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A bilateral exchange model: The paradox of quantifying the linguistic values of qualitative characteristics

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ABSTRACT

You can't always get what you want but (with a bit of luck) you may get what you expect. We study a bilateral exchange model where decision makers (DMs) perceive subjectively the characteristics of the products they initially own. They use a common language to communicate with each other while four requirements are imposed to prevent them from purposely trying to manipulate the exchange process. We illustrate how, even if these requirements are satisfied, the product that each DM receives from the exchange is possibly quite different from the one (or ones) that each had envisioned based on the reports provided by the other DM. In particular, the products received may deliver a utility higher or lower than that of the product originally owned by each DM which may be a direct consequence of the DMs using linguistic values to describe the qualitative characteristics of their products. However, we show that DMs may agree to exchange and turn out to be worse off even when they are asked to express their qualitative evaluations using real values belonging to a normalized interval. Paradoxically enough, we will argue that quantifying the linguistic values of qualitative characteristics creates more misunderstanding than using the corresponding linguistic values.

1. Introduction

The formal modelization of the imprecision and incompleteness inherent in the preferences and beliefs of decision makers (DMs) has constituted one of the main research interests among economists for quite some time [6,19,26,34]. At the same time, fuzzy set theorists [29,42,43] and intuitionist philosophers [4,5,20] have tackled and shed light on the imprecision inherent in the behavior of DMs through their respective disciplines. The current paper studies the modelization of the behavior of rational DMs within a bilateral exchange setting in such a way that can be useful to any of these disciplines.

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We study the bilateral exchange of products between rational DMs. Products are defined through characteristics distributed among different categories. At the same time, DMs perceive subjectively the characteristics of the products while sharing a common language to communicate with each other. We will illustrate how *absent strategic considerations* based on the capacity of DMs to report falsely for their own benefit, commonly assumed in the economic literature [1,11], DMs may agree to exchange products and end up with a less preferred product than the one they were initially endowed with. This provides novel insights into the suboptimality inherent to the DMs interacting within a given exchange system.

The strategic approach to fuzziness from the economics literature concentrates on the topological properties and the selection process determining the existence of an equilibrium through a fixed point (see [24,33]), as well as analyzing the properties of the core in the determination of the equilibrium [22]. While we have previously dealt with the properties of fuzzy preferences under uncertainty (see [14,15]), in the current paper we focus on the effects that the subjective perceptions of DMs have when evaluating product alternatives and reporting the information *as they perceive it*.

Consider a system composed of two DMs, each endowed with a particular product that allows them to potentially engage in bilateral trade. Each DM must decide whether or not to exchange his product with the one of the other DM and will only do so if he expects to attain a higher utility level through the exchange. In order to be able to evaluate the expected utility derived from the exchange, each DM must communicate, that is, each DM must describe his own product through a report.

Four requirements are imposed to prevent the DMs from purposely trying to manipulate the exchange process (for their own benefit). After the DMs communicate their reports in a way that the manipulation-free requirements are satisfied, each DM calculates his exchange expected utility on the basis of his subjective beliefs regarding the product expected to be received. Each DM will agree to trade only if the value obtained from the exchange expected utility is higher than the utility value of the product that he owns. Hence, exchange will occur only if both DMs perceive this to be the case.

Suppose that the DMs agree to exchange under the manipulation-free requirements. Even if all the requirements are satisfied, the product that each DM receives from the exchange is quite possibly completely different from the one (or ones) that he had envisioned on the basis of the reports provided by the other DM. This difference is due to the subjective perceptions that DMs have of the products. Nevertheless, the newly acquired product may still deliver a utility higher than the one originally owned. Thus, even if the DM does not get exactly what he was thinking of, he may still get what he was expecting in terms of utility, i.e. an acceptable product.

However, it is also conceivable that one or both DMs get an unacceptable product delivering a lower utility than the one they were initially endowed with. This result may be considered a direct consequence of the fact that DMs use linguistic values to describe the qualitative characteristics of their products. Thus, one may intuitively conjecture that this suboptimal situation would not arise if the DMs were allowed to express themselves using a real scale [0, 10] instead of through linguistic evaluations – a quite common assumption in the standard microeconomic literature [12,27,30].

The main contribution of our paper is a formal proof of the fact that the above conjecture is false and the DMs can agree to exchange and turn out to be worse off even when they are asked to express their qualitative evaluations via real values belonging to a normalized interval. Paradoxically enough, we will actually argue that quantifying the linguistic values of qualitative characteristics may create more misunderstanding than using the corresponding linguistic values.

Fuzzy methods are commonly used to evaluate the assessments made by DMs when describing linguistically the behavior or value of variables relating to the risk inherent in the decisions they must make, see [8,25]. In particular, the use of equally distributed (symmetric) triangular fuzzy numbers (TFNs) is quite prevalent in the literature on expert and knowledge based systems, see [37–40]. Our approach will consider asymmetric TFNs and it is in this sense more general. At the same time, it allows for a more natural formalization of subjective fuzzy descriptions whose ranges of variations are not imposed on the DMs but are autonomously constructed by them. Asymmetric fuzzy numbers constructed in this way are generally used in control, expert and knowledge-based systems [7], particularly when dealing with genetic and neural fuzzy systems (see [10,16,28]), but remain foreign to the economic literature. The subjective construction of asymmetric fuzzy numbers by DMs within a bilateral exchange setting does not only provide a novel approach to the corresponding economic analysis but also opens the way for the creation of multiple links and potential complementarities between the systems and economic literatures.

The paper proceeds as follows. In Section 2, we introduce the basic notations. Section 3 defines the shared language used by DMs, introduces the manipulation-free requirements and the exchange expected utilities, and discusses the distinction among expected, acceptable and unacceptable products. We use fuzzy numbers to objectify linguistic variables in Section 4. The main results are presented in Section 5. Section 6 provides managerial implications and highlights potential extensions. Section 7 concludes.

2. Basic concepts and initial assumptions

Let $\wp$ be the set of all products. Let $\{C_j: j \in J\}$ be the partition of $\wp$ whose elements are subsets of $\wp$ formed by products of the same type or category. That is, $\wp = \cup_{j \in J} C_j$, where $C_j \cap C_{j'} = \varnothing$ for every $j \neq j'$. For instance, one may think of C_1 as the set of all cars, C_2 as the set of all wine bottles, C_3 as the set of all shoes, and so on. Following the standard set-theoretical notations, the symbol J will be used to denote both the cardinality of the set of all categories to which products may belong and the cardinal number indexing this very set.

For every $j \in J$, let Δ_j be the set indexing all the characteristics of the products in the category C_j and let X_{δ_j}, where $\delta_j \in \Delta_j$, be the set of all possible evaluations that a DM can use to describe the (δ_j)-th characteristic of a product in C_j.

The set X_{δ_j} can consist of either quantitative or qualitative evaluations for the (δ_j)-th characteristic of a product in C_j. Suppose, for example, that the product to describe is a wine bottle. The % *vol* and the *price* are quantitative characteristics, but the *color*, the *flavor*, and the *smell* are qualitative characteristics.

If X_{δ_j} is a set of numerical values, we can assume it to coincide with an interval of real numbers. Otherwise, X_{δ_j} plays the role of a linguistic variable, that is, it contains all possible qualitative evaluations of a certain adjective that applies to the product in question. Thus, we assume the following:

Assumption 1. For every $j \in J$ and every $\delta_j \in \Delta_j$,

- if X_{δ_j} consists of quantitative evaluations, then
$$X_{\delta_j} = [m_{\delta_j}, M_{\delta_j}] \tag{1}$$
where $m_{\delta_j},\ M_{\delta_j} \geqslant 0$, with $m_{\delta_j} < M_{\delta_j}$;
- if X_{δ_j} consists of qualitative evaluations, then
$$\begin{aligned} X_{\delta_j} = \{&not\ a(\delta_j),\ almost\ not\ a(\delta_j),\ lowly\ a(\delta_j),\ not\ very\ a(\delta_j),\\ &regularly\ a(\delta_j),\ very\ a(\delta_j),\ highly\ a(\delta_j),\ extremely\ a(\delta_j)\} \end{aligned} \tag{2}$$

where $a(\delta_j)$ stands for adjective describing the (δ_j) – characteristic of a generic product in C_j.

Linguistic values similar to the ones used above are common in the literature on fuzzy decision making. See, among others, [8,25].

Assumption 2. There are two DMs, D_1 and D_2, each of them endowed with one particular product, $\diamondsuit_1$ and $\diamondsuit_2$.

Assumption 3. DMs share the same language L, which allows the DMs to create sentences to describe any product P in $\wp$.

Assumption 4. Each DM has a subjective perception of each product P in $\wp$.

Assumptions 3 and 4 imply that, independently from the possibility of directly observing the products, when DMs communicate, they are forced to use the shared language to express their own subjective perceptions.

3. The communication-exchange-verification process

The purpose of the paper is to analyze the case of two DMs who meet to decide whether to exchange the product they are endowed with. In this section we formalize the process of communication-exchange-verification faced by the DMs.

3.1. Communication between the DMs

We start by formalizing the descriptions that the DMs give of their products. The key idea we build on is that these descriptions, henceforth called *reports*, must depend on the way the DMs perceive the products and can be, therefore, very subjective.

The language L shared by the DMs (Assumption 3) can be used to create both simple and complex descriptions that we formalize through the following definitions.

Definition 3.1. Let $P \in \wp$. A *simple sentence* (or *simple report*) *about* P, in the language L is any sentence that can be rephrased as follows: "P belongs to the category C_j and its (δ_j)-th characteristic takes the value x_{δ_j} in X_{δ_j}". ■

Clearly, in the definition above, X_{δ_j} can be both a set of numerical values and a set of qualitative evaluations relative to a particular adjective $a(\delta_j)$. Examples of simple sentences are: "*the color of the car is regular blue*", "*the smell of the wine is very intense*", "*the size of the TV screen is 15 inches*".

Indeed, each of these sentences contains information about the category to which the product described belongs and about the value (quantitative or qualitative) of one of its characteristics.

Henceforth, we will denote by $t^P_{\delta_j}$ the simple sentence expressing the category C_j and the (δ_j)-th characteristic of product P and denote by T the set of all simple sentences about all products. Thus:

$$T = \left\{ t^P_{\delta_j} : P \in \wp \ and\ \delta_j \in \Delta_j\ with\ j\ s.t.\ P \in C_j \right\} \tag{3}$$

Definition 3.2. Let $P \in \wp$. An *n-complex report about P* is a set of *n* simple reports about *P*. In symbols:

$$R_n^P = \left\{ t_{\delta_j}^P : \delta_j \in F,\ F \subseteq \Delta_j \text{ and } card(F) = n \right\}. \blacksquare \tag{4}$$

Since each DM has his own perception of each product $P \in \wp$ (see Assumption 4), he describes each product $P \in \wp$ using a set of simple sentences (that is, a complex report) in general different from the one that the other DM would use.

Henceforth, we will use the following notation: for every $i = 1,2$, the sub-index $-i$ will refer to the other DM, that is, to the DM different from the i-th one. This will allow us to refer to the two DMs as D_i and D_{-i} whenever we need to express an interaction between them. In particular, the reader should keep in mind that $D_{-1} = D_2, D_{-2} = D_1,\ \diamondsuit_{-1} = \diamondsuit_2$ and $\diamondsuit_{-2} = \diamondsuit_1$.

For every $P \in \wp,\ P_i^*$ will stand for the product P as it is perceived by the DM D_i, with $i = 1,2$. By Assumptions 3 and 4, we can formally identify P_i^* with the set of all the simple reports that D_i may use to describe P.

Let $\left[t_{\delta_j}^P\right]_i$ denote the simple sentence that D_i would use to describe the (δ_j)-th characteristic of product P, that is, the sentence stating either the value of x_{δ_j} or the evaluation of $a(\delta_j)$ from D_i's point of view. The following definition formalizes P_i^* as a subset of T.

Definition 3.3. Let $i = 1,2$ and $P \in \wp$. The *subjective perception* that D_i has of the product P, denoted by P_i^*, is the set of all simple sentences that D_i uses to describe the product P, that is:

$$P_i^* = \left\{ \left[t_{\delta_j}^P\right]_i : \delta_j \in \Delta_j \text{ and } j \text{ is s.t. } P \in C_j \right\}. \blacksquare \tag{5}$$

For $i = 1, 2$, let S_i be the set of all subjective perceptions that D_i has of the products in $\wp$. That is:

$$S_i = \{P_i^* : P \in \wp\}. \tag{6}$$

For $i = 1, 2$, we introduce a nonempty set-valued map Ψ_i from the power set of T, $Pow(T)$, into S_i, that is, a map from the set of all subsets of T into the set of all subsets of subjective perceptions of D_i, $Pow(S_i)$. Note that $Pow(S_i) \subseteq Pow(Pow(T))$. We will use this map to formalize the correspondence between the subjective perceptions of D_i and the subsets of simple reports used by D_i to describe them.

Definition 3.4. For $i = 1, 2$, let $\Psi_i : Pow(T) \overset{\rightarrow}{\rightarrow} S_i$ be the set-valued map defined as follows:

$$\forall R \in Pow(T), \quad \Psi_i(R) = \{P_i^* \in S_i : R \subseteq P_i^*\} = \bigcap_{t \in R} \{P_i^* \in S_i : t \in P_i^*\}. \tag{7}$$

In particular, for every simple sentence $t \in T$, we have:

$$\Psi_i(t) = \{P_i^* \in S_i : \{t\} \subseteq P_i^*\} = \{P_i^* \in S_i : t \in P_i^*\}. \blacksquare \tag{8}$$

Also, since every element of S_i is a subset of T, we have:

$$\forall P_i^* \in S_i, \quad \Psi_i(P_i^*) = \{Q_i^* \in S_i : P_i^* \subseteq Q_i^*\}, \tag{9}$$

hence,

$$\forall P_i^* \in S_i, \quad P_i^* \in \Psi_i(P_i^*). \tag{10}$$

At the same time, considering P_i^* as an element of S_i, we have:

$$\begin{aligned} \forall P_i^* \in S_i, \quad & \Psi_i^{-1}(P_i^*) = \{R \in Pow(T) : \Psi_i(R) \cap \{P_i^*\} \neq \varnothing\} \\ &= \{R \in Pow(T) : (\cap_{t \in R}\{Q_i^* \in S_i : t \in Q_i^*\}) \cap \{P_i^*\} \neq \varnothing\} = \{R \in Pow(T) : P_i^* \in (\cap_{t \in R}\{Q_i^* \in S_i : t \in Q_i^*\})\} \\ &= \{R \in Pow(T) : \forall t \in R, P_i^* \in \{Q_i^* \in S_i : t \in Q_i^*\}\} = \{R \in Pow(T) : \forall t \in R, t \in P_i^*\} = \{R \in Pow(T) : R \subseteq P_i^*\}. \end{aligned} \tag{11}$$

Thus,

$$\forall P_i^* \in S_i, \quad \Psi_i^{-1}(P_i^*) = Pow(P_i^*). \tag{12}$$

Interpretation of Ψ_i. For every simple report t [resp. complex report R], $\Psi_i(t)$ [resp. $\Psi_i(R)$] contains all the products that D_i perceives as satisfying the requirement described by t [resp. by R]. At the same time, the fact that a simple report t, or a complex report R, belongs to $\Psi_i^{-1}(P_i^*)$ means that this report can be used by D_i to describe P. Thus, whenever D_i hears the simple report t or a complex report R, he becomes aware of the type of product that D_{-i} is describing and selects all the products of the same type that he would describe using t or R.$\blacksquare$

Example 1. Suppose that D_i must describe a wine bottle. D_i has a subjective perception of each bottle and collects these perceptions in the set S_i. More precisely, given a bottle P, D_i perceives this bottle via the set of all simple reports that he would use to actually describe it. We identify this set with the perception itself that D_i has of P, that is P_i^*. Thus, the perceived products are elements of S_i and, at the same time, subsets of T.

If D_i were to describe P, he must use any t belonging to the set of simple reports P_i^* or a complex report R belonging to $\Psi_i(P_i^*) \subseteq Pow(P_i^*)$. For example, "*the color is* ...", "*the aftertaste is* ...", "*the smell is* ...", "*the alcohol content is* ...", "*the flavor is* ...".

On the other hand, when D_i hears the values of a certain list of characteristics, i.e. "*the color is* ...", "*the aftertaste is* ...", "*the smell is* ...", "*the alcohol content is* ...", "*the flavor is* ...", he thinks of a set of possible bottles that, again according to his perception, fits the provided description, such a set being the following:

$$\Psi_i(\textit{the color is}\ldots) \cap \Psi_i(\textit{the aftertaste is}\ldots) \cap \Psi_i(\textit{the smell is}\ldots) \cap \Psi_i(\textit{the alcohol content is}\ldots) \cap \Psi_i(\textit{the flavor is}\ldots).$$

The set R_5^P ={"*the color is* ...", "*the aftertaste is* ...", "*the smell is* ...", "*the alcohol content is* ...", "*the flavor is* ..."} is a 5-complex report about the bottle P.■

Remark 1. Unless both DMs have the same subjective perceptions of all the products, the sets $\Psi_1(t)$ and $\Psi_2(t)$ of subjective perceptions that D_1 and D_2, respectively, associate to the same simple sentence t are in general different. These sets could be even disjoint. Suppose, for example, that both DMs hear the simple sentence t = "*the wine has a very strong* ...". Then, one DM may have in mind a set of wine bottles totally different from the one that the other DM is thinking of. Hence, $\Psi_1(t) \neq \Psi_2(t)$ or even $\Psi_1(t) \cap \Psi_2(t) = \emptyset$.■

3.2. Exchange between the DMs

DMs are endowed with a particular product (Assumption 2), which allows them to potentially engage in bilateral trade. The product of the i-th DM D_i is denoted by $\diamondsuit_i$, while $\diamondsuit_i^*$ denotes the subjective perception that D_i has of $\diamondsuit_i$ (Definition 3.3).

Each DM must decide whether or not to exchange his product with that of the other DM. Clearly, DMs will only exchange their products if they expect to better off after the exchange takes place. That is, D_i will agree to exchange $\diamondsuit_i$ with $\diamondsuit_{-i}$ if the expected utility he derives from the exchange is higher than the utility he derives from $\diamondsuit_i$. Recall that $D_{-1} = D_2$, $D_{-2} = D_1$, $\diamondsuit_{-1} = \diamondsuit_2$ and $\diamondsuit_{-2} = \diamondsuit_1$. In order to evaluate and compare these values, D_i must be already endowed with a preference relation on the set S_i.

Assumption 5. For $i = 1, 2$, D_i is endowed with a strict preference relation $>_i$ on S_i, represented by a utility function $u_i : S_i \to \Re$.

A binary relation $>_i$ on S_i is a *strict preference relation* on S_i [30,32] if it satisfies irreflexivity ($\forall x \in S_i$, $x >_i x$ does not hold), completeness ($\forall x, y \in S_i$, $x >_i y$ or $y >_i x$ and transitivity ($\forall x, y, z \in S_i$, $x >_i y$ and $y >_i z$ imply $x >_i z$). Clearly, strict preference relations are complete strict preorders.

A *utility function* representing a strict preference relation $>_i$ on S_i is a function $u_i : S_i \to \Re$ such that $\forall x, y \in S_i$, $x >_i y \iff u_i(x) > u_i(y)$.

Note that by Assumption 5, D_i also has a strict preference relation on each image set $\Psi_i(t)$, namely, the restriction of $>_i$ to $\Psi_i(t)$.

At the same time, given a simple report t [resp. a complex report R] about a certain product, we need each DM to assign a probability function on $\Psi_i(t)$ [resp. $\Psi_i(R)$]. The value taken by this function at P_i^* must allow D_i to quantify how much he believes P_i^* to actually be the product described by the simple report t [resp. a complex report R]. Thus, we assume:

Assumption 6. For $i = 1, 2$ and for every $t \in T$, D_i defines a subjective probability function $\mu_i(\cdot|t)$ on $\Psi_i(t)$. The value $\mu_i(P_i^*|t)$ is the probability assigned by D_i to P_i^* actually being the product described by the simple sentence t. Similarly, for every $R \subseteq T$, D_i defines a subjective probability function $\mu_i(\cdot|R)$ on $\Psi_i(R)$ such that $\mu_i(P_i^*|R)$ expresses D_i's subjective belief that P_i^* actually is the product described by the complex report R.

Definition 3.5. For every $i = 1, 2$ and every $t \in T$, let

$$E(u_i, \mu_i, t) = \sum_{P_i^* \in \psi_i(t)} u_i(P_i^*)\mu_i(P_i^*|t). \tag{13}$$

For every n-complex report $R_n \subset T$, let

$$E(u_i, \mu_i, R_n) = \sum_{P_i^* \in \psi_i(R_n)} u_i(P_i^*)\mu_i(P_i^*|R_n). \tag{14}$$

These sums measure D_i's *exchange expected utility* induced by the simple report t and the n-complex report R_n, respectively.■

By the last definition, it is clear that the expected utility value that D_i must calculate in order to decide whether or not to accept to exchange $\diamondsuit_i$ with $\diamondsuit_{-i}$ depends on the simple or complex report that D_{-i} uses to describe the product $\diamondsuit_{-i}$.

Suppose, for example, that $\diamondsuit_{-i}$ is a wine bottle. The set of potential bottles that D_i will consider to evaluate the expected utility if provided with the 2-complex report R_2 = {"*the wine is white*", "*the wine is very sparkly*"} is different from the one he will consider to evaluate the expected utility if provided with the 3-complex report R_3 = {"*the wine is white*", "*the alcohol content is 14% vol*", "*the wine is almost not tasty*"}.

The fact that the exchange expected utility of each DM depends on the specific report provided can be used by one of the DMs to strategically manipulate the choice of the other (see [13]).

In order to provide a manipulation-free environment for both DMs to agree to exchange, we assume the communication of simple/complex reports between them to satisfy the following requirements.

R.1. For $i = 1, 2$ and $j \in J$, D_i define a *strict* preference relation $\rhd_{i,j}$ on the index set Δ_j, which, in turn, induces a strict preference relation on each subjective perception P_i^*. (That is, D_i is also endowed with a strict preference relation on the set of simple reports that he would use to describe a particular product.)

R.2. For $i = 1, 2$, D_i reveals the category $C_{j(i)}$ to which his product $\diamondsuit_i$ belongs and the value of some of its quantitative characteristics. D_i chooses which characteristics to describe to D_{-i} following the preference relation $\rhd_{i,j(i)}$ on $\Delta_{j(i)}$. Both DMs reveal the same amount of information.

R.3. For $i = 1, 2$, D_i chooses which qualitative characteristics of $\diamondsuit_{-i}$ must be described by D_{-i}. D_i chooses which characteristics he wants to be described by D_{-i}, who must follow the preference relation $\rhd_{i,j(-i)}$ on $\Delta_{j(-i)}$ when describing the qualitative characteristics. Both DMs reveal the same amount of information.

R.4. Both DMs report truthfully their perceptions, both in R.2 and R.3.

About Requirements R.1–R.4. These requirements provide a common framework for DMs to exchange information before trading. Requirement R.2 states that each DM reveals a subset of quantitative characteristics, which are not subject to a subjective interpretation. For example, a DM may describe the alcohol content of a bottle of wine, its year of production, and the percentage of different varieties of grapes used to produce the wine. Both DMs communicate to each other the same amount of characteristics. Each DM decides which characteristics to communicate based on his subjective preferences. At the same time, requirement R.3 describes how subjective qualitative reports will be transmitted between the DMs. In this case, each DM is allowed to ask a given number of qualitative questions to the other one. Both DMs are allowed to ask the same amount of questions before trading. Requirement R.4 guarantees that the reports provided by each DM are truthful descriptions of the *perceptions* that each DM has of his own product.■

Remark 2. There is an essential difference between the preference relations $>_i$ and $\rhd_{i,j(i)}$. While the first is defined by D_i to order the products (as he subjectively perceives them), the latter is defined by D_i to express his preferences on the set of all characteristics that a generic product in the j-th category may have, independently from the value that these characteristics take. Suppose, for example, that C_j is the set of all wine bottles. Then, D_i will use $>_i$ to express his preferences among the bottles, which are identified with complex reports, but he will need to use $\rhd_{i,j(i)}$ to order the set {*color*, *aftertaste, smell*, *alcohol content, flavor*} of all the characteristics that any wine bottle has. In other words, complex reports of the form {"*the color is* ...", "*the aftertaste is* ...", "*the smell is* ...", "*the alcohol content is* ...", "*the flavor is* ..."} describe and are identified with one specific wine bottle. On the other hand, the set of all the characteristics of any wine, {*color*, *aftertaste, smell*, *the alcohol content, flavor*}, is not a complex report and does not represent any specific bottle in C_j. The necessity of introducing both preference relations is clear when considering the overall communication-exchange-verification process. See also Fig. 1 below.■

After the DMs have communicated reports to each other in a way that R.2–R.4 are satisfied, each DM can calculate his exchange expected utility on the basis of his subjective beliefs (see Definition 3.5). Each DM will then compare the value obtained from the exchange expected utility with the utility value of the product that he already owns. He shall trade only if the former is higher than the latter. Hence, exchange will occur only if both DMs perceive this to be the case. We introduce the following definition to formalize the exchange necessary condition.

Definition 3.6. For $i = 1, 2$, let $r_i \subset \diamondsuit_i^*$ be the report used by D_i to describe the product $\diamondsuit_i$ to D_{-i}. Let r_1 and r_2 be both simple or both n-complex, where n is a positive integer. We say that *DMs agree to exchange* if:

$$E(u_1, \mu_1, r_2) - u_1(\diamondsuit_1^*) = \sum_{P_1^* \in \psi_1(r_2)} u_1(P_1^*)\mu_1(P_1^*|r_2) - u_1(\diamondsuit_1^*) > 0 \tag{15}$$

and

$$E(u_2, \mu_2, r_1) - u_2(\diamondsuit_2^*) = \sum_{P_2^* \in \psi_2(r_1)} u_2(P_2^*)\mu_2(P_2^*|r_1) - u_2(\diamondsuit_2^*) > 0.\blacksquare \tag{16}$$

In order to simplify the presentation of the main results (Theorem 5.1 and Paradox in Section 5), we will focus our attention on DMs who agree to exchange only if there is no risk inherent to the trade. That is, D_i agrees to exchange only if none of the products that he perceives as satisfying the properties described by the n-complex report r_{-i} provided by D_{-i}, and that D_i believes to be possible, may turn out to be worse than the product he already owns.

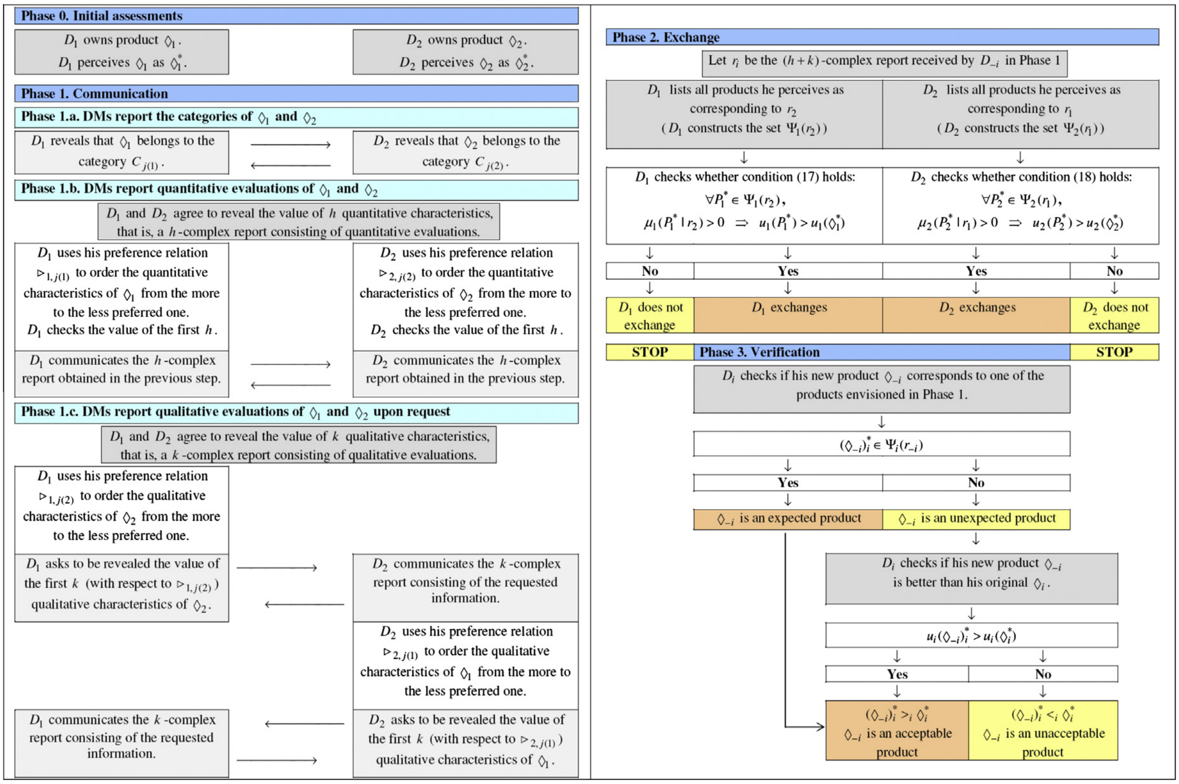

Fig. 1. The communication-exchange-verification process.

In other words, we concentrate on the case where each DM agrees to exchange only if, given both his perceptions of the products and his beliefs about understanding the other DM (see Assumption 6), he is *completely sure* to be better off after the exchange.

Henceforth, we assume that the DMs agree to exchange if the following conditions hold:

$$\forall P_1^* \in \Psi_1(r_2), \quad \mu_1(P_1^*|r_2) > 0 \Rightarrow u_1(P_1^*) > u_1(\diamondsuit_1^*) \tag{17}$$

and

$$\forall P_2^* \in \Psi_2(r_1), \quad \mu_2(P_2^*|r_1) > 0 \Rightarrow u_2(P_2^*) > u_2(\diamondsuit_2^*) \tag{18}$$

Clearly, conditions (17) and (18) are stronger than, and hence imply, conditions (15) and (16), respectively. Further comments about these conditions are provided in the next section.

3.3. Verification: expected, acceptable and unacceptable products

Suppose that the DMs agree to exchange under the manipulation-free requirements R.1–R.4 described above. By conditions (17) and (18), if the product that a DM obtains after the exchange takes place is actually one of those he had thought of based on the report received, then the DM will be better off and gets what he was expecting from trading. On the other hand, it is also possible that the product that a DM gets after the exchange is completely different from the one (or the ones) that he had envisioned on the basis of the reports provided by the other DM. Nevertheless, the newly acquired product can still deliver a utility higher than the one originally owned. Thus, even if the DM does not get exactly what he was thinking of (a wanted/expected product), he still gets what he was expecting in terms of utility (an acceptable product). However, it is also reasonable to think of a situation where either one or both DMs get an unacceptable product, that is, a product that provides the DMs with a utility lower than that of the one they originally owned. The distinction among *expected*, *acceptable* and *unacceptable* products can be easily formalized in terms of subjective perceptions and post-exchange utility values as follows:

Definition 3.7. For $i = 1, 2$, let $r_i \subset \diamondsuit_i^*$ be the report used by D_i to describe the product $\diamondsuit_i$ to D_{-i}. Let r_1 and r_2 be both simple or both n-complex, where n is a positive integer. Assume DMs agree to exchange. We say that:

- D_i gets an *expected product* (or a product he expects) if $(\diamondsuit_{-i})_i^* \in \Psi_i(r_{-i})$.
- D_i gets an *unexpected product* if $(\diamondsuit_{-i})_i^* \notin \Psi_i(r_{-i})$.

- D_i gets an *acceptable product* if $u_i((\diamondsuit_{-i})_i^*) > u_i(\diamondsuit_i^*)$.
- D_i gets an *unacceptable product* if $u_i((\diamondsuit_{-i})_i^*) < u_i(\diamondsuit_i^*)$.■

Definition 3.7 proposes a classification of the products obtained after the exchange. In particular, $(\diamondsuit_{-i})_i^* \in \Psi_i(r_{-i})$ means that the newly acquired product $\diamondsuit_{-i}$ is one of the products that D_i had envisioned after receiving the report r_{-i}. That is, the actual perception that D_i has of the newly acquired product $\diamondsuit_{-i}$ coincides with the one he was induced to think of by the report of the other DM. In other words, $(\diamondsuit_{-i})_i^* \in \Psi_i(r_{-i})$ is equivalent to say that $\diamondsuit_{-i}$ is one of the products that D_i was expecting to receive after the exchange.

Note that since exchange is assumed to happen only if both DMs are completely sure to be better off after it (refer to conditions (17) and (18)), condition $(\diamondsuit_{-i})_i^* \in \Psi_i(r_{-i})$ being satisfied implies that D_i actually receives a product that he prefers to the one originally owned. The following proposition expresses this fact.

Proposition 3.8. *For* $i = 1, 2$, *let* $r_i \subset \diamondsuit_i^*$ *be the report used by* D_i *to describe the product* $\diamondsuit_i$ *to* D_{-i}. *Let* r_1 *and* r_2 *be both simple or both n-complex, where n is a positive integer. If DMs agree to exchange, then, for* $i = 1, 2$,

$$(\diamondsuit_{-i})_i^* \in \Psi_i(r_{-i}) \Rightarrow u_i((\diamondsuit_{-i})_i^*) > u_i(\diamondsuit_i^*). \blacksquare \tag{19}$$

It follows that, after the exchange, an expected product is also an acceptable product.

Remark 3. Conditions (17) and (18) imply that DMs do not want to face any risk when exchanging products, which constitutes a strong requirement. Without these conditions, DMs may understand each other perfectly but end up with an unacceptable product due to the risk inherent in the trade. Thus, in our setting, the suboptimal outcome where DMs receive an unacceptable product may only follow from the differences in perception existing between them. However, the analysis performed and the results obtained remain completely valid if these conditions are not imposed.■

Fig. 1 provides a graphical representation of the phases composing the communication-exchange-verification process faced by the DMs. In particular, the figure highlights how requirements R.1–R.4 apply and at which stage each DM D_i needs to use the preference relations $\triangleright_{i,j(i)}$ and $\triangleright_{i,j(-i)}$. Note that the last phase of the process takes place only if the DMs agree to exchange and consists of each DM verifying whether the newly obtained product is actually the one envisioned in the communication phase. The preference relation $>_i$ is used in this last stage, where the exchanged products may be acceptable even if they are not among the expected ones.

4. Does quantifying linguistic values avoid unacceptable products?

It is conceivable that either one or both DMs get an unacceptable product as a consequence of the fact that DMs are allowed to use linguistic values to describe the qualitative characteristics of their products. Thus, one may intuitively conjecture that this issue is solved by modifying Assumption 1 so as to allow all the sets X_δ to be identified with a real interval – a quite common assumption in the microeconomic literature [12,27].

Conjecture. For $i = 1, 2$, let $r_i \subset \diamondsuit_i^*$ be the *n*-complex report (with *n* positive integer) used by D_i to describe the product $\diamondsuit_i$ to D_{-i}. If the DMs also describe the qualitative characteristics of their products using a real interval scale, then, for $i = 1, 2$, $(\diamondsuit_{-i})_i^* \in \Psi_i(r_{-i})$, that is, the DMs get an expected product whenever they agree to exchange.■

Our main result (Theorem 5.1) provides a formal proof of the fact that the above conjecture is false and that the DMs can agree to exchange and turn out to be worse off even when they are asked to express their qualitative evaluations using real values belonging to a normalized interval.

Paradoxically enough, we will actually argue that quantifying the linguistic values of qualitative characteristics may create more misunderstanding than using their linguistic values.

Following the recent and increasing literature on fuzziness, we assume the DMs to be able to describe linguistic values by means of fuzzy numbers.

4.1. Fuzzy numbers and linguistic variables

For every $j \in J$ and $\delta_j \in \Delta_j$, we have assumed X_{δ_j} to be made of either quantitative or qualitative evaluations for the (δ_j)-th characteristic of a product in C_j (see Assumption 1). In particular, when being a set of qualitative evaluations, X_{δ_j} has been identified with a linguistic variable that indicates not only the adjective corresponding to the (δ_j)-characteristic of the product described, but also how much this certain adjective characterizes the (δ_j)-characteristic itself. Thus,

$$\begin{aligned} X_{\delta_j} = \{ & \textit{not } a(\delta_j), \textit{almost not } a(\delta_j), \textit{lowly } a(\delta_j), \textit{not very } a(\delta_j), \\ & \textit{regularly } a(\delta_j), \textit{ very } a(\delta_j), \textit{ highly } a(\delta_j), \textit{extremely } a(\delta_j)\} \end{aligned} \tag{20}$$

where $a(\delta_j)$ stands for adjective that must be used to describe the (δ_j)-characteristic of a generic product in C_j.

Given a product $P \in C_j \subset \wp$, we can define a triangular fuzzy number (TFN) representation for every set X_{δ_j} of linguistic evaluations as in Eq. (15). More importantly, the TFN representation associated with a linguistic variable X_{δ_j} does not need to be symmetric.

In order to associate a TFN to each of the seven values that a linguistic variable X_{δ_j} can take in our setting, we assume that each DM scales the linguistic values of X_{δ_j} within a standard range such as [0, 10] and [0, 100]. Up to a normalization, all these intervals can be identified with [0, 1]. For every $i = 1, 2$, $j \in J$ and $\delta_j \in \Delta_j$ such that X_{δ_j} consists of linguistic values as in Eq. (20), let

$$k_{\delta_j}^{i,0} = 0, \quad k_{\delta_j}^{i,1}, \quad \ldots, \quad k_{\delta_j}^{i,5}, \quad k_{\delta_j}^{i,6} = 1 \tag{21}$$

be an increasing sequence of values in [0, 1].

The family $\left\{\left[k_{\delta_j}^{i,s}, k_{\delta_j}^{i,s+1}\right] : s = 0, \ldots, 5\right\}$ is clearly a cover of [0, 1], that allows the i-th DM D_i to define a TFN and the corresponding membership function for each of the linguistic values taken by X_{δ_j} as follows:

Definition 4.1. Let $a(\delta_j)$ be the adjective describing the (δ_j)-characteristic of a generic product in C_j. Let X_{δ_j} be the set of linguistic values relative to $a(\delta_j)$. The D_i's *TFN representation of* X_{δ_j} *determined by the sequence* $\left\{k_{\delta_j}^{i,0} = 0,\ k_{\delta_j}^{i,1}, \ldots, k_{\delta_j}^{i,5}, k_{\delta_j}^{i,6} = 1\right\}$ is the set of TFNs with which the linguistic values of X_{δ_j} are identified, that is, the set formed by the following triples:

$$\begin{array}{l} \left(0, k_{\delta_j}^{i,0}, k_{\delta_j}^{i,1}\right) = \left(0, 0, k_{\delta_j}^{i,1}\right) \\ \left(k_{\delta_j}^{i,n-2}, k_{\delta_j}^{i,n-1}, k_{\delta_j}^{i,n}\right) \quad \text{for every} \quad n = 2,3,4,5,6 \\ \left(k_{\delta_j}^{i,5}, k_{\delta_j}^{i,6}, 1\right) = \left(k_{\delta_j}^{i,5}, 1, 1\right). \blacksquare \end{array} \tag{22}$$

The membership functions corresponding to the TFNs of Definition 4.1 are defined as follows.

$$\varphi_{\delta_j}^{i,1}(x) = \begin{cases} 1 - \frac{1}{k_{\delta_j}^{i,1}} x, & 0 \leqslant x \leqslant k_{\delta_j}^{i,1} \\ 0, & k_{\delta_j}^{i,1} \leqslant x \leqslant 1 \end{cases} \tag{23}$$

$$\varphi_{\delta_j}^{i,n}(x) = \begin{cases} 0, & 0 \leqslant x < k_{\delta_j}^{i,n-2} \\ \frac{1}{k_{\delta_j}^{i,n-1} - k_{\delta_j}^{i,n-2}} x - \frac{1}{k_{\delta_j}^{i,n-1} - k_{\delta_j}^{i,n-2}} k_{\delta_j}^{i,n-2}, & k_{\delta_j}^{i,n-2} \leqslant x < k_{\delta_j}^{i,n-1} \\ \frac{1}{k_{\delta_j}^{i,n} - k_{\delta_j}^{i,n-1}} k_{\delta_j}^{i,n} - \frac{1}{k_{\delta_j}^{i,n} - k_{\delta_j}^{i,n-1}} x, & k_{\delta_j}^{i,n-1} \leqslant x \leqslant k_{\delta_j}^{i,n} \\ 0, & k_{\delta_j}^{i,n} \leqslant x \leqslant 1 \end{cases} \tag{24}$$

where $n = 2, 3, 4, 5, 6$ and

$$\varphi_{\delta_j}^{i,7}(x) = \begin{cases} 0, & 0 \leqslant x < k_{\delta_j}^{i,5} \\ \frac{1}{1 - k_{\delta_j}^{i,5}} x - \frac{1}{1 - k_{\delta_j}^{i,5}} k_{\delta_j}^{i,5}, & k_{\delta_j}^{i,5} \leqslant x \leqslant 1 \end{cases} \tag{25}$$

Table 1 describes in detail the linguistic values of a generic adjective $a(\delta_j)$ characterizing the variable X_{δ_j} and the TFNs associated with each one of them. Fig. 2 depicts the corresponding membership functions.

Table 1
Linguistic values of X_{δ_j} and corresponding TFNs.

D_i's subjective descriptions of $a(\delta_j)$	Triangular fuzzy number
Almost not $a(\delta_j)$	$\left(0, 0, k_{\delta_j}^{i,1}\right)$
Lowly $a(\delta_j)$	$\left(0, k_{\delta_j}^{i,1}, k_{\delta_j}^{i,2}\right)$
Not very $a(\delta_j)$	$\left(k_{\delta_j}^{i,1}, k_{\delta_j}^{i,2}, k_{\delta_j}^{i,3}\right)$
Regularly $a(\delta_j)$	$\left(k_{\delta_j}^{i,2}, k_{\delta_j}^{i,3}, k_{\delta_j}^{i,4}\right)$
Very $a(\delta_j)$	$\left(k_{\delta_j}^{i,3}, k_{\delta_j}^{i,4}, k_{\delta_j}^{i,5}\right)$
Highly $a(\delta_j)$	$\left(k_{\delta_j}^{i,4}, k_{\delta_j}^{i,5}, 1\right)$
Extremely $a(\delta_j)$	$\left(k_{\delta_j}^{i,5}, 1, 1\right)$

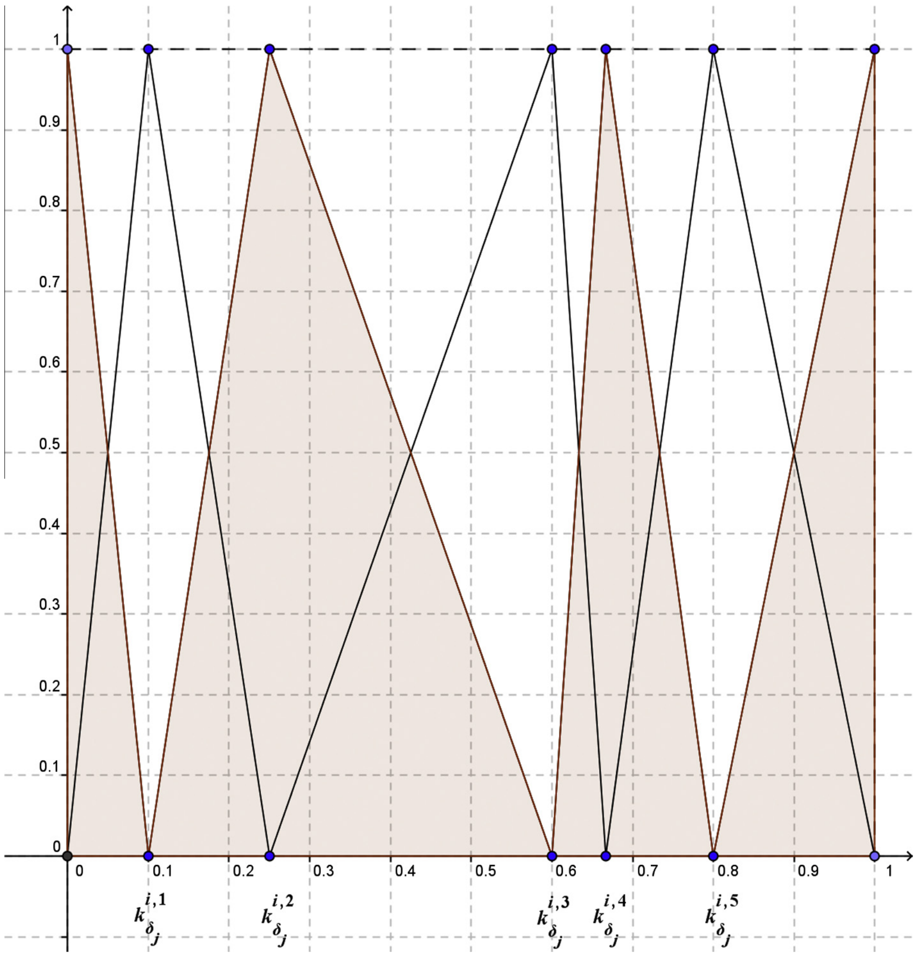

Fig. 2. D_i's generic TFN representation of the linguistic values of X_{δ_j}.

If, in particular, the sequence of values in [0, 1] determining D_i's TFN representation of the linguistic values that can be assigned to the (δ_j)-characteristic is

$$k_{\delta_j}^{i,s} = \frac{s}{6} \quad \text{for every} \quad s = 0, 1, 2, 3, 4, 5, 6 \tag{26}$$

then, the DM D_i identifies the linguistic values with equally distributed (symmetric) TFNs. Table 2 specifies the TFNs used by D_i in this case. The membership functions associated with these TFNs can be easily derived from those in Definition 4.1. These functions are represented in Fig. 3.

$$\varphi_{\delta_j}^{i,1}(x) = \begin{cases} 1 - 6x, & 0 \leqslant x \leqslant \frac{1}{6} \\ 0, & \frac{1}{6} \leqslant x \leqslant 1 \end{cases} \tag{27}$$

Table 2
Linguistic values of X_{δ_j} and equally distributed TFNs.

D_i's subjective descriptions of $a(\delta_j)$	Triangular fuzzy number
Almost not $a(\delta_j)$	(0, 0, 0.167)
Lowly $a(\delta_j)$	(0, 0.167, 0.333)
Not very $a(\delta_j)$	(0.167, 0.333, 0.5)
Regularly $a(\delta_j)$	(0.333, 0.5, 0.667)
Very $a(\delta_j)$	(0.5, 0.667, 0.833)
Highly $a(\delta_j)$	(0.667, 0.833, 1)
Extremely $a(\delta_j)$	(0.833, 1, 1)

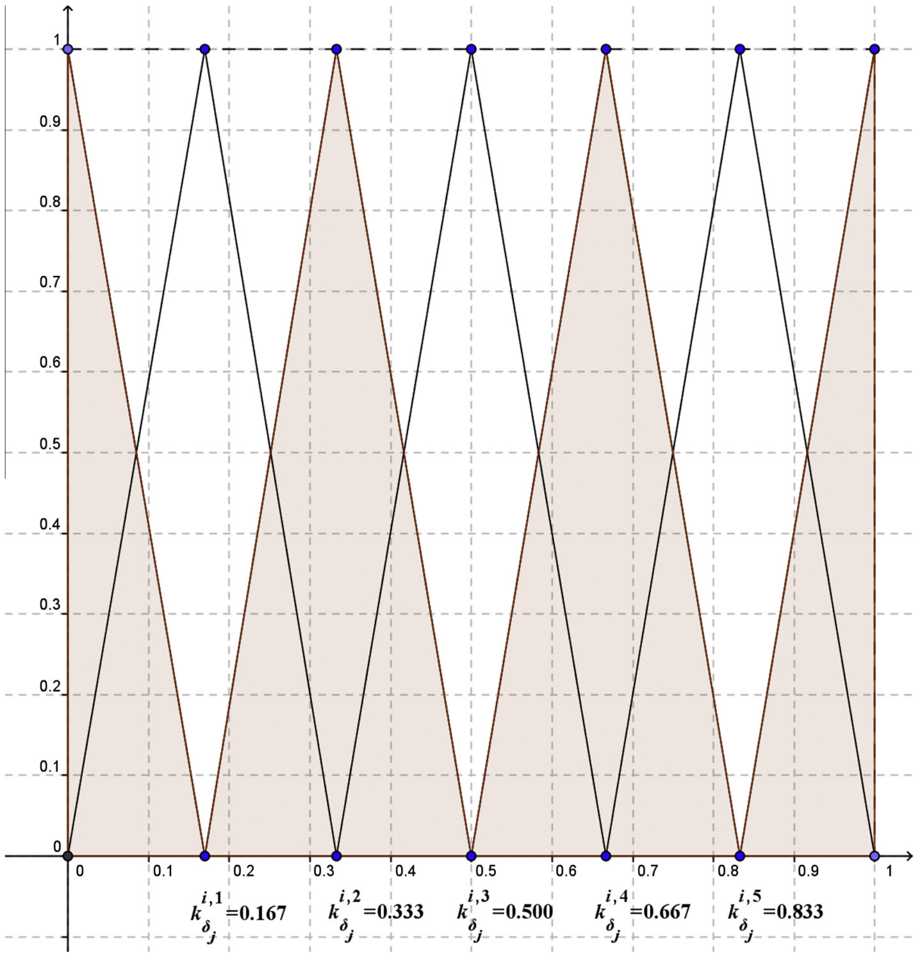

Fig. 3. D_i's symmetric TFN representation of the linguistic values of X_{δ_j}.

$$\varphi_{\delta_j}^{i,n}(x) = \begin{cases} 0, & 0 \leqslant x < \frac{n-2}{6} \\ 6x-(n-2), & \frac{n-2}{6} \leqslant x < \frac{n-1}{6} \\ n-6x, & \frac{n-1}{6} \leqslant x \leqslant \frac{n}{6} \\ 0, & \frac{n}{6} \leqslant x \leqslant 1 \end{cases} \tag{28}$$

with $n = 2, 3, 4, 5, 6$ and

$$\varphi_{\delta_j}^{i,7}(x) = \begin{cases} 0, & 0 \leqslant x < \frac{5}{6} \\ 6x-5, & \frac{5}{6} \leqslant x \leqslant 1 \end{cases} \tag{29}$$

Remark 4. Given $P \in C_j \subset \wp$, the fact that D_i assigns the value $x_{\delta_j}^{iP} = not\ a(\delta_j)$ to the (δ_j)-characteristic of P is not subject to fuzziness. That is, if either one of the DMs, D_i, communicates the value $x_{\delta_j}^{iP} = not\ a(\delta_j)$ about the product P, then the other DM can be sure that the adjective $a(\delta_j)$ does not apply to P, even though it may apply to other products in C_j. For instance, suppose that D_1's product $\diamondsuit_1$ is a cup of coffee. If D_1's simple report says that $\left[t_{\delta_j}^{\diamondsuit_1}\right]_i = \diamondsuit_1$ *has no sugar*, D_2 cannot interpret this information in a fuzzy manner. When evaluating whether or not to agree to exchange, D_2 will consider only the cups of coffee that he likes to drink without sugar. That is, D_2 will evaluate the expected utility $E(u_2, \mu_2, \diamondsuit_1\ has\ no\ sugar) = \sum_{P_2^* \in \psi_2(\diamondsuit_1\ has\ no\ sugar)} u_2(P_2^*)\mu_2(P_2^*|\diamondsuit_1\ has\ no\ sugar)$, where $\Psi_2(\diamondsuit_1\ has\ no\ sugar)$ contains the cups of coffee that he likes to drink despite the fact that they have no sugar, hence because of the size, the flavor, the smell, the cream, etc. ■

The use of equally distributed (symmetric) TFNs is quite diffuse in the literature (see [36–40]). Our approach considers asymmetric TFNs and it is in this sense more general. At the same time, it allows for a more natural formalization of subjective fuzzy descriptions whose ranges of variation are not imposed on the DMs but autonomously constructed by them.

Remark 5. We would like to emphasize the fact that the values of the parameters $k_{\delta_j}^{i,0},\ k_{\delta_j}^{i,1},\ \ldots\ ,\ k_{\delta_j}^{i,5},\ k_{\delta_j}^{i,6}$ used in Definition 4.1 are given by any increasing sequence of real values between 0 and 1 such that the first and last values, $k_{\delta_j}^{i,0}$ and $k_{\delta_j}^{i,6}$, always coincide with 0 and 1, respectively. These parameters provide a (not necessarily symmetric) ranking of the linguistic values that can be assigned to the specific characteristic δ_j of the product that D_i is describing. Thus, technically speaking, the numbers $k_{\delta_j}^{i,0},\ k_{\delta_j}^{i,1},\ \ldots\ ,\ k_{\delta_j}^{i,5},\ k_{\delta_j}^{i,6}$ do not satisfy any particular requirement. They are chosen in a completely subjective manner by the DM, the only restriction being the fact that both DMs must use the same range, [0, 1].■

4.2. DMs don't always get what they expect

To simplify notations, suppose that each DM D_i knows the category to which the product $\diamondsuit_{-i}$ of the other DM D_{-i} belongs. This allows us to drop the sub-index j.

For i = 1, 2, suppose that D_i uses the TFNs determined by the sequence $\left\{k_\delta^{i,0}=0, k_\delta^{i,1},\ldots,k_\delta^{i,5},k_\delta^{i,6}=1\right\}$ to describe the (δ)-characteristic of his product $\diamondsuit_i$ to the other DM D_{-i}, where this characteristic is represented by a linguistic variable. That is, D_i identifies his evaluation of each (δ)-characteristic, whose values are specified by the corresponding linguistic variable X_δ, with one of the TFNs determined by the sequence $\left\{k_\delta^{i,0}=0, k_\delta^{i,1},\ldots,k_\delta^{i,5},k_\delta^{i,6}=1\right\}$. Then, he uses this TFN when communicating either a simple or a complex report to D_{-i}.

At the same time, D_i using a triple of the form $\left(k_\delta^{i,s-1},k_\delta^{i,s},k_\delta^{i,s+1}\right)$ implicitly means that D_i believes the *intermediate* value $k_\delta^{i,s}$ to be the most reasonable value in the range $\left[k_\delta^{i,s-1},k_\delta^{i,s+1}\right]$ to express his perception of how much the quality $a(\delta)$ applies to his product.

Thus, we can assume that whenever D_i communicates a TFN $\left(k_\delta^{i,s-1},k_\delta^{i,s},k_\delta^{i,s+1}\right)$ to D_{-i}, D_{-i} considers the position of the value $k_\delta^{i,s}$ with respect to the sequence $\left\{k_\delta^{-i,0}=0,k_\delta^{-i,1},\ldots,k_\delta^{-i,5},k_\delta^{-i,6}=1\right\}$ determining the TFN representation that he uses to rank the linguistic values of the same quality $a(\delta)$.

As a consequence, from D_{-i}'s point of view, the value of quality $a(\delta)$ described by D_i is the one that he identifies with the TFN $\left(k_\delta^{-i,s'-1},k_\delta^{-i,s'},k_\delta^{-i,s'+1}\right)$, where the intermediate value $k_\delta^{-i,s'}$ is the closest one in $\{k_\delta^{-i,s}:s=0,\ldots,6\}$ to the value $k_\delta^{i,s}$. (See Examples 2–4 below.)

To summarize:

- D_i communicating a TFN $\left(k_\delta^{i,s-1},k_\delta^{i,s},k_\delta^{i,s+1}\right)$ is equivalent to D_i communicating the intermediate value $k_\delta^{i,s}$; and
- D_{-i} receiving a value $k_\delta^{i,s}$ is equivalent to D_{-i} creating his own TFN $\left(k_\delta^{-i,s'-1},k_\delta^{-i,s'},k_\delta^{-i,s'+1}\right)$, where the intermediate value $k_\delta^{-i,s'}$ is the closest one in $\left\{k_\delta^{-i,s}:s=0,\ldots,6\right\}$ to the value $k_\delta^{i,s}$.

As discussed earlier, the value of the (δ)-characteristic of product $\diamondsuit_i$ that D_{-i} is induced to believe after having receivedD_i's report does not need to coincide with the value that D_{-i} would assign to the same characteristic if he could observe the product $\diamondsuit_i$ directly. Thus, should the DMs decide to exchange their products, D_{-i} will not necessarily get the product that he expects to. This may be the case independently of the fact that the exchanged product provides an improvement or a loss with respect to the product initially owned by D_{-i}.

More importantly, and contrary to the intuitive conjecture discussed at beginning of this section, D_{-i} may not receive the product envisioned either when D_i decides to use linguistic values or when D_i decides to report via TFNs. We show below that, unless the TFNs used by the two DMs are somehow alike ("synchronized"), using TFN representations may paradoxically create even more misunderstanding than using linguistic values.

We need first to introduce the concepts of synchronized and unsynchronized TFN representations.

Definition 4.2. The TFN representations of the linguistic values of X_δ used by D_1 and D_2 will be called:

- *synchronized* if both representations use the same TFNs. (In particular, if both use symmetric TFNs.)
- *slightly unsynchronized* if $\forall s\in\{1,\ldots,5\}$,

$$k_\delta^{1,s}\neq k_\delta^{2,s}\Rightarrow\begin{cases}\left|k_\delta^{1,s}-k_\delta^{2,s}\right|<\min\left\{\left|k_\delta^{1,s-1}-k_\delta^{2,s}\right|,\left|k_\delta^{1,s+1}-k_\delta^{2,s}\right|\right\}\\ \qquad\qquad \textit{and}\\ \left|k_\delta^{1,s}-k_\delta^{2,s}\right|<\min\left\{\left|k_\delta^{1,s}-k_\delta^{2,s-1}\right|,\left|k_\delta^{1,s}-k_\delta^{2,s+1}\right|\right\}\end{cases}\tag{30}$$

- *unsynchronized* if $\exists s \in \{1, \dots, 5\}$ such that

$$\begin{aligned} & k_\delta^{1,s} \neq k_\delta^{2,s}; \\ & \left|k_\delta^{1,s} - k_\delta^{2,s}\right| > \min\left\{\left|k_\delta^{1,s} - k_\delta^{2,\tilde{s}}\right| : \tilde{s} \neq s\right\}; \\ & \left|k_\delta^{1,s} - k_\delta^{2,s}\right| > \min\left\{\left|k_\delta^{1,\tilde{s}} - k_\delta^{2,s}\right| : \tilde{s} \neq s\right\}. \end{aligned} \tag{31}$$

- *completely unsynchronized* if $\forall s \in \{1, \dots, 5\}$,

$$\begin{aligned} & k_\delta^{1,s} \neq k_\delta^{2,s}; \\ & \left|k_\delta^{1,s} - k_\delta^{2,s}\right| > \min\left\{\left|k_\delta^{1,s} - k_\delta^{2,\tilde{s}}\right| : \tilde{s} \neq s\right\}; \\ & \left|k_\delta^{1,s} - k_\delta^{2,s}\right| > \min\left\{\left|k_\delta^{1,\tilde{s}} - k_\delta^{2,s}\right| : \tilde{s} \neq s\right\}. \end{aligned} \tag{32}$$

If the TFN representations of the linguistic values of X_δ used by D_1 and D_2 are synchronized or slightly unsynchronized, then every value $k_\delta^{i,s}$ reported by D_i is translated by D_{-i} either as the same TFN or as a very close TFN.

Example 2. Suppose that D_1 and D_2 use the TFNs determined by the finite sequences reported in Table 3 to represent the linguistic values of variable X_δ. Fig. 4 provides a graphical comparison of the membership functions associated with these TFNs.

Table 3
DMs' sequences of values determining slightly unsynchronized TFNs for X_δ.

	$k_\delta^{1,0}$	$k_\delta^{1,1}$	$k_\delta^{1,2}$	$k_\delta^{1,3}$	$k_\delta^{1,4}$	$k_\delta^{1,5}$	$k_\delta^{1,6}$
D_1	0	0.1	0.25	0.6	0.67	0.8	1
D_2	0	0.1	0.27	0.6	0.7	0.76	1
	$k_\delta^{2,0}$	$k_\delta^{2,1}$	$k_\delta^{2,2}$	$k_\delta^{2,3}$	$k_\delta^{2,4}$	$k_\delta^{2,5}$	$k_\delta^{2,6}$

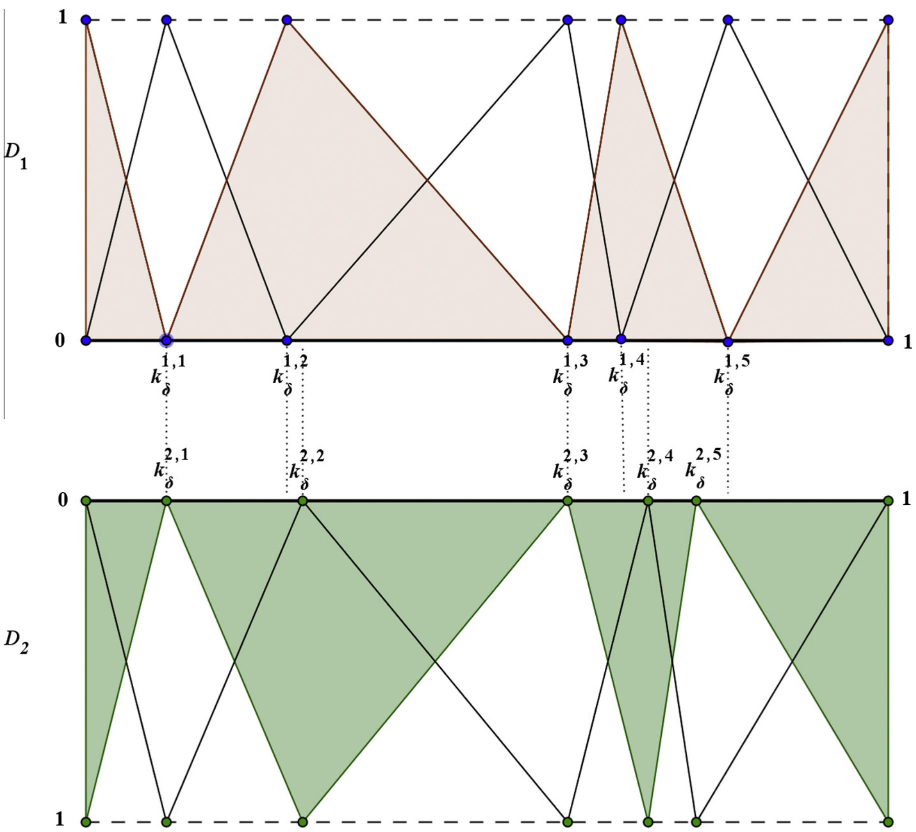

Fig. 4. Example of slightly unsynchronized TFN representations of X_δ.

Table 4
DMs' sequences of values determining unsynchronized TFNs for X_δ.

	$k_\delta^{1,0}$	$k_\delta^{1,1}$	$k_\delta^{1,2}$	$k_\delta^{1,3}$	$k_\delta^{1,4}$	$k_\delta^{1,5}$	$k_\delta^{1,6}$
D_1	0	0.1	**0.25**	0.6	0.67	0.8	1
D_2	0	0.1	**0.52**	0.6	0.7	0.76	1
	$k_\delta^{2,0}$	$k_\delta^{2,1}$	$k_\delta^{2,2}$	$k_\delta^{2,3}$	$k_\delta^{2,4}$	$k_\delta^{2,5}$	$k_\delta^{2,6}$

Table 5
DMs' sequences of values determining completely unsynchronized TFNs for X_δ.

	$k_\delta^{1,0}$	$k_\delta^{1,1}$	$k_\delta^{1,2}$	$k_\delta^{1,3}$	$k_\delta^{1,4}$	$k_\delta^{1,5}$	$k_\delta^{1,6}$
D_1	0	0.05	0.25	0.5	0.6	0.75	1
D_2	0	0.2	0.45	0.7	0.82	0.93	1
	$k_\delta^{2,0}$	$k_\delta^{2,1}$	$k_\delta^{2,2}$	$k_\delta^{2,3}$	$k_\delta^{2,4}$	$k_\delta^{2,5}$	$k_\delta^{2,6}$

If D_1 uses the TFN $\left(k_\delta^{1,3}, k_\delta^{1,4}, k_\delta^{1,5}\right)$ to describe the value of X_δ, then D_2 thinks of the TFN $\left(k_\delta^{2,3}, k_\delta^{2,4}, k_\delta^{2,5}\right)$ as the one describing the value in his representation, and vice versa. The same happens with every other TFN describing the values of X_δ. Thus, both DMs basically share the same criterion when describing the values of the variable X_δ.■

On the other hand, saying that the TFN representations of the linguistic values of X_δ used by D_1 and D_2 are unsynchronized means that whenever D_i uses the TFN $(k_\delta^{i,s-1}, k_\delta^{i,s}, k_\delta^{i,s+1})$, where s is the index of the pair of values $\left(k_\delta^{1,s}, k_\delta^{2,s}\right)$ causing the unsynchronization, the other DM D_{-i} will think of a TFN different from $\left(k_\delta^{-i,s-1}, k_\delta^{-i,s}, k_\delta^{-i,s+1}\right)$. That is, D_{-i} will never think of the product $\diamondsuit_i$ as satisfying the linguistic value corresponding to $\left(k_\delta^{-i,s-1}, k_\delta^{-i,s}, k_\delta^{-i,s+1}\right)$, even if D_{-i} would actually use this very same linguistic value to describe the (δ)-characteristic of $\diamondsuit_i$ should he observe it directly.

Example 3. Suppose that D_1 and D_2 use the TFNs determined by the finite sequences reported in Table 4 to represent the linguistic values of variable X_δ. The membership functions associated with these TFNs are compared in Fig. 5.

Suppose that D_1 assigns the value *not very* $a(\delta)$ to the characteristic X_δ of his product $\diamondsuit_1$. Then, D_1 uses $\left(k_\delta^{1,1}, k_\delta^{1,2}, k_\delta^{1,3}\right)$ to describe $\diamondsuit_1$. Following his own TFN representation, D_2 understands that the linguistic value being described is *lowly* $a(\delta)$, since the value $k_\delta^{2,1}$ is the closest one to $k_\delta^{1,2}$ among those in the sequence used by D_2.■

Example 4. Suppose that D_1 and D_2 use the TFNs determined by the finite sequences reported in Table 5 to represent the linguistic values of variable X_δ. The membership functions associated with these TFNs are compared in Fig. 6.

Whatever is the value that D_1 assigns to the characteristic X_δ of his product $\diamondsuit_1$ and, hence, the TFN that D_1 uses to describe $\diamondsuit_1$, the other DM D_2 will never think of the same linguistic value.■

5. Main results

Theorem 5.1. *Let requirements R.1–R.4 be satisfied. For i = 1, 2, let j(i) be the index of the category to which $\diamondsuit_i$ belongs and $r_i \subseteq \diamondsuit_i^*$ be D_i's report to describe the product $\diamondsuit_i$ to D_{-i}.*

(a) *If:*
- (a.1) *for every linguistic variable $X_{\delta_{j(i)}}$ such that $t_{\delta_{j(i)}} \in r_i$, DMs' TFN representations of $X_{\delta_{j(i)}}$ are synchronized or slightly unsynchronized;*
- (a.2) *DMs agree to exchange;*

  then, D_{-i} gets an acceptable product.
  If, moreover, $r_i \subseteq (\diamondsuit_i)^*_{-i}$, then D_{-i} gets the product he expects.

(b) *If:*
- (b.1) *there exists a linguistic variable $X_{\delta_{j(i)}}$ such that $t_{\delta_{j(i)}} \in r_i$ and DMs' TFN representations of $X_{\delta_{j(i)}}$ are unsynchronized or completely unsynchronized;*
- (b.2) *DMs agree to exchange;*

  then D_{-i} may or may not get an acceptable product.

Proof. In both statement (a) and (b), it is assumed that D_i is the DM reporting one or more values to D_{-i} through the report r_i. Fix a linguistic variable $X_{\delta_{j(i)}}$ such that $t_{\delta_{j(i)}} \in r_i$.

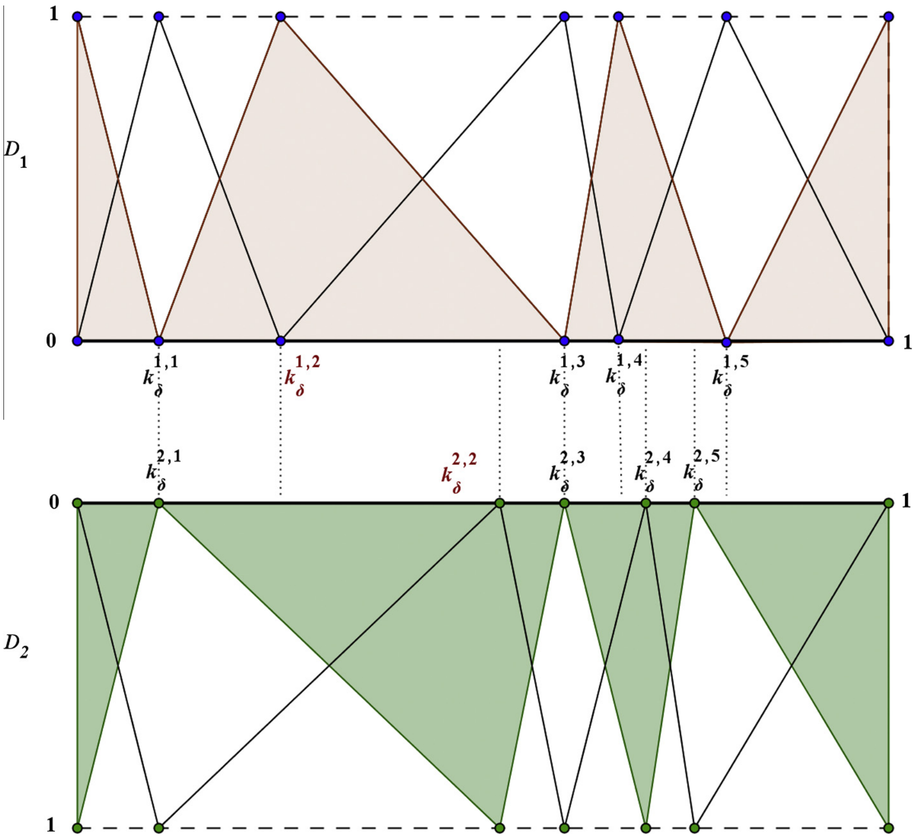

Fig. 5. Example of unsynchronized TFN representations of X_δ.

If DMs' TFN representations of $X_{\delta_{j(i)}}$ are synchronized or slightly unsynchronized, the exchange expected utility (Definition 3.5) that D_{-i} calculates considering all the products in $\Psi_{-i}(r_i)$ will actually coincide with the utility value that D_{-i} assigns to the exchanged product. Thus, even if the product received is not exactly the one that D_{-i} had envisioned on the basis of the description r_i, it is still an acceptable one (Definition 3.7).

If DMs' TFN representations of $X_{\delta_{j(i)}}$ are unsynchronized or completely unsynchronized, the set $\Psi_{-i}(r_i)$ contains products whose utility may be very far from the one that D_{-i} would actually assign to the product $\diamondsuit_i$. In particular, the exchange expected utility calculated considering all the products in $\Psi_{-i}(r_i)$ could be higher than the real utility that $\diamondsuit_i$ delivers to D_{-i}. Thus, the exchange expected utility of D_{-i} being higher than $u_{-i}(\diamondsuit_{-i}^*)$ does not guarantee that $\diamondsuit_i$ will be an acceptable product for D_{-i}.■

Finally, letting the DMs to express qualitative characteristics via TFNs yields the following paradoxical situation.

Paradox. Suppose that, for $i = 1, 2$, D_{-i}'s subjective perception $(\diamondsuit_i)_{-i}^*$ of the product $\diamondsuit_i$ belonging to D_i coincides with D_i's subjective perception $\diamondsuit_i^*$ of $\diamondsuit_i$. That is, if D_{-i} would be allowed to directly observe the product $\diamondsuit_i$, then he would assign to its characteristics, both qualitative and quantitative, the same values as those assigned by D_i.

Thus, if the DMs communicate their reports using linguistic values for the qualitative characteristics they are asked to describe (see requirements R.1–R.4), then agreeing to exchange implies that both DMs do actually get what they expect.

On the other hand, if the DMs communicate their reports using their TFN representations of the values that they assign to the qualitative characteristics, then, even if the DMs agree to exchange, one or both of them may get an unacceptable product.■

The following example illustrates the paradox.

Example 5. Suppose that, for $i = 1, 2$, $(\diamondsuit_i)_{-i}^* = \diamondsuit_i^*$ and that each D_i is requested to describe one qualitative characteristic of $\diamondsuit_i$.

Suppose D_1 is asked to communicate the value he assigns to the (δ)-characteristic and that the DMs' TFN representations of the possible values of this characteristic are those determined by the sequences reported in Table 5. This is an instance of completely unsynchronized TFN representations (see Example 4 above).

Finally, suppose that D_1 assigns the value x_δ = *highly* $a(\delta)$ to the (δ)-characteristic of the product $\diamondsuit_1$.

Whenever D_1 is asked to express this value with a TFN, he reports $(k_\delta^{1,4} = 0.6,\ k_\delta^{1,5} = 0.75,\ k_\delta^{1,6} = 1)$. D_2 considers the intermediate value $k_\delta^{1,5} = 0.75$ with respect to his own TFN representation and, hence, thinks of the triple $\left(k_\delta^{2,2} = 0.45, k_\delta^{2,3} = 0.7, k_\delta^{2,4} = 0.82\right)$, which for D_2 means that x_δ = *regularly* $a(\delta)$.

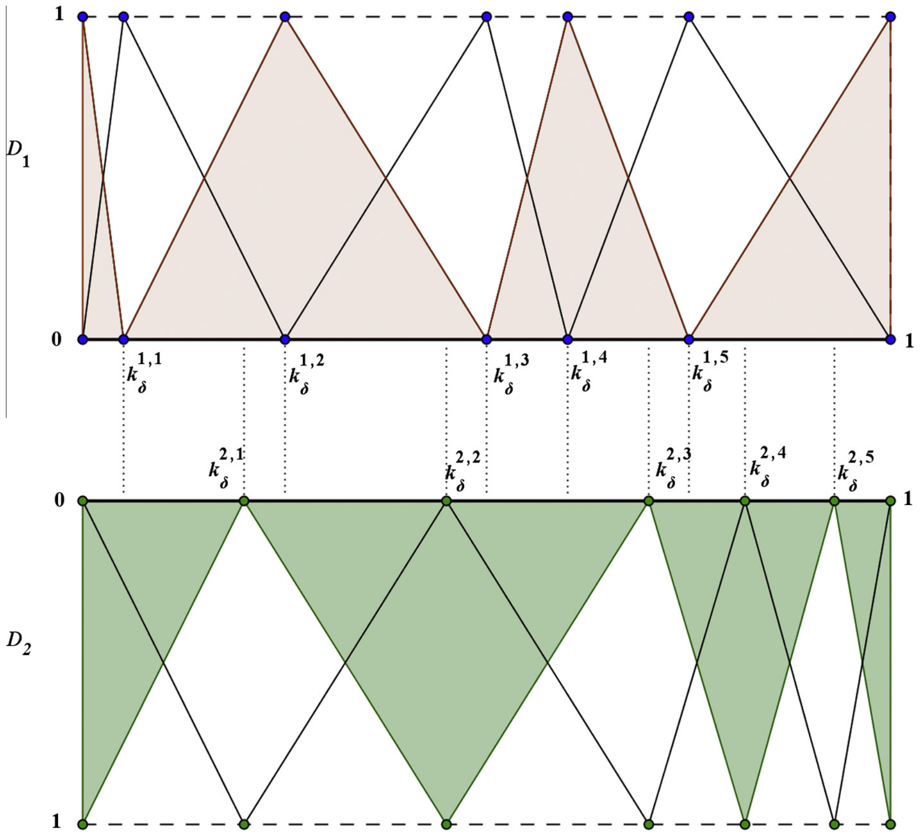

Fig. 6. Example of completely unsynchronized TFN representations of X_δ.

Thus, letting D_1 communicate via a TFN instead of the linguistic value that this TFN represents induces D_2 to an evaluation which is totally different form the one he would make if he could observe the product directly.■

6. Managerial implications

In this section we describe four main empirical-based implications that follow from the current model. We concentrate on the economics and operational research branches of the literature, which account for most of the empirical-related research.

6.1. Economics and psychology

The current paper provides an additional research source to study the findings reported by economists and psychologists regarding the choice of undesired and disappointing outcomes by DMs. The empirical evidence on the suboptimal choices due to the subjective evaluation errors incurred by DMs when predicting the utility they may derive from the available choice options is reviewed by [21,23]. In this regard, even if directly observable, it is widely acknowledged that most of the characteristics of a product are hard to assess, see [31]. This evaluation constraint intensifies when exchanging products online based on the description provided by an unknown DM. The current model accounts for the distortions that follow from the perception and evaluation differences existing among DMs.

However, while perception and evaluation factors play a fundamental role in our model, we also account for an additional source of frictions arising from the communication process between DMs. That is, even though utilities may be defined precisely and perceptions be identical for both DMs, the communication of subjective evaluations may still differ between them. This result opens a whole new set of possibilities when analyzing trade frictions and the trendy topic of happiness, see [17].

Finally, our model emphasizes the fact that suboptimality is, to a great extent, inherent to the DMs interacting within an economic/exchange system and cannot be fully eliminated, though its effects may be contained.

6.2. Fuzzy decision making

When considering the potential applicability of the current model within a fuzzy environment generated by the imprecise perceptions of DMs several potential scenarios arise. For example, the data retrieved by [2,3] on iPhone usage and acceptance relies on evidence derived from netnographic linguistic variables. Similarly, vague evaluations are generally received by DMs when making subjective risky decisions at the initial stages of a given project (see [41,44]). The effect of these linguistic reports on the expectations of DMs, their degree of trust on a firm or project and acceptance of a novel technological product, see [9], can all be studied using the current formal setting.

6.3. Game theory and strategic behavior

The communication process taking place between DMs extents our model into the game theoretical branch of the economic literature and foresees several strategic implications. This is particularly the case when considering the capacity of DMs to misrepresent the characteristics observed for their own benefit. Our model introduces an additional variable that must be approximated by the subjective beliefs of DMs, i.e. the synchronization in the perception of the characteristics of the product. Thus, two sources of uncertainty interact and should be accounted for by the economic signaling literature analyzing the strategic transmission of information between DMs [11,18,32]: the standard uncertainty related to the type of DMs playing the game together with the degree of synchronization in their respective perceptions. The capacity of DMs to estimate correctly the degree of synchronization with the perception of other DMs will play an essential role in determining the set of equilibria of the corresponding games.

6.4. Expert systems and project selection at NASA

Consider the assessment of advanced-technology projects at NASA provided by [37]. In order to rank the projects, experts are asked to give subjective evaluations on each project using linguistic variables. The evaluations are afterward translated in fuzzy numbers through a unified fuzzy scale. This scale imposes a consistency constraint among the linguistic evaluations of the experts. According to our results, the ranking obtained may either be the result of an artificial consistency constraint or follow from the "good enough" synchronization existing among experts. That is, experts whose numerical evaluations for a given linguistic report are relatively close should lead to more robust rankings. Therefore, if experts were asked to provide, together with their linguistic evaluation, a numerical equivalent using either a real value or in the form of a fuzzy interval, the robustness of the resulting ranking could be analyzed. As a result, a consistency test based on the synchronicity existing among experts could be calculated and included with the corresponding ranking. This type of consistency analysis could be applied within standard group decision environments or used to validate the rankings supplied by committees of experts (see [35,37]).

7. Conclusion

We have studied a bilateral exchange model where the subjective perception of the characteristics of the products with which DMs are endowed may lead to exchange agreements that would be deemed as unacceptable by either one or both DMs after being performed. This is the case even when both DMs share a common language and within an imposed manipulation-free environment.

The synchronization problem derived from the perception differences existing between DMs persists even when they are allowed to express their qualitative evaluations using real values belonging to a normalized interval instead of through linguistic variables. Indeed, we have shown how quantifying the linguistic values of qualitative characteristics may actually create more misunderstanding than using the corresponding linguistic values.

Finally, we have discussed several managerial implications ranging from economics and psychology to expert systems and group decision making.

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