \beta)_{LR}$ be a fuzzy number and λ be a real number in the interval $[0, 1]$, then the crisp set, $\tilde{A}_\lambda = \{x \in R : \mu_{\tilde{A}}(x) \geq \lambda\} = [A^L_\lambda, A^R_\lambda] = [m - \alpha L^{-1}(\lambda), m + \beta R^{-1}(\lambda)]$, is said to be a λ-cut of $\tilde{A}$. A^L_λ and A^R_λ are called the left- and right-extreme points of $\tilde{A}$ at level λ.

Definition 6. Let $\tilde{A}_1 = (m_1, \alpha_1, \beta_1)_{LR}$ and $\tilde{A}_2 = (m_2, \alpha_2, \beta_2)_{LR}$ be two $L - R$ fuzzy numbers and k be a nonzero real number. Then, the formula for the extended addition and the scalar multiplication is defined as follows:

(i) $(m_1, \alpha_1, \beta_1)_{LR} + (m_2, \alpha_2, \beta_2)_{LR} = (m_1 + m_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)_{LR}$,
(ii) $k > 0$, $k(m_1, \alpha_1, \beta_1)_{LR} = (km_1, k\alpha_1, k\beta_1)_{LR}$,
(iii) $k < 0$, $k(m_1, \alpha_1, \beta_1)_{LR} = (km_1, -k\beta_1, -k\alpha_1)_{LR}$.

Definition 7. (Extension principle) Let X be a Cartesian product of universes $X = X_1 \times \cdots \times X_r$, $\tilde{A}_1, \ldots, \tilde{A}_r$ be r fuzzy sets in $X_1, \ldots, X_r$, respectively, and f is a mapping from X to universe $Y = f(x_1, \ldots, x_r)$. Then the extension principle allows us to define a fuzzy set $\tilde{B}$ in Y by $\tilde{B} = \{(Y, M_{\tilde{B}}(Y)) \mid Y = f(x_1, \ldots, x_r), (x_1, \ldots, x_r) \in X\}$ where

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup_{(x_1,\ldots,x_r) \in f^{-1}(y)} \min\{\mu_{\tilde{A}_1}(x_1), \ldots, \mu_{\tilde{A}_r}(x_r)\}, & \text{if } f^{-1}(u) \neq 0 \\ 0, & \text{otherwise} \end{cases},$$

where f^{-1} is the inverse of f.

Definition 8. Let $(\Theta, P(\Theta), \mathrm{Pos})$ be a possibility space where Θ is a nonempty set involving all possible events, and $P(\Theta)$ is the power set of Θ. For every $A \in P(\Theta)$, there is a non-negative number $\mathrm{Pos}(A)$, a so-called possibility measure, satisfying the following axioms:

(i) $P(\emptyset) = 0$, $P(\Theta) = 1$,
(ii) for every $A, B \in P(\Theta)$, $A \subseteq B$ implies $\mathrm{Pos}(A) \leq \mathrm{Pos}(B)$,
(iii) for every subset $\{A_w : w \in W\} \subseteq P(\Theta)$, $\mathrm{Pos}(\bigcup_w A_w) = \mathrm{Sup}_w \mathrm{Pos}(A_w)$.
The elements of $P(\Theta)$ are also called fuzzy events.

Definition 9. A probability space is defined as a triplet $(\Omega, \Sigma, \mathrm{Pr})$, where Ω is a sample space, Σ is the s-algebra of subsets of Ω (i.e., the set of all possible potentially interesting events), and a probability measure on Ω, denoted by Pr, satisfies:

(1) $\mathrm{Pr}(\emptyset) = 0$, $\mathrm{Pr}(\Omega) = 1$
(2) $\mathrm{Pr}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathrm{Pr}(A_i)$, for any countable and mutually disjoint events $A_i \in \sum, i = 1, 2, \ldots$.

Definition 10. Let us assume $(\Omega, \Sigma, \mathrm{Pr})$ is a probability space and Ω is a sample space. Let us further assume that Σ is the s-algebra of subsets of Ω (i.e., the set of all possible interesting events) and Pr is a probability measure on Ω. Then, a fuzzy random variable is a function ξ from a probability space $(\Omega, \Sigma, \mathrm{Pr})$ to the set of fuzzy variables such that for every Borel set B of $\Re$, $\mathrm{Pos}\{\xi(w), w \in B\}$ is a measurable function of ω.

3. Extended Normal Distribution

A fuzzy random variable can be characterized as a general random variable taking on values that may be fuzzy but not limited to crisp ones. Hence, we extend the concept of stochastic distributions for these variables using the following definitions:

$\tilde{\tilde{X}} = (\tilde{\lambda}, \alpha, \beta)$:	Fuzzy stochastic variable
$\tilde{\lambda}$:	Fuzzy mean of $\tilde{\tilde{X}}$ as a stochastic variable
α, β:	Left and right spread, respectively
$\bar{\Omega}$:	Triangular fuzzy number set
$\bar{\Sigma} = \{\bar{\omega} \mid \bar{\omega} = (\omega, \alpha, \beta), \omega \in \tilde{\lambda}\}$:	S-algebra of subsets of $\bar{\Omega}$
$E(.)$ and $\mathrm{Var}(.)$:	Expectation and variance, respectively
$f(.)$:	Density function
$\mathrm{Pr}(.)$:	Probability measure
$\Phi(.)$:	cdf of standard normal distribution
$N(\mu, \sigma)$:	Normal distribution

Definition 11. Let $\tilde{\tilde{X}} = (\tilde{\lambda}, \alpha, \beta)$ be a fuzzy stochastic variable. We say $(\tilde{\lambda}, \alpha, \beta) \sim \bar{N}(\bar{\mu}, \sigma)$ with $\bar{\mu} = (\mu, \alpha, \beta)$ has an extended normal distribution, with the following components:

$$\begin{aligned} \bar{f}(\bar{\omega}) &= f(\omega), \\ \bar{\Phi}(\bar{\omega}) &= \overline{\Pr}(\tilde{\tilde{X}} \leq \bar{\omega}) = \Pr(\tilde{\lambda} \leq \omega). \end{aligned} \tag{1}$$

where $\bar{f}(.)$ and $\bar{\Phi}(.)$ are the density function and the cumulative distribution function, respectively. In addition, we define fuzzy variable $\bar{x} \notin \bar{\Sigma}$ as:

$$\begin{aligned} \overline{\Pr}(\tilde{\tilde{X}} \leq \bar{x}) &= \sup_{\bar{\omega} \leq \bar{x}} \overline{\Pr}(\tilde{\tilde{X}} \leq \bar{\omega}), \\ \overline{\Pr}(\tilde{\tilde{X}} \geq \bar{x}) &= \sup_{\bar{\omega} \geq \bar{x}} \overline{\Pr}(\tilde{\tilde{X}} \geq \bar{\omega}). \end{aligned} \tag{2}$$

In the following, we use the notations $\overline{E}(\tilde{\tilde{X}})$ and $\overline{\mathrm{Var}}(\tilde{\tilde{X}})$ in representation of $\bar{\mu}$ and σ, respectively. Afterwards, $\overline{E}\ (\tilde{\tilde{X}}) = \bar{\mu} = (E(\tilde{\lambda}), \alpha, \beta)$, $\overline{\mathrm{Var}}(\tilde{\tilde{X}}) = \mathrm{Var}(\tilde{\lambda}) = \sigma^2$. It is notable that $\overline{E}(.)$ and $\overline{\mathrm{Var}}(.)$ are not necessarily the expectation and variance.

Proposition 1. *If* $\tilde{\tilde{X}} = (\tilde{\lambda}, \alpha, \beta) \sim \bar{N}(\bar{\mu}, \sigma)$ *with* $\bar{\mu} = (\mu, \alpha, \beta)$, *then*

$$\tilde{\tilde{Z}} = \frac{\tilde{\tilde{X}} - \overline{E}(\tilde{\tilde{X}})}{\sqrt{\overline{\mathrm{Var}}(\tilde{\tilde{X}})}} \sim \bar{N}(\bar{0}, 1) \quad \text{with} \quad \bar{0} = \left(0, \frac{\alpha+\beta}{\sigma}, \frac{\alpha+\beta}{\sigma}\right). \tag{3}$$

Proof. Let

$$\tilde{\tilde{Z}} = \frac{\tilde{\tilde{X}} - \bar{\mu}}{\sigma} = \frac{(\tilde{\lambda}, \alpha, \beta) - (\mu, \alpha, \beta)}{\sigma}.$$

By fuzzy arithmetic and assuming $\sigma > 0$, we have

$$\tilde{\tilde{Z}} = \frac{(\tilde{\lambda} - \mu, \alpha+\beta, \beta+\alpha)}{\sigma} = \left(\frac{\tilde{\lambda} - \mu}{\sigma}, \frac{\alpha+\beta}{\sigma}, \frac{\alpha+\beta}{\sigma}\right).$$

According to Definition 11 and the standard normal distribution properties, the components of the last fuzzy random variable will be:

$$\begin{aligned} \bar{E}(\tilde{\tilde{Z}}) &= \left(E\left(\left(\frac{\tilde{\lambda} - \mu}{\sigma}\right), \frac{\alpha+\beta}{\sigma}, \frac{\alpha+\beta}{\sigma}\right)\right) = \left(0, \frac{\alpha+\beta}{\sigma}, \frac{\alpha+\beta}{\sigma}\right), \\ \overline{\mathrm{Var}}(\tilde{\tilde{Z}}) &= \mathrm{Var}\left(\frac{\tilde{\lambda} - \mu}{\sigma}\right) = 1. \end{aligned}$$

□

Proposition 2. *If* $\tilde{\tilde{Z}} \sim \bar{N}(\bar{0}, 1)$ *with* $\bar{0} = (0, \alpha, \beta)$, *then* $-\tilde{\tilde{Z}} \sim \bar{N}(\bar{0}, 1)$ *where* $\bar{0} = (0, \beta, \alpha)$.

Proof. Consider $\tilde{\tilde{z}} = (\tilde{z}, \alpha, \beta); E(\tilde{z}) = 0$, $\mathrm{var}(\tilde{z}) = 1$, hence $-\tilde{\tilde{z}} = (-\tilde{z}, \beta, \alpha)$; $E(-\tilde{z}) = -E(\tilde{z}) = 0$, $\mathrm{var}(-\tilde{z}) = (-1)^2\, \mathrm{var}(\tilde{z}) = 1$. As a result, $-\tilde{\tilde{Z}} \sim \bar{N}(\bar{0}, 1)$ where $\bar{0} = (0, \beta, \alpha)$. □

Proposition 3. *If* $\tilde{\bar{Z}} \sim \bar{N}(\bar{0}, 1)$ *with* $\bar{0} = (0, \beta, \alpha)$, *then*

$$\overline{\Pr}(\tilde{\bar{Z}} \leq \bar{z}_0) > \gamma \quad \text{iff} \quad (\Phi^{-1}(\gamma), \alpha, \beta) \leq \bar{z}_0. \tag{4}$$

$$\overline{\Pr}(\tilde{\bar{Z}} \geq \bar{z}_0) > \gamma \quad \text{iff} \quad (\Phi^{-1}(1-\gamma), \alpha, \beta) \geq \bar{z}_0. \tag{5}$$

Proof. We first prove Eq. (4). From Eq. (2), we have

$$\overline{\Pr}(\tilde{\bar{Z}} \leq \bar{z}_0) > \gamma \Longleftrightarrow \left(\sup_{\bar{\omega} \leq \bar{z}_0} \overline{\Pr}(\tilde{\bar{Z}} \leq \bar{\omega})\right) > \gamma.$$

So,

$$\exists\, \bar{\omega} = \bar{\omega}_0 = (\omega_0, \alpha, \beta) \in \bar{\Sigma}, \quad \overline{\Pr}(\tilde{\bar{Z}} \leq \bar{\omega}_0) = \Pr(\tilde{Z} \leq \omega_0) > \gamma.$$

On the other hand, by Definition 11 and the properties of $\Phi^{-1}(.)$, we have:

$$\overline{\Pr}(\tilde{\bar{Z}} \leq (\Phi^{-1}(\gamma), \alpha, \beta)) = \Pr(\tilde{Z} \leq \Phi^{-1}(\gamma)) = \gamma.$$

The combination of two last relations results in $\Phi^{-1}(\gamma) \leq \omega_0$. So, we have $(\Phi^{-1}(\gamma), \alpha, \beta) \leq \bar{\omega}_0$. Finally, $\bar{\omega}_0 \leq \bar{z}_0$ completes the proof (3).

Now consider Eq. (5). From Eq. (2), we have

$$\overline{\Pr}(\tilde{\bar{Z}} \geq \bar{z}_0) > \gamma \Longleftrightarrow \left(\sup_{\bar{\omega} \geq \bar{z}_0} \overline{\Pr}(\tilde{\bar{Z}} \geq \bar{\omega})\right) > \gamma.$$

Thus,

$$\exists \bar{\omega} = \bar{\omega}_0 = (\omega_0, \alpha, \beta) \in \bar{\Sigma}, \quad \overline{\Pr}(\tilde{\bar{Z}} \geq \bar{\omega}_0) = \Pr(\tilde{Z} \geq \omega_0) > \gamma.$$

On the other hand, by Definition 11 and the properties of $\Phi^{-1}(.)$, we have:

$$\overline{\Pr}(\tilde{\bar{Z}} \geq (\Phi^{-1}(1-\gamma), \alpha, \beta)) = \Pr(\tilde{Z} \geq \Phi^{-1}(1-\gamma)) = \gamma.$$

The combination of two last relations results in $\Phi^{-1}(1-\gamma) \geq \omega_0$. This means that $(\Phi^{-1}(1-\gamma), \alpha, \beta) \geq \bar{\omega}_0$. Finally, $\bar{\omega}_0 \geq \bar{z}_0$ completes the proof (4). □

Theorem 1. *If* $\tilde{\bar{X}} \sim \bar{N}(\bar{\mu}, \sigma)$ *with* $\bar{\mu} = (\mu, \alpha, \beta)$, *then:*

$$\overline{\Pr}(\tilde{\bar{X}} \leq r) > \gamma \text{ iff } \left(\Phi^{-1}(\gamma), \frac{\alpha+\beta}{\sigma}, \frac{\alpha+\beta}{\sigma}\right) \leq \frac{r-\bar{\mu}}{\sigma}. \tag{6}$$

$$\overline{\Pr}(\tilde{\bar{X}} \geq r) > \gamma \text{ iff } \left(\Phi^{-1}(1-\gamma), \frac{\alpha+\beta}{\sigma}, \frac{\alpha+\beta}{\sigma}\right) \geq \frac{r-\bar{\mu}}{\sigma}. \tag{7}$$

Proof. First we prove (6). For the case of

$$\frac{\tilde{\bar{X}} - \bar{\mu}}{\sigma} \sim \bar{N}(\bar{0}, 1),$$

we have Proposition 1. Also,

$$\overline{\Pr}(\tilde{\bar{X}} \leq r) = \Pr\left(\frac{\tilde{\bar{X}} - \bar{\mu}}{\sigma} \leq \frac{r-\bar{\mu}}{\sigma}\right) \quad \bar{0} = \left(0, \frac{\alpha+\beta}{\sigma}, \frac{\alpha+\beta}{\sigma}\right).$$

In addition, according to Preposition 3, we have:

$$\overline{\mathrm{Pr}}\left(\frac{\tilde{\bar{X}}-\bar{\mu}}{\sigma} \leq \frac{r-\bar{\mu}}{\sigma}\right) > \gamma \Longleftrightarrow \left(\Phi^{-1}(\gamma), \frac{\alpha+\beta}{\sigma}, \frac{\alpha+\beta}{\sigma}\right) \leq \frac{r-\bar{\mu}}{\sigma}.$$

The proof of (7) is similar to (6) by using the relation (5). □

It is notable that, to solve an inequality in possibility space, we use the following Lemma to convert the fuzzified variable into a deterministic one.

Lemma 1 (Ref. 57). *Let $\bar{\lambda}_1$ and $\bar{\lambda}_2$ be two fuzzy numbers with continuous membership functions. For a given confidence level $\alpha \in [0,1]$, $\mathrm{Pos}\{\bar{\lambda}_1 \geq \bar{\lambda}_2\} \geq \alpha$ if and only if $\lambda^R_{1,\alpha} \geq \lambda^R_{2,\alpha}$, where $\lambda^L_{1,\alpha}$, $\lambda^R_{1,\alpha}$ and $\lambda^L_{2,\alpha}$,$\lambda^R_{2,\alpha}$ are the left- and right-side extreme points of the α-level sets $\bar{\lambda}_1$ and $\bar{\lambda}_2$, respectively, and $\mathrm{Pos}\{\bar{\lambda}_1 \geq \bar{\lambda}_2\} \geq \alpha$ presents the degree of possibility.*

Remark 2. We have extended the normal distribution for stochastic variables to fuzzy stochastic variables. We named such a distribution as the extended normal distribution. It is worthwhile to note that the extended normal distribution is not a kind of stochastic distribution for random variables. This is a fuzzy stochastic distribution. Moreover, the existing approach for solving DEA models in the presence of fuzzy random variables such as the chance-constrained programming approach and the measure-based approach are not able to determine a unique measure in fuzzy random space. In the following section, we show that the proposed extended normal distribution allows us to define a unique measure on the fuzzy stochastic space.

4. Fuzzy Stochastic DEA-CCR Model

4.1. *Proposed DEA-CCR model*

Consider a set of n DMUs, where DMU_j has a production plan (x_j, y_j) and using m inputs $x_j = (x_{1j}, x_{2j}, \ldots, x_{mj})$ produces s outputs $y_j = (y_{1j}, y_{2j}, \ldots, y_{sj})$. The technical efficiency of a given DMU_k under a constant return to scale can be obtained by utilizing the following problem called the input-oriented CCR primal model:

$$\begin{aligned}
E_k = {} & \mathrm{Min}\, \theta \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{ik}, \quad i = 1, 2, \ldots, m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rk}, \quad r = 1, 2, \ldots, s \\
& \lambda_j \geq 0, \quad j = 1, 2, \ldots, n.
\end{aligned} \tag{8}$$

The DMU_k is (technically) efficient if $E_k = 1$, otherwise if $0 < E_k < 1$, it is (technically) inefficient.

Substituting $\hat{x}_{ij} = \lambda_j x_{ij}$ and $\hat{y}_{rj} = \lambda_j y_{rj}$ into Model (8), the following equivalent model is obtained:

$$\begin{aligned}
& E_k = \text{Min}\,\theta \\
& \text{s.t.} \\
& \sum_{j=1}^{n} \hat{x}_{ik} \leq \theta x_{ik} \quad i = 1, 2, \ldots, m \\
& \sum_{j=1}^{n} \hat{y}_{rj} \geq y_{rk} \quad r = 1, 2, \ldots, s \\
& \lambda_j y_{rj} \leq \hat{y}_{rj} \leq \lambda_j y_{rj} \quad \forall r, j \\
& \lambda_j x_{ij} \leq \hat{x}_{ij} \leq \lambda_j x_{ij} \quad \forall i, j \\
& \lambda_j \geq 0, \quad j = 1, 2, \ldots, n
\end{aligned} \tag{9}$$

The above substitution is similar to the approach proposed in Ref. 16 to solve a parametric fuzzy DEA model.

4.2. *Fuzzy stochastic DEA-CCR model: An extended probability approach*

The aim of this section is to propose a DEA method for evaluating the efficiencies of DMUs with fuzzy stochastic input and output data. An input or output variable in a DEA problem can be a fuzzy variable, and its mean values can still be normally distributed. In this study, we consider fuzzy stochastic variables in DEA problems. To this end, consider n DMUs where each DMU using m fuzzy stochastic inputs, $\tilde{\bar{x}}_{ij} = (x_{ij}^m, x_{ij}^\alpha, x_{ij}^\beta)_{LR}, i = 1, \ldots, m, j = 1, \ldots, n$, produces s fuzzy stochastic outputs, denoted by $\tilde{\bar{y}}_{rj} = (y_{rj}^m, y_{rj}^\alpha, y_{rj}^\beta)_{LR}, r = 1, \ldots, s$, $j = 1, \ldots, n$. Let x_{ij}^m and y_{rj}^m, denoted by $x_{ij}^m \sim N(x_{ij}, \sigma_{ij}^2)$ and $y_{rj}^m \sim N(y_{rj}, \sigma_{rj}^2)$, be normally distributed. Therefore, $x_{ij}(y_{rj})$ and $\sigma_{ij}^2(\sigma_{rj}^2)$ are the mean and the variance of x_{ij}^m (y_{rj}^m) for DMU$_j$, respectively. According to Definition 11, each of them has an extended normal distribution as $\tilde{\bar{x}}_{ij} \sim \bar{N}(\bar{x}_{ij}, \sigma_{ij})$ with $\bar{x}_{ij} = (x_{ij}, x_{ij}^\alpha, x_{ij}^\beta)$ and $\tilde{\bar{y}}_{rj} \sim \bar{N}(\bar{y}_{rj}, \sigma_{rj})$ with $\bar{y}_{rj} = (y_{rj}, y_{rj}^\alpha, y_{rj}^\beta)$.

The chance-constrained programming[56] is a stochastic optimization model suitable for solving optimization problems with uncertain data. We build on this model and propose the following extended CCR model:

$$\begin{aligned}
& E_k^* = \text{Min}\,\theta \\
& \text{s.t.} \\
& \overline{\Pr}\left(\sum_{j=1}^{n} \hat{x}_{ik} \leq \theta \tilde{\bar{x}}_{ik}\right) \geq \gamma, \quad i = 1, 2, \ldots, m, \qquad \text{(i)} \\
& \overline{\Pr}\left(\sum_{j=1}^{n} \hat{y}_{rj} \geq \tilde{\bar{y}}_{rk}\right) \geq \gamma, \quad r = 1, 2, \ldots, s, \qquad \text{(ii)} \\
& \overline{\Pr}\left(\frac{\hat{x}_{ij}}{\lambda_j} \leq \tilde{\bar{x}}_{ij} \leq \frac{\hat{x}_{ij}}{\lambda_j}\right) \geq \gamma \quad \forall i, j, \qquad \text{(iii)}
\end{aligned}$$

$$\begin{aligned}&\overline{\Pr}\left(\frac{\hat{y}_{rj}}{\lambda_j} \le \tilde{\bar{y}}_{rj} \le \frac{\hat{y}_{rj}}{\lambda_j}\right) \ge \gamma \quad \forall r, j, \qquad \text{(iv)}\\ &\lambda_j \ge 0, \quad j = 1, 2, \ldots, n,\end{aligned} \tag{10}$$

where γ is the predetermined threshold defined by the DM.

In what follows we show that the extended probability CCR Model (10) can be equivalently transformed into a linear programing model.

Based on Theorem 1, the constraint (i) in Model (10) is equivalent to the following equations:

$$\overline{\Pr}\left(\sum_{j=1}^{n} \hat{x}_{ik} \le \theta\tilde{\bar{x}}_{ik}\right) \ge \gamma \quad \Longleftrightarrow \overline{\Pr}\left(\tilde{\bar{x}}_{ik} \ge \frac{\sum_{j=1}^{n} \hat{x}_{ij}}{\theta}\right) \ge \gamma$$

$$\Longleftrightarrow \left(\Phi^{-1}(1-\gamma), \frac{x_{ik}^{\alpha} + x_{ik}^{\beta}}{\sigma_{ik}}, \frac{x_{ik}^{\alpha} + x_{ik}^{\beta}}{\sigma_{ik}}\right) \ge \frac{\sum_{j=1}^{n} \hat{x}_{ij}/\theta - \bar{x}_{ik}}{\sigma_{ik}}$$

Based on Lemma 1, the above relation, at given γ threshold, becomes:

$$\left(\left(\Phi^{-1}(1-\gamma), \frac{x_{ik}^{\alpha} + x_{ik}^{\beta}}{\sigma_{ik}}, \frac{x_{ik}^{\alpha} + x_{ik}^{\beta}}{\sigma_{ik}}\right)\right)_{\delta}^{R} \ge \left(\frac{\sum_{j=1}^{n} \hat{x}_{jk}/\theta - \bar{x}_{ik}}{\sigma_{ik}}\right)_{\delta}^{R}$$

$$\Longleftrightarrow \Phi^{-1}(1-\gamma) + R^{-1}(\delta)\frac{x_{ik}^{\alpha} + x_{ik}^{\beta}}{\sigma_{ik}} \ge \frac{\sum_{j=1}^{n} \hat{x}_{jk}/\theta - (x_{ik} - L^{-1}(\delta)x_{ik}^{\alpha})}{\sigma_{ik}}$$

Therefore,

$$\sum_{j=1}^{n} \hat{x}_{jk} \le \theta(x_{ik} - L^{-1}(\delta)x_{ik}^{\alpha} + R^{-1}(\delta)(x_{ik}^{\alpha} + x_{ik}^{\beta}) + \sigma_{ik}\Phi^{-1}(1-\gamma)).$$

Similarly, constraint (ii) in Model (10) is converted to the following deterministic constraint:

$$\sum_{j=1}^{n} \hat{y}_{rj} \ge (y_{rk} + R^{-1}(\delta)y_{rk}^{\beta} - L^{-1}(\delta)(y_{rk}^{\alpha} + y_{rk}^{\beta}) - \sigma_{rk}\Phi^{-1}(1-\gamma)).$$

Constraint (iii) in Model (10) can be transformed into two constraints

$$\overline{\Pr}\left(\frac{\hat{x}_{ij}}{\lambda_j} \le \tilde{\bar{x}}_{ij}\right) \ge \gamma \quad \text{and} \quad \overline{\Pr}\left(\tilde{\bar{x}}_{ij} \le \frac{\hat{x}_{ij}}{\lambda_j}\right) \ge \gamma.$$

These constraints can be rewritten as the following constraints based on Theorem 1, where $r = \frac{\hat{x}_{ij}}{\lambda_j}$ and $\tilde{\bar{X}} = \tilde{\bar{x}}_{ij}$:

$$\overline{\Pr}(\tilde{\bar{X}} \ge r) \ge \gamma \Longleftrightarrow \left(\Phi^{-1}(1-\gamma), \frac{x_{ij}^{\alpha} + x_{ij}^{\beta}}{\sigma_{ij}}, \frac{x_{ij}^{\alpha} + x_{ij}^{\beta}}{\sigma_{ij}}\right) \ge \frac{r - \bar{x}_{ij}}{\sigma_{ij}}$$

and

$$\overline{\mathrm{Pr}}(\tilde{\tilde{X}} \leq r) \geq \gamma \Longleftrightarrow \left(\Phi^{-1}(\gamma), \frac{x_{ij}^{\alpha}+x_{ij}^{\beta}}{\sigma_{ij}}, \frac{x_{ij}^{\alpha}+x_{ij}^{\beta}}{\sigma_{ij}}\right) \leq \frac{r-\bar{x}_{ij}}{\sigma_{ij}},$$

where $r = \frac{\hat{x}_{ij}}{\lambda_j}$ and $\tilde{X} = \tilde{\tilde{x}}_{ij}$.

Based on Lemma 1, the constraint (iii) in Model (10) is equivalent to the following equations:

$$\left(\Phi^{-1}(1-\gamma), \frac{x_{ij}^{\alpha}+x_{ij}^{\beta}}{\sigma_{ij}}, \frac{x_{ij}^{\alpha}+x_{ij}^{\beta}}{\sigma_{ij}}\right)_{\delta}^{R} \geq \left(\frac{r-x_{ij}}{\sigma_{ij}}\right)_{\delta}^{R}$$
$$\Longleftrightarrow \Phi^{-1}(1-\gamma)+R^{-1}(\delta)\frac{x_{ij}^{\alpha}+x_{ij}^{\beta}}{\sigma_{ij}} \geq \frac{r-(x_{ij}-L^{-1}(\delta)x_{ij}^{\alpha})}{\sigma_{ij}}$$

and

$$\left(\Phi^{-1}(\gamma), \frac{x_{ij}^{\alpha}+x_{ij}^{\beta}}{\sigma_{ij}}, \frac{x_{ij}^{\alpha}+x_{ij}^{\beta}}{\sigma_{ij}}\right)_{\delta}^{L} \leq \left(\frac{r-x_{ij}}{\sigma_{ij}}\right)_{\delta}^{L}$$
$$\Longleftrightarrow \Phi^{-1}(\gamma)-L^{-1}(\delta)\frac{x_{ij}^{\alpha}+x_{ij}^{\beta}}{\sigma_{ij}} \leq \frac{r-(x_{ij}+R^{-1}(\delta)x_{ij}^{\beta})}{\sigma_{ij}}.$$

Hence

$$r \leq x_{ij} - L^{-1}(\delta)x_{ij}^{\alpha} + R^{-1}(\delta)(x_{ij}^{\alpha}+x_{ij}^{\beta}) + \sigma_{ij}\Phi^{-1}(1-\gamma)$$

and

$$x_{ij} + R^{-1}(\delta)x_{ij}^{\beta} - L^{-1}(\delta)(x_{ij}^{\alpha}+x_{ij}^{\beta}) + \sigma_{ij}\Phi^{-1}(\gamma) \leq r.$$

Finally, with the substitutions $r = \hat{x}_{ij}/\lambda_j$ and $\Phi^{-1}(\gamma) = -\Phi^{-1}(1-\gamma)$, constraint (iii) in Model (10), $\overline{\mathrm{Pr}}(\hat{x}_{ij}/\lambda_j \leq \tilde{\tilde{x}}_{ij} \leq \hat{x}_{ij}/\lambda_j) \geq \gamma$, is converted to the following deterministic constraints:

$$\hat{x}_{ij} \leq \lambda_j(x_{ij} - L^{-1}(\delta)x_{ij}^{\alpha} + R^{-1}(\delta)(x_{ij}^{\alpha}+x_{ij}^{\beta}) + \sigma_{ij}\Phi^{-1}(1-\gamma))$$

and

$$\lambda_j(x_{ij} + R^{-1}(\delta)x_{ij}^{\beta} - L^{-1}(\delta)(x_{ij}^{\alpha}+x_{ij}^{\beta}) - \sigma_{ij}\Phi^{-1}(1-\gamma)) \leq \hat{x}_{ij}.$$

Similarly, for constraint (iv) in Model (10),

$$\overline{\mathrm{Pr}\left(\frac{\hat{y}_{rj}}{\lambda_j} \leq \tilde{\tilde{y}}_{rj} \leq \frac{\hat{y}_{rj}}{\lambda_j}\right)} \geq \gamma,$$

we will have

$$\hat{y}_{rj} \leq \lambda_j(y_{rj} - L^{-1}(\delta)y_{rj}^{\alpha} + R^{-1}(\delta)(y_{rj}^{\alpha}+y_{rj}^{\beta}) + \sigma_{rj}\Phi^{-1}(1-\gamma))$$

and

$$\lambda_j(y_{rj} + R^{-1}(\delta)y^{\beta}_{rj} - L^{-1}(\delta)(y^{\alpha}_{rj} + y^{\beta}_{rj}) - \sigma_{rj}\Phi^{-1}(1-\gamma)) \le \hat{y}_{rj}.$$

Therefore, the deterministic equivalent for Model (10) can be derived as follows:

$$\begin{aligned}
&E_k(\delta,\gamma) = \min \varphi\\
&\text{s.t.}\\
&\sum_{j=1}^{n} \hat{x}_{ij} \le \theta(\tilde{\bar{x}}_{ik} + R^{-1}(\delta)x^{\beta}_{ik} + \sigma_{ik}\Phi^{-1}(1-\gamma)), \quad i = 1,2,\ldots,m\\
&\sum_{j=1}^{n} \hat{y}_{rj} \ge (y_{rk} + R^{-1}(\delta)y^{\beta}_{rk} - L^{-1}(\delta)(y^{\alpha}_{rk} + y^{\beta}_{rk}) - \sigma_{rk}\Phi^{-1}(1-\gamma)), \quad r = 1,2,\ldots,s\\
&\lambda_j(y_{rj} + R^{-1}(\delta)y^{\beta}_{rj} - L^{-1}(\delta)(y^{\alpha}_{rj} + y^{\beta}_{rj}) - \sigma_{rj}\Phi^{-1}(1-\gamma)) \le \hat{y}_{rj}, \quad \forall r,j\\
&\hat{y}_{rj} \le \lambda_j(y_{rj} - L^{-1}(\delta)y^{\alpha}_{rj} + R^{-1}(\delta)(y^{\alpha}_{rj} + y^{\beta}_{rj}) + \sigma_{rj}\Phi^{-1}(1-\gamma)), \quad \forall r,j\\
&\lambda_j(x_{ij} + R^{-1}(\delta)x^{\beta}_{ij} - L^{-1}(\delta)(x^{\alpha}_{ij} + x^{\beta}_{ij}) - \sigma_{ij}\Phi^{-1}(1-\gamma)) \le \hat{x}_{ij}, \quad \forall i,j\\
&\hat{x}_{ij} \le (x_{ij} - L^{-1}(\delta)x^{\alpha}_{ij} + R^{-1}(\delta)(x^{\alpha}_{ij} + x^{\beta}_{ij}) + \sigma_{ij}\Phi^{-1}(1-\gamma)), \quad \forall i,j\\
&\lambda_j \ge 0, \quad \forall j.
\end{aligned} \tag{11}$$

In case that $L^{-1}(\delta) = R^{-1}(\delta)$, Model (11) is converted to the following Model (12). An example for this case is the triangular fuzzy number with $L^{-1}(\delta) = R^{-1}(\delta) = 1 - \delta$.

$$\begin{aligned}
&E^{T}_k(\delta,\gamma) = \min \varphi\\
&\text{s.t.}\\
&\sum_{j=1}^{n} \hat{x}_{ij} \le \theta(\tilde{\bar{x}}_{ik} + R^{-1}(\delta)x^{\beta}_{ik} + \sigma_{ik}\Phi^{-1}(1-\gamma)), \quad i = 1,2,\ldots,m\\
&\sum_{j=1}^{n} \hat{y}_{rj} \ge (y_{rk} - L^{-1}(\delta)y^{\alpha}_{rk} - \sigma_{rk}\Phi^{-1}(1-\gamma)), \quad r = 1,2,\ldots,s\\
&\lambda_j(y_{rj} - L^{-1}(\delta)y^{\alpha}_{rj} - \sigma_{rj}\Phi^{-1}(1-\gamma))\\
&\qquad \le \hat{y}_{rj} \le \lambda_j(y_{rj} + R^{-1}(\delta)y^{\beta}_{rj} + \sigma_{rj}\Phi^{-1}(1-\gamma)), \quad \forall r,j\\
&\lambda_j(x_{ij} - L^{-1}(\delta)x^{\alpha}_{ij} - \sigma_{ij}\Phi^{-1}(1-\gamma))\\
&\qquad \le \hat{x}_{ij} \le \lambda_j(x_{ij} + R^{-1}(\delta)x^{\beta}_{ij} + \sigma_{ij}\Phi^{-1}(1-\gamma)), \quad \forall i,j\\
&\lambda_j \ge 0, \quad \forall j.
\end{aligned} \tag{12}$$

The above model is obviously a linear program. It should be noted that the corresponding deterministic model obtained by Ref. 41 is a nonlinear program.

Lemma 2. *Consider the following problem*:

$$\begin{aligned}
T(\lambda) = \min\ & C(x)\\
& f_j(\lambda; x) \ge 0, \quad j = 1,2,\ldots,m,\\
& x \in S
\end{aligned} \tag{13}$$

where T is a function related to λ, and x is a decision variable that belongs to $S \subseteq R^n$. If f_j is a decreasing function related to λ, then T will be increasingly related to λ.

Proof. Given $\lambda_2 \geq \lambda_1$, we show that $T(\lambda_2) \geq T(\lambda_1)$. A direct conclusion of the increasing function f_j is $f_j(\lambda_1; x) \geq f_j(\lambda_2; x)$, $\forall x \in S$. As a result, each feasible solution x, with $f_j(\lambda_2; x) \geq 0$, from model related to $T(\lambda_2)$ is also feasible, with $f_j(\lambda_1; x) \geq 0$, for the model related to $T(\lambda_1)$. As the problem is minimized, $T(\lambda_2) \geq T(\lambda_1)$ and the proof is complete. □

The following theorem shows that the objective function of Model (11), $E_k(\delta, \gamma)$, is monotonously decreasing related to each level of δ and γ.

Theorem 2. *If $E_k(\delta, \gamma)$ is the optimum objective function value of Model* (11), *then $E_k(\delta_1, \gamma) \leq E_k(\delta_2, \gamma)$ and $E_k(\delta, \gamma_1) \leq E_k(\delta, \gamma_2)$ where $\delta_1 \leq \delta_2$ and $\gamma_1 \leq \gamma_2$.*

Proof. Let $T(\lambda) = E_k(\delta, \gamma)$ with $\lambda = \delta$ in Lemma 2. Consider each constraint in Model (11) as $f_j(\lambda; x) \geq 0$. As $L^{-1}(\delta)$ and $R^{-1}(\delta)$ are decreasing functions, the corresponding function f_j related to each constraint is decreasing related to $\lambda = \delta$. Hence, $E_k(\delta_2, \gamma) = T(\lambda_2) \geq T(\lambda_1) = E_k(\delta_1, \gamma)$. Let $T(\lambda) = E_k(\delta, \gamma)$ with $\lambda = \gamma$ as in Lemma 1 and $\Phi^{-1}(\gamma) = \Phi_\gamma^{-1}$. As $\Phi^{-1}(\gamma)$ is increasing, the function $\Phi^{-1}(1 - \gamma)$ would be decreasing. Hence, a similar reasoning can be applied to show that $E_k(\delta, \gamma_2) \geq E_k(\delta, \gamma_1)$. This completes the proof. □

We present the following definition to define the efficiency of each DMU.

Definition 12. For the given level δ and γ, we define $E_k^T(\delta, \gamma) = E_k(\delta, \frac{\gamma}{2})$ as the efficiency score of DMU_k in the fuzzy random DEA model.

The corresponding model with $E_k^T(\delta, \gamma)$ is as follows:

$$
\begin{aligned}
& E_k^T(\delta, \gamma) = \min \varphi \\
& \text{s.t.} \\
& \sum_{j=1}^{n} \hat{x}_{ij} \leq \theta\Big(\tilde{\bar{x}}_{ik} + R^{-1}(\delta) x_{ik}^{\beta} + \sigma_{ik} \Phi^{-1}\Big(1 - \frac{\gamma}{2}\Big)\Big) \quad i = 1, 2, \ldots, m \\
& \sum_{j=1}^{n} \hat{y}_{rj} \geq \Big(y_{rk} - L^{-1}(\delta) y_{rk}^{\alpha} - \sigma_{rk} \Phi^{-1}\Big(1 - \frac{\gamma}{2}\Big)\Big) \quad r = 1, 2, \ldots, s \\
& \lambda_j\Big(y_{rj} - L^{-1}(\delta) y_{rj}^{\alpha} - \sigma_{rj} \Phi^{-1}\Big(1 - \frac{\gamma}{2}\Big)\Big) \\
& \qquad \leq \hat{y}_{rj} \leq \lambda_j\Big(y_{rj} + R^{-1}(\delta) y_{rj}^{\beta} + \sigma_{rj} \Phi^{-1}\Big(1 - \frac{\gamma}{2}\Big)\Big), \quad \forall r, j \\
& \lambda_j\Big(x_{ij} - L^{-1}(\delta) x_{ij}^{\alpha} - \sigma_{ij} \Phi^{-1}\Big(1 - \frac{\gamma}{2}\Big)\Big) \\
& \qquad \leq \hat{x}_{ij} \leq \lambda_j\Big(x_{ij} + R^{-1}(\delta) x_{ij}^{\beta} + \sigma_{ij} \Phi^{-1}\Big(1 - \frac{\gamma}{2}\Big)\Big), \quad \forall i, j \\
& \lambda_j \geq 0, \quad \forall j.
\end{aligned}
\tag{14}
$$

As $E_k(\delta, \gamma)$ is monotonically increasing relative to each level, especially γ, there is a bijection correspondence between $E_k(\delta, \gamma)$ and $E_k(\delta, \frac{\gamma}{2})$. Hence, $E_k(\delta, \frac{\gamma}{2})$ is a suitable substitution of the efficiency value $E_k(\delta, \gamma)$. On the contrary, the choice of $E_k(\delta, \gamma)$

for efficiency scores of Model (11) may lead to infeasible results for values $\gamma > 0.5$. Hence, the well-defined transition $E_k^T(\delta, \gamma) = E_k(\delta, \frac{\gamma}{2})$ solves the problem.

Theorem 3. *Consider $E_k^T(\delta, \gamma)$ as the optimum objective function value of Model (14) for DMU_k, then*

(1) *$E_k^T(\delta_1, \gamma) \geq E_k^T(\delta_2, \gamma)$ and $E_k^T(\delta, \gamma_1) \geq E_k^T(\delta, \gamma_2)$ where $\delta_1 \leq \delta_2$ and $\gamma_1 \leq \gamma_2$.*
(2) *Model (14) is feasible for any δ and γ.*

Proof. The proof of (1) is straightforward using Theorem 1 and Definition 12. According to part (1), $E_k^T(\delta, \gamma)$ is increasing with respect to both the δ and γ thresholds, and so $E_k^T(\delta, \gamma) \leq E_k^T(1, 1)$. Let $\delta = 1$ and $\gamma = 1$, then $L^{-1}(1) = R^{-1}(1) = 0$ and $\Phi^{-1}(0.5) = 0$. Hence, we have $\hat{x}_{ij} = \lambda_j x_{ij}$, $\hat{y}_{rj} = \lambda_j y_{rj}$ in Model (14). Therefore, the corresponding model with $E_k^T(1, 1)$ will be as follows:

$$
\begin{aligned}
E_k(1,1) = {} & \operatorname{Min} \theta \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{ik}, \quad i = 1, 2, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rk}, \qquad r = 1, 2, \ldots, s, \\
& \lambda_j \geq 0, \qquad\qquad\quad j = 1, 2, \ldots, n.
\end{aligned} \tag{15}
$$

As seen the above model is same as the traditional CCR-DEA model given in Eq. (8). Denote the feasible space of Model (14) by $S_{\delta,\gamma}^T$. According to the proof of Theorem 2, $S_{1,1}^T \subseteq S_{\delta,\gamma}^T$. Therefore, it is sufficient to show that the feasible space $S_{1,1}^T$ is nonempty. As said earlier, $E_k^T(1, 1)$ is given by Model (15) and this model is always feasible as the traditional CCR-DEA model. This completes the proof of part (2). □

Remark 3. For the stochastic threshold value $\gamma > 1/2$, the model related to $E_k(\delta, \gamma)$ will be infeasible. So, the efficiency cannot be defined in this case. Hence, by Definition 12, we have defined a new representation of the efficiency, $E_k^T(\delta, \gamma/2)$, in the proposed fuzzy random DEA model. This concept of efficiency is well defined according to Theorem 2 since $E_k(\delta, \gamma)$ is increasing with respect to threshold γ.

4.3. *Advantages of the proposed approach*

The chance-constrained programming approach and the measure-based approach are two main approaches for solving DEA models with fuzzy random variables. But, these approaches are not able to determine a unique measure in fuzzy random space. Introducing such measure is the main contribution of this study. The other advantages of the proposed models can be summarized as follows:

- The main advantage of the proposed model is to introduce an extension of normal distribution to support some fuzzy random data, and this is the first attempt in this area.

- In contrast to traditional DEA models, the existing fuzzy stochastic DEA models (Ref. 41) do not provide the efficiency score for each DMU within the specified range (0, 1]. The extended fuzzy stochastic DEA models in this paper, similar to the conventional DEA models, provide the efficiency score for each DMU within the specified range (0, 1].
- The existing method proposed by Ref. 41 is infeasible for some cases, whereas the-fuzzy stochastic DEA model proposed here is always feasible, as proved in Theorem 3.
- In the proposed approaches, several DMUs may be assessed as FSDEA-efficient. Similar to the conventional DEA ranking models, the FSDEA cross efficiency model can be extended to distinguish the performance of FSDEA-efficient DMUs.
- It is worthwhile to note that using the Pr(pos(.)) approach for solving fuzzy stochastic DEA models, the model under consideration is converted to a quadratic programming model, while using the $\overline{\mathrm{Pr}}(.)$ approach for solving DEA models with fuzzy stochastic data results in a linear programming model.

4.4. *Numerical example*

The efficiency of four farms D_1, D_2, D_3, and D_4, with areas of 5, 5, 4, and 7 acres, respectively, is to be evaluated. The crop is wheat in all of the farms. The amount of the yield is a random variable normally distributed with mean 2.5, 3, 5, and 3.5. The variance is 1 for all. The amount of rainfall is estimated as a fuzzy random variable. The data are listed in Table 1.

The corresponding model 14 with D_1 is as follows:

$$
\begin{aligned}
&E_1^T(\delta,\gamma) = \min \varphi\\
&\text{s.t.}\\
&\theta \le \hat{y}_{11}\\
&\hat{x}_{11} + \hat{x}_{12} + \hat{x}_{13} + \hat{x}_{14} \le \theta\Big(\tilde{\bar{x}}_{11} + R^{-1}(\delta)x_{11}^{\beta} + \sigma_{11}\Phi^{-1}\Big(1-\frac{\gamma}{2}\Big)\Big)\\
&\hat{x}_{21} + \hat{x}_{22} + \hat{x}_{23} + \hat{x}_{24} \le \theta\Big(\tilde{\bar{x}}_{21} + R^{-1}(\delta)x_{21}^{\beta} + \sigma_{21}\Phi^{-1}\Big(1-\frac{\gamma}{2}\Big)\Big)\\
&\hat{y}_{11} + \hat{y}_{12} + \hat{y}_{13} + \hat{y}_{14} \ge \Big(y_{11} - L^{-1}(\delta)y_{11}^{\alpha} - \sigma_{11}\Phi^{-1}\Big(1-\frac{\gamma}{2}\Big)\Big)\\
&\lambda_j\Big(y_{1j} - L^{-1}(\delta)y_{1j}^{\alpha} - \sigma_{1j}\Phi^{-1}\Big(1-\frac{\gamma}{2}\Big)\Big)\\
&\qquad \le \hat{y}_{1j} \le \lambda_j\Big(y_{1j} + R^{-1}(\delta)y_{1j}^{\beta} + \sigma_{1j}\Phi^{-1}\Big(1-\frac{\gamma}{2}\Big)\Big), \quad \forall j
\end{aligned}
$$

Table 1. The data for the numerical example.

	D_1	D_2	D_3	D_4
I_1	$(N(6,1),1)$	$(N(2,1),1)$	$(N(4,1),1)$	$(N(1.5,1),1)$
I_2	5	5	4	7
O_1	$N(2.5,1)$	$N(3,1)$	$N(5,1)$	$N(3.5,1)$

$$\lambda_j\Big(x_{1j} - L^{-1}(\delta)x_{1j}^{\alpha} - \sigma_{1j}\Phi^{-1}\Big(1-\frac{\gamma}{2}\Big)\Big)$$
$$\leq \hat{x}_{1j} \leq \lambda_j\Big(x_{1j} + R^{-1}(\delta)x_{1j}^{\beta} + \sigma_{1j}\Phi^{-1}\Big(1-\frac{\gamma}{2}\Big)\Big), \quad \forall j$$
$$\lambda_j\Big(x_{2j} - L^{-1}(\delta)x_{2j}^{\alpha} - \sigma_{2j}\Phi^{-1}\Big(1-\frac{\gamma}{2}\Big)\Big)$$
$$\leq \hat{x}_{2j} \leq \lambda_j\Big(x_{2j} + R^{-1}(\delta)x_{2j}^{\beta} + \sigma_{2j}\Phi^{-1}\Big(1-\frac{\gamma}{2}\Big)\Big), \quad \forall j$$
$$\lambda_j \geq 0. \tag{16}$$

If we consider $(\delta = 0.25, \gamma = 0.75)$ as threshold values, then $L^{-1}(\delta) = 1 - \delta = 0.75$ and $\Phi^{-1}(1-\gamma/2) = \Phi^{-1}(0.625) = 0.32$. Therefore, the corresponding efficiency value of D_1 given by Model (14) is $E_1^T(\delta,\gamma) = 0.3278$. In this way, we obtain $E_2^T(\delta,\gamma) = 0.4499$, $E_3^T(\delta,\gamma) = 0.8797$, and $E_4^T(\delta,\gamma) = 0.5087$. The efficiency scores related to probability-credibility model proposed by Tavana *et al.*[41] will be infeasible for DMU3.

5. Case Study

In this section, we present a case study in the banking industry to exhibit the applicability of the proposed method and feasibility of the obtained solutions. The International Bank of Iran (IBI)[a] is the first international bank established in 1927 by the Iranian government. The bank has 25 branches in the city of Qaemshahr in southern Iran. The input and output data used in this study belong to these 25 IBI branches from May 2012 to February 2013. We considered a personnel score (reflecting quantity and quality of the personnel at each branch), total short-term and long-term deposits (TD), and nonperforming assets or loans (NPA). The output data included: interest and fee revenues. Each branch uses three inputs (personnel score, deposits, and delinquency) to produce two outputs (interests and fees). Table 2 presents the crisp input–output data for each branch.

However, there is some degree of uncertainty associated with these input and output data which could be represented by fuzzy stochastic numbers. In banks, the source of uncertainty is the discrepancy between the actual and available data. The two inputs (TDs and NPAs) and all outputs are represented by TFNs. The collected crisp data in Table 1 related to TDs, NPAs, and outputs are considered as the mean of the TFNs. The left and right spreads of the inputs and outputs are calculated by 1% of the mean value. On the contrary, the inputs and outputs corresponding to these 10 consecutive months are assumed to be random variables that need to be estimated. By using goodness of fit tests, normal distributions have been fit on the random variables. The corresponding expected value is the observed input (output) data and the standard deviation of the components is 1. Hence, each DMU in this case is considered as a fuzzy variable with randomized mean.

[a] The name is changed to protect the anonymity of the bank.

Table 2. The fuzzy random input and output data.

DMU	Input 1 (personnel rate)	Input 2 (total of deposits)	Input 3 (nonperforming loans)	Output 1 (received interest)	Output 2 (fee)
1	36	2,696,995	87,243	108,634.8	965.97
2	18	340,377	9,945	32,396.65	304.67
3	25	1,027,546	47,575	96,842.33	2,285.03
4	20	1,145,235	19,292	32,362.8	207.98
5	14	390,902	3,428	12,662.71	63.32
6	19	988,115	13,929	53,591.3	480.16
7	27	144,906	27,827	40,507.97	176.58
8	24	408,163	9,070	56,260.09	4,654.71
9	20	335,070	412,036	176,436.8	560.26
10	14	700,842	8,638	662,725.2	58.89
11	20	641,680	500	17,527.58	1,070.81
12	22	453,170	16,148	35,757.83	375.07
13	18	553,167	17,163	45,652.24	438.43
14	22	309,670	17,918	8,143.79	936.62
15	39	286,149	51,582	106,798.6	1,203.79
16	25	321,435	20,975	89,971.47	200.36
17	27	618,105	41,960	33,036.79	2,781.24
18	13	248,125	18,641	9,525.6	240.04
19	27	640,890	19,500	66,097.16	961.56
20	28	119,948	31,700	21,991.53	282.73
21	19	513,463	4,545	30,263.6	789.64
22	22	576,434	21,521	495,947.2	402.36
23	19	789,994	18,014	40,336.46	346.25
24	15	375,744	5,383	18,582.89	135.73
25	23	750,555	193,359	128,680.1	1,595.12

Five different (δ, γ)-threshold levels of $(\delta = 0.5, \gamma = 0.25)$, $(\delta = 0.75, \gamma = 0.25)$, $(\delta = 0.5, \gamma = 0.5)$, $(\delta = 0.25, \gamma = 0.5)$, and $(\delta = 0.25, \gamma = 0.75)$ were considered according to the previous DM performance evaluation studies.

In Table 2, we present the efficiency values associated with the bank branches for five specified threshold levels given by Model (14). As shown in this table, DMUs 8, 10, 11, and 22 have the best efficiency scores at each given level, respectively.

Generally from Table 3, we can see the applicability of Theorems 2 and 3 for Model (12) when the efficiency scores of the DMUs are improved when the level δ increases from $(\delta = 0.5, \gamma = 0.25)$ to $(\delta = 0.75, \gamma = 0.25)$ and the level γ increases from $(\delta = 0.25, \gamma = 0.5)$ to $(\delta = 0.25, \gamma = 0.75)$. In addition to the decreasing properties of efficiency scores, efficiency scores are between 0 and 1 for all DMUs with the ability of distinguishing between efficient and inefficient DMUs.

To compare the proposed Model (14) with the approach given in Ref. 41, we run the probability-possibility model of their approach. As we see from Table 4, the efficiency of DMU8 and 10 is greater than 1 for all levels.

Figure 1 illustrates a comparison of these two approaches in the average index.

The final rankings of the 25 bank branches are presented in Table 5.

The main reason for using the proposed approach in this study in comparison with the approach given by Tavana *et al.*[41] is for performance evaluation. First, the model

Table 3. The fuzzy random efficiency scores (Model 14).

	($\delta = 0.5$, $\gamma = 0.25$)	($\delta = 0.75$, $\gamma = 0.25$)	($\delta = 0.5$, $\gamma = 0.5$)	($\delta = 0.25$, $\gamma = 0.5$)	($\delta = 0.25$, $\gamma = 0.75$)	Average 1
1	0.193	0.193	0.199	0.197	0.194	0.195
2	0.147	0.149	0.147	0.147	0.147	0.147
3	0.523	0.525	0.543	0.541	0.540	0.534
4	0.102	0.103	0.112	0.110	0.108	0.107
5	0.121	0.125	0.124	0.121	0.118	0.121
6	0.173	0.174	0.173	0.173	0.169	0.172
7	0.357	0.363	0.357	0.355	0.352	0.356
8	0.998	0.999	0.998	0.998	0.997	0.998
9	0.641	0.642	0.642	0.641	0.641	0.641
10	0.997	0.997	0.997	0.996	0.995	0.996
11	0.612	0.612	0.621	0.611	0.609	0.613
12	0.126	0.125	0.126	0.126	0.125	0.125
13	0.163	0.164	0.164	0.162	0.162	0.163
14	0.217	0.227	0.227	0.213	0.208	0.218
15	0.612	0.612	0.702	0.624	0.623	0.634
16	0.326	0.326	0.326	0.326	0.326	0.326
17	0.539	0.539	0.539	0.524	0.521	0.532
18	0.130	0.131	0.131	0.129	0.127	0.129
19	0.226	0.236	0.235	0.221	0.219	0.227
20	0.315	0.316	0.315	0.306	0.306	0.311
21	0.543	0.543	0.521	0.543	0.543	0.538
22	0.940	0.940	0.953	0.938	0.937	0.941

Table 4. The fuzzy random efficiency scores (the approach given in Ref. 41).

	($\delta = 0.5$, $\gamma = 0.25$)	($\delta = 0.75$, $\gamma = 0.25$)	($\delta = 0.5$, $\gamma = 0.5$)	($\delta = 0.25$, $\gamma = 0.5$)	($\delta = 0.25$, $\gamma = 0.75$)	Average 2
1	0.217	0.216	0.219	0.219	0.218	0.217
2	0.148	0.148	0.148	0.149	0.152	0.149
3	0.592	0.581	0.580	0.583	0.583	0.583
4	0.125	0.124	0.131	0.141	0.141	0.132
5	0.112	0.108	0.108	0.118	0.119	0.113
6	0.203	0.203	0.201	0.202	0.202	0.202
7	0.359	0.359	0.341	0.341	0.342	0.348
8	1.033	1.032	1.032	1.032	1.032	1.032
9	0.462	0.451	0.451	0.453	0.461	0.455
10	1.074	1.074	1.073	1.073	1.073	1.073
11	0.614	0.612	0.612	0.615	0.616	0.613
12	0.139	0.138	0.136	0.137	0.139	0.137
13	0.192	0.192	0.191	0.191	0.191	0.191
14	0.247	0.237	0.235	0.245	0.245	0.241
15	0.614	0.614	0.613	0.613	0.614	0.613
16	0.328	0.328	0.328	0.329	0.329	0.328
17	0.605	0.604	0.601	0.604	0.615	0.605
18	0.145	0.143	0.143	0.143	0.144	0.143
19	0.257	0.257	0.256	0.256	0.257	0.256
20	0.317	0.317	0.308	0.310	0.311	0.312
21	0.544	0.543	0.541	0.542	0.542	0.542
22	0.944	0.944	0.943	0.943	0.943	0.943

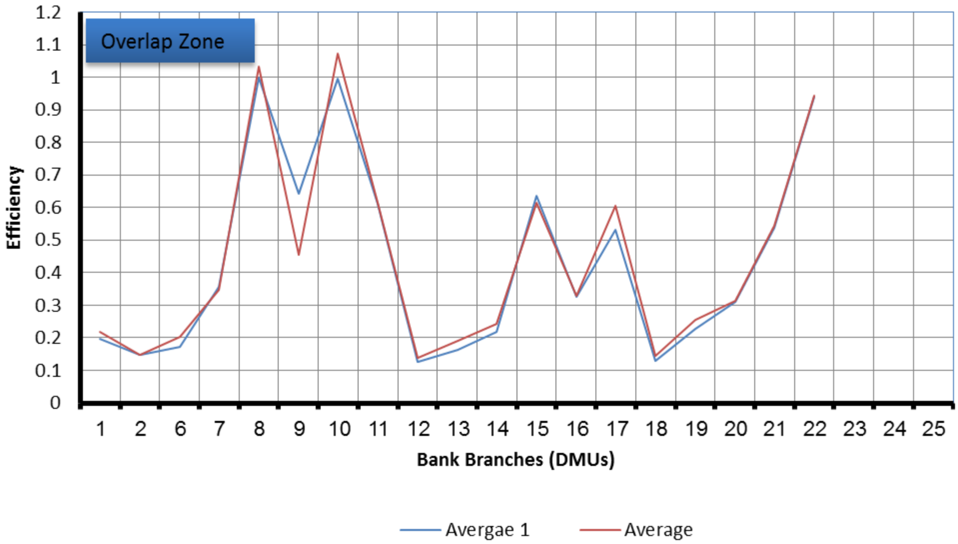

Fig. 1. A comparison illustration.

proposed in Ref. 41 cannot divide the DMUs into two groups, namely, inefficient and efficient DMUs. Secondly, that approach extended the multiplier DEA models to a fuzzy stochastics environment. Thus, based on these models, it is not possible to determine virtual unit on the efficient surface as a target unit for the inefficient unit. While our proposed stochastics DEA models not only provide fuzzy stochastics efficiency scores that can be used in practical applications as performance indicators of the DMUs, but also divide the DMUs into two different groups, inefficient and efficient DMUs, by providing efficiency scores less than and equal to one, respectively. Moreover, since we have extended the envelopment DEA model to a fuzzy stochastics environment, it is practical to find a target unit for each inefficient DMU that can inform the decision maker of the amount (%) by which an inefficient DMU

Table 5. Complete rankings of the DMUs.

δ	γ	Complete rankings of the DMUs
0.50	0.25	$10 > 8 > 11 > 22 > 15 > 9 > 17 > 3 > 25 > 7 > 20 > 21 > 16 > 14 > 19 > 1 > 2 > 6 > 13 > 12 > 23 > 18 > 24 > 4 > 5$
0.75	0.25	$10 > 8 > 11 > 22 > 15 > 9 > 17 > 3 > 25 > 7 > 20 > 21 > 16 > 14 > 19 > 1 > 2 > 6 > 13 > 12 > 23 > 18 > 24 > 4 > 5$
0.50	0.50	$10 > 8 > 11 > 22 > 15 > 9 > 17 > 3 > 25 > 7 > 20 > 21 > 16 > 14 > 19 > 1 > 6 > 2 > 13 > 12 > 23 > 18 > 24 > 4 > 5$
0.25	0.50	$10 > 8 > 11 > 22 > 15 > 9 > 17 > 3 > 25 > 7 > 20 > 21 > 16 > 14 > 19 > 1 > 6 > 2 > 13 > 12 > 23 > 18 > 24 > 4 > 5$
0.25	0.75	$10 > 8 > 11 > 22 > 15 > 9 > 17 > 3 > 25 > 7 > 20 > 21 > 16 > 14 > 19 > 1 > 6 > 13 > 2 > 12 > 23 > 18 > 24 > 4 > 5$

should decrease its inputs and/or increase its outputs to become efficient. Finally, the proposed model in Ref. 41 is nonlinear which makes it difficult and time-consuming to obtain the optimal solutions, whereas the proposed model in this study is linear which can be solved by the standard linear programming algorithm in a simple manner.

6. Conclusions

In this paper, we formulated a DEA model in a fuzzy random environment. In this way, we applied an extended normal distribution to illustrate a normal random with fuzzy values. This extension provides a measure of fuzzy random variables. Furthermore, the methodology uses chance-constrained programming to solve such a DEA model with the proposed measure. In comparison with the proposed model in Ref. 41, our proposed approach not only results in a linear programming model, but also gives efficiency scores within the range of zero to one for DMUs similar to traditional input-oriented DEA models. Moreover, the feasibility of the proposed model is proved for different fuzzy and random thresholds. Also, a case study for the banking industry has been utilized to analyze the performance of some commercial bank branches in Iran. Therefore, future research can develop global measures for fuzzy stochastic programming using chance-constrained programming. Also this work can be extended to solve other mathematical programming problems.[58–61]

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