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An extended stochastic VIKOR model with decision maker's attitude towards risk

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A B S T R A C T

We propose a risk-based stochastic VIKOR (RB-VIKOR) model that accounts for differences in the risk attitudes of the decision makers (DMs) when ranking stochastic alternatives. Our proposed RB-VIKOR model is designed to solve multi-criteria problems characterized by stochastic data and DMs categorized by their risk averse or risk seeking behavior. These differences in risk attitudes determine the subjective beliefs of the DMs regarding the evaluation of each alternative per decision criterion and the resulting rankings. We present a case study in the banking industry to illustrate how differences in the risk attitudes of the DMs condition the rankings obtained. Moreover, we compare our results with those derived from a stochastic super-efficiency data envelopment analysis (DEA) model to demonstrate the applicability and efficacy of RB-VIKOR. The proposed method has a considerable amount of potential applications to diverse research areas ranging from economics to knowledge based and decision support systems.

1. Introduction

Multi-criteria decision making (MCDM) techniques are used to rank sets of alternatives characterized by multiple and often conflicting criteria [7]. Classical MCDM methods generally assume that the ratings of the alternatives and the weights of the criteria are known precisely. However, decision makers (DMs) face different degrees of risk and uncertainty throughout their decision making processes when dealing with real-world data [22].

MCDM models have been adapted to account for different types of information frictions, which are generally modeled through stochastic analysis or fuzzy set theory. The former approach applies when a probabilistic data set represents the risk faced by the DMs, while the latter approach is more appropriate when the observations retrieved are vague and ambiguous [29].

The VIKOR method is a MCDM technique designed to rank a set of alternatives in the presence of conflicting criteria by proposing a compromise solution [14,17]. This method has been used to solve different types of MCDM problems both

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in crisp [3] and fuzzy [2,15] environments. In particular, there has been an increasing interest regarding its applicability to multi-criteria group decision making scenarios characterized by different types of fuzzy information such as intuitionistic [5] and dual hesitant [21]. Moreover, VIKOR has also been integrated with other MCDM techniques such as the fuzzy analytic hierarchy process (AHP) [9]. A more detailed literature review regarding recent applications of VIKOR can be found in Tavana et al. [25].

These latter authors extended the basic structure of VIKOR and developed a method to solve MCDM problems with stochastic data. Their model considered several stochastic criteria, whose weights were determined via the fuzzy AHP. The decision framework developed by Tavana et al. [25] constitutes the base on which the current model is built. In particular, we define an extended version of the VIKOR method introduced by Tavana et al. [25] that accounts for differences in the risk attitudes of the DMs when ranking stochastic alternatives.

Our formal framework allows for modifications in the rating behavior of the DMs that depend both on whether they are risk seekers or averters and the coefficient of variation exhibited by each alternative. That is, risk-based stochastic VIKOR (RB-VIKOR) allows the DMs to select the alternative that is more in accordance with their subjective preferences and risk attitudes while accounting for the uncertainty inherent to the data retrieved to evaluate each alternative.

Tavana et al. [25] also compared empirically their method with a stochastic version of the super-efficiency data envelopment analysis (DEA) model of Khodabakhshi et al. [10]. We use the same banking industry study to illustrate how differences in the risk attitudes of the DMs condition the rankings obtained. We conclude that if the DMs do not know the exact distribution from which the observations are being drawn or are not neutral to the inherent risk, their resulting rankings will differ from the ones provided by more neutral models such as super-efficiency DEA.

It should be emphasized that the rankings obtained are determined by the volatility exhibited by the data retrieved from the different alternatives together with the risk attitude of the DMs. That is, RB-VIKOR improves upon any other MCDM method that imposes a given probability density on the data or does not account for the subjective risk preferences of the DMs. Thus, there is a substantial amount of potential applications of the current model to diverse research areas ranging from economics [6,13,28] to knowledge based [20,26,27] and decision support systems [4,12,18].

The rest of the paper is organized as follows. Section 2 provides a brief introduction to the VIKOR method. Section 3 summarizes the stochastic VIKOR model proposed by Tavana et al. [25]. Section 4 describes the main characteristics of RB-VIKOR and Section 5 completes the model focusing on the effect that the coefficients of variation have on the expected performance of the alternatives. Section 6 compares the ranking obtained using RB-VIKOR with the one derived from applying the super-efficiency stochastic DEA model. Section 7 concludes and suggests future research directions.

2. The VIKOR method

The VIKOR method is a MCDM technique introduced by Opricovic [14] that defines positive and negative ideal points and determines the relative distance of each alternative. After computing each relative distance, the method obtains a weighted compromise ranking determining the importance of each x_j alternative, with $j = 1, 2, \ldots, m$. The steps composing the compromise ranking algorithm are:

1. Define the rating functions f_{ij} that describe the value achieved by alternative x_j, $i = 1, 2, \ldots, n$, when considering the $i-th$ criterion. Compute the best, f_i^+, and the worst, f_i^-, values for all the rating functions. If the criterion considered is a positive one, then the corresponding extreme values are given by:

$$f_i^+ = \max\left[(f_{ij}) | \, j = 1, 2, \ldots, m\right] \tag{1}$$

$$f_i^- = \min\left[(f_{ij}) | \, j = 1, 2, \ldots, m\right] \tag{2}$$

2. Calculate the values of S_j and R_j, $j = 1, 2, \ldots, m$, as follows

$$S_j = \sum_{i=1}^{n} w_i \frac{(f_i^+ - f_{ij})}{\left(f_i^+ - f_i^-\right)} \tag{3}$$

$$R_j = \max_i \left[w_i \frac{\left(f_i^+ - f_{ij}\right)}{\left(f_i^+ - f_i^-\right)} \right] \tag{4}$$

where w_i are the weights reflecting the relative importance of the criteria. S_j represents the group utility measure and R_j the individual regret measure defined for each alternative x_j.

3. Compute the values of Q_j, $j = 1, 2, \ldots, m$, as follows:

$$Q_j = v\left[\frac{(S_j - S^+)}{(S^- - S^+)}\right] + (1 - v)\left[\frac{(R_j - R^+)}{(R^- - R^+)}\right] \tag{5}$$

where

$$S^{+} = Min\left[\left(S_j\right) \mid j = 1, 2, \ldots, m\right] \tag{6}$$

$$S^{-} = Max\left[\left(S_j\right) \mid j = 1, 2, \ldots, m\right] \tag{7}$$

$$R^{+} = Min\left[\left(R_j\right) \mid j = 1, 2, \ldots, m\right] \tag{8}$$

$$R^{-} = Max\left[\left(R_j\right) \mid j = 1, 2, \ldots, m\right] \tag{9}$$

and v is the subjectively defined weight supporting either maximum group utility ($\nu > 1/2$), individual regret ($\nu < 1/2$), or consensus between both strategies ($\nu = 1/2$).

4. Rank the alternatives using the values of (S_j, R_j, Q_j), which give place to three different rankings that can be used to define and validate a compromise solution [16].

3. VIKOR method with stochastic data

We restate the model of Tavana et al. [25] in order to provide some basic intuition and allow for direct comparisons based on the degree of risk aversion of the DM. These authors defined the following version of VIKOR so as to account for stochastic realizations of the different rating functions. Their model is based on the sample mean of the rating functions per criterion and alternative together with their associated coefficients of variation. Given a sample of l realizations, y_k, $k = 1, \ldots, l$, the coefficient of variation (cv) is defined as $cv = \frac{\sigma}{\bar{y}}$, i.e. the ratio of the standard deviation, $\sigma = \sqrt{\frac{1}{l}\sum_{k=1}^{l}(y_k - \bar{y})^2}$, to the sample mean, $\bar{y} = \frac{\sum y_k}{l}$.

Consider the following decision matrix where the stochastic data are described in terms of rating means and coefficients of variation:

	C_1	C_2	$\ldots$	C_n
A_1	$[f_{11}, cv_{11}]$	$[f_{21}, cv_{21}]$	$\ldots$	$[f_{n1}, cv_{n1}]$
A_2	$[f_{12}, cv_{12}]$	$[f_{22}, cv_{22}]$	$\ldots$	$[f_{n2}, cv_{n2}]$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
A_m	$[f_{1m}, cv_{1m}]$	$[f_{2m}, cv_{2m}]$	$\ldots$	$[f_{nm}, cv_{nm}]$
		$W = [w_1, w_2, \ldots, w_n]$		

where $A_1, A_2, \ldots, A_m$ represent the alternatives among which the DM has to choose, $C_1, C_2, \ldots, C_n$ are the different criteria measuring the performance of the alternatives, and $w_i (i = 1, 2, \ldots, n)$ are the weights representing the relative importance assigned by the DM to each criterion. The f_{ij} entries of the matrix correspond to the sample means of the rating functions for the $i-th$ criterion and alternative x_j. The coefficient of variation associated to the $i-th$ criterion and alternative x_j is denoted by cv_{ij}.

Specifying a domain for a stochastic entry in the decision matrix implies that the realizations of the corresponding variables can take any value within the interval based on their associated probability densities. In addition, when stochastic data are used to infer the range of the domain of potential values that an entry can take, their coefficient of variation can be used to measure the uncertainty associated with a set of realizations.

The steps of the stochastic version of VIKOR proposed by Tavana et al. [25] are the following ones:

1. Determine the best f_i^{+} and the worst f_i^{-} values for all the rating functions considering two different types of criteria, i.e. positive and negative. In their model, Tavana et al. [25] used the maximum coefficient of variation obtained per criterion, i.e. $\max_j cv_{ij}$, in order to define the extremes of the rating functions as follows:

$$f_i^{+}$$

$$\text{Positive criterion} \quad max\ f_{ij} \times \left(1 + \max_j cv_{ij}\right),\ \forall j:\ j = 1, 2, \ldots, m \tag{10}$$

$$\text{Negative criterion} \quad min\ f_{ij} \times \left(1 - \max_j cv_{ij}\right),\ \forall j: j = 1, 2, \ldots, m \tag{11}$$

$$f_i^{-}$$

$$\text{Positive criterion} \quad min\ f_{ij} \times \left(1 - \max_j cv_{ij}\right), \forall j: j = 1, 2, \ldots, m \tag{12}$$

$$\text{Negative criterion} \quad max\ f_{ij} \times \left(1 + \max_j cv_{ij}\right), \forall j: j = 1, 2, \ldots, m \tag{13}$$

2. Compute the values of S_j and R_j, $j = 1, 2, 3, \ldots, m$

S_j

Positive criterion
$$\sum_{i=1}^{n} w_i \frac{max\, f_{ij} \times \left(1 + \max_j cv_{ij}\right) - f_{ij}}{max\, f_{ij} \times \left(1 + \max_j cv_i\right) - min\, f_{ij} \times \left(1 - \max_j cv_i\right)} \tag{14}$$

Negative criterion
$$\sum_{i=1}^{n} w_i \frac{min\, f_{ij} \times \left(1 - \max_j cv_i\right) - f_{ij}}{min\, f_{ij} \times \left(1 - \max_j cv_i\right) - max\, f_{ij} \times \left(1 + \max_j cv_i\right)} \tag{15}$$

R_j

Positive criterion
$$max_i \left\{ w_i \frac{max\, f_{ij} \times \left(1 + \max_j cv_i\right) - f_{ij}}{max\, f_{ij} \times \left(1 + \max_j cv_i\right) - min\, f_{ij} \times \left(1 - \max_j cv_i\right)} \right\} \tag{16}$$

Negative criterion
$$max_i \left\{ w_i \frac{min\, f_{ij} \times \left(1 - \max_j cv_i\right) - f_{ij}}{min\, f_{ij} \times \left(1 - \max_j cv_i\right) - max\, f_{ij} \times \left(1 + \max_j cv_i\right)} \right\} \tag{17}$$

We should emphasize that one of the main differences between our model and that of Tavana et al. [25] can be observed in the computation of the values of S_j, and R_j. In particular, note that the f_{ij} terms subtracted in the numerators of Eqs. (14)–(17) do not depend on the coefficient of variation of the respective observations. Thus, the variability inherent to each alternative per criterion is not considered by the authors when computing the ranking positions. In the current paper, we account for this variability together with the effect that the risk attitude of the DMs has on the corresponding rankings.

3. Calculate the values of Q_j, $j = 1, 2, \ldots, m$, using the relation:

$$Q_j = \nu\left[\frac{(S_j - S^+)}{(S^- - S^+)}\right] + (1 - \nu)\left[\frac{(R_j - R^+)}{(R^- - R^+)}\right] \tag{18}$$

Tavana et al. [25] interpreted $(0 < \nu \le 1)$ as a proxy for the optimism level of the DM, with optimistic DMs assigning higher values to ν than pessimistic ones. Rational DMs were assumed to choose $\nu = 0.5$, since the resulting rankings would be similar to those derived from comparing the sample means of each alternative.

4. Rank the alternatives, sorting by the values (S_j, R_j, Q_j), and propose a compromise solution.

4. VIKOR method with stochastic data and risk considerations

The stochastic data retrieved from different choice alternatives provide a highly intuitive framework to account for the risk and uncertainty inherent to real-life decision making problems. The volatility exhibited by the alternatives for the different selection criteria, which can be directly inferred from the data retrieved, determines the risk faced by the DMs and has a direct effect on their evaluations.

Our extended version of the stochastic VIKOR model developed by Tavana et al. [25] accounts for this real-life feature. That is, evaluations are performed by DMs, who are endowed with different risk attitudes when choosing among the alternatives. The same sample mean value of the rating functions for a given pair of alternatives does not lead to the same ranking position when volatility differs, an idea considered by economists and explored further by psychologists [8].

We consider different domains to account for the values that can be taken by the stochastic realizations of the alternatives based on their associated probability densities. Tavana et al. [25] assumed and verified that their data followed a normal distribution. This was done to compare their extended VIKOR ranking with the one obtained from the stochastic super-efficiency data envelopment analysis (DEA) model of Khodabakhshi et al. [10]. In this regard, the ranking structure introduced in the current paper does not depend on the density functions determining the realizations of the data but on the observed variability of the latter. Following Tavana et al. [25], we will use the coefficient of variation to measure the spread of the domain associated with a given set of realizations.

Two distinct frameworks will be defined to introduce the risk attitude of the DMs into the equations defining the stochastic VIKOR process. Our definitions of risk averse and risk seeker DMs follow directly from the economic decision-theoretic ones. That is, risk averse DMs are endowed with concave utility functions, leading them to prefer alternatives with lower variability over those with a higher one. Risk seekers, endowed with convex utilities, prefer the more variable alternatives. Finally, risk neutral DMs are endowed with linear utilities and, therefore, indifferent to the variability exhibited by the alternatives, as is the case in the model of Tavana et al. [25].

4.1. The risk averse setting

The modifications implemented to the extended VIKOR model defined by Tavana et al. [25] will focus on the definitions of the best f_i^+ and the worst f_i^- values of the rating functions and the resulting values of S_j and R_j. In all cases, we must differentiate the effects of positive and negative criteria in the corresponding definitions. Consider first the limit values of the rating functions:

$$f_i^+$$

$$\text{Positive criterion} \quad \mathit{max}\, f_{ij} \times \left(1 - \min_j cv_{ij}\right), \forall j: \quad j = 1, 2, \ldots, m \tag{19}$$

$$\text{Negative criterion} \quad \mathit{min}\, f_{ij} \times \left(1 + \min_j cv_{ij}\right), \forall j: \quad j = 1, 2, \ldots, m \tag{20}$$

$$f_i^-$$

$$\text{Positive criterion} \quad \mathit{min}\, f_{ij} \times \left(1 - \max_j cv_{ij}\right), \forall j: \quad j = 1, 2, \ldots, m \tag{21}$$

$$\text{Negative criterion} \quad \mathit{max}\, f_{ij} \times \left(1 + \max_j cv_{ij}\right), \forall j: \quad j = 1, 2, \ldots, m \tag{22}$$

In their model, Tavana et al. [25] defined the extremes of the rating functions using the maximum coefficient of variation obtained per criterion, i.e. $\max_j cv_{ij}$. However, when considering explicitly the risk attitudes of the DMs, the extremes of the reference domains must be modified. The main differences between the model of Tavana et al. [25] and the current approach in terms of the limit values defined for f_i^+ and f_i^- are presented in Fig. 1.

For example, when defining f_i^+ for a positive criterion, risk averse DMs do not only require the maximum value of the rating function, but also the minimum coefficient of variation. Similarly, the definition of f_i^- for a positive criterion by risk averse DMs involves both the minimum value of the rating function and the maximum coefficient of variation. The same intuition applies to the remaining cases, with risk seeking DMs requiring the maximum coefficient of variation when defining f_i^+ and the minimum one when defining f_i^-.

Moreover, within the current framework, the coefficient of variation per criterion and alternative, cv_{ij}, will be compared to the limit values of the corresponding rating functions, determining the resulting ranking of the alternatives. That is, in order to account directly for their variability, the coefficients of variation will be introduced in the relative evaluation of the corresponding alternatives. The values of S_j (and R_j), $j = 1, 2, 3, \ldots, m$, are therefore given by:

$$S_j$$

$$\text{Positive criterion} \quad \sum_{i=1}^{n} w_i \frac{\mathit{max}\, f_{ij} \times \left(1 - \min_j cv_{ij}\right) - f_{ij} \times \left(1 - cv_{ij}\right)}{\mathit{max}\, f_{ij} \times \left(1 - \min_j cv_{ij}\right) - \mathit{min}\, f_{ij} \times \left(1 - \max_j cv_{ij}\right)} \tag{23}$$

$$\text{Positive criterion} \quad \sum_{i=1}^{n} w_i \frac{\mathit{max}\, f_{ij} \times \left(\sqrt{N-1} - \min_j cv_{ij}\right) - f_{ij} \times \left(\sqrt{N-1} - cv_{ij}\right)}{\mathit{max}\, f_{ij} \times \left(\sqrt{N-1} - \min_j cv_{ij}\right) - \mathit{min}\, f_{ij} \times \left(\sqrt{N-1} - \max_j cv_{ij}\right)} \tag{23$'$}$$

$$\text{Negative criterion} \quad \sum_{i=1}^{n} w_i \frac{\mathit{min}\, f_{ij} \times \left(1 + \min_j cv_{ij}\right) - f_{ij} \times \left(1 + cv_{ij}\right)}{\mathit{min}\, f_{ij} \times \left(1 + \min_j cv_{ij}\right) - \mathit{max}\, f_{ij} \times \left(1 + \max_j cv_{ij}\right)} \tag{24}$$

Note that we have provided two definitions of S_j for the positive criterion. The second one, presented in Eq. (23′), has been adapted to account for the maximum potential spread of the coefficient of variation. That is, when retrieving a finite sample of N non-negative numbers with a real zero, the coefficient of variation is bounded within the $[0, \sqrt{N-1}]$ interval, with its maximum value being attained when all the realizations except one are equal to zero.

Thus, in order to avoid potential negative values of the positive criterion for some realizations of the alternatives, we have shifted the reference point to the maximum value of the coefficient of variation in Eq. (23′). This guarantees that all the evaluations obtained for both types of criteria have a positive sign and can be directly compared. The same analysis and conclusions follow from the resulting R_j values and have therefore been omitted.

If the data retrieved do not lead to coefficients of variation whose values are higher than one, then the reference points can be shifted back to those defined in Eq. (23). This is the case since, as we will show later, the rankings obtained experience small variations when shifting the corresponding reference points.

There is a substantial difference between this approach and the one of Tavana et al. [25], where the coefficients of variation do not play a direct role in the behavior of the DMs, other than adjusting the intervals of potential realizations for the different criteria. In the current setting, if the DMs are risk averse, we have a direct negative effect that follows from the coefficient of variation when determining their evaluation structure.

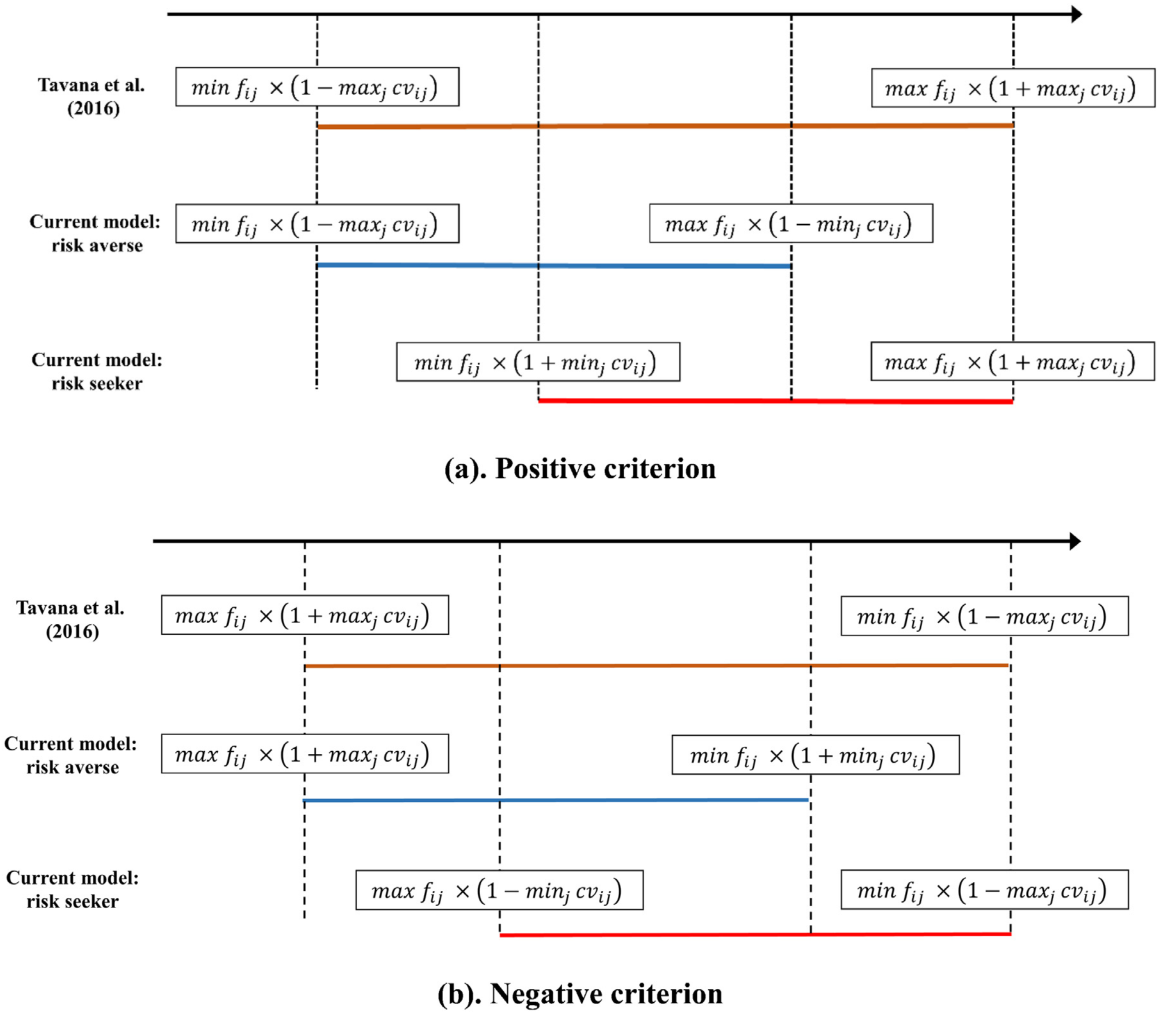

Fig. 1. f_i^+ and f_i^- limit values defined by the DMs based on their attitudes towards risk.

4.2. The risk seeker setting

We maintain a consistent approach through both settings to the effect that the coefficient of variation has on the ranking behavior of the DMs. In the risk averse case, the DMs preferred the alternatives exhibiting a lower coefficient of variation. In the current setting, the maximum coefficient of variation will be used to define the highest rating achievable by an alternative, while the minimum one determines the lowest rating.

Thus, given a positive criterion, the alternative with the highest rating will only achieve the best f_i^+ value if it has also the highest coefficient of variation. Once again, the main differences between our settings and the one of Tavana et al. [25] for both types of criteria are intuitively described in Fig. 1. Note how risk averse and risk seeker DMs share each one limit rating value with the framework introduced by Tavana et al. [25], who shift between both our settings when defining f_i^+ and f_i^-.

The best and worst values of the rating function within the risk seeker setting are:

$$f_i^+$$

Positive criterion $$max\ f_{ij} \times \left(1 + \max_j cv_{ij}\right), \forall j: \quad j = 1, 2, \ldots, m \tag{25}$$

Negative criterion $$min\ f_{ij} \times \left(1 - \max_j cv_{ij}\right), \forall j: \quad j = 1, 2, \ldots, m \tag{26}$$

$$f_i^-$$

Positive criterion $$min\ f_{ij} \times \left(1 + \min_j cv_{ij}\right), \forall j: \quad j = 1, 2, \ldots, m \tag{27}$$

$$\text{Negative criterion} \qquad max\ f_{ij}\ \times \left(1 - \min_j cv_{ij}\right), \forall j: \quad j = 1, 2, \ldots, m \tag{28}$$

The values of S_j, $j = 1, 2, 3, \ldots, m$, are defined as follows:

$$S_j$$

$$\text{Positive criterion} \qquad \sum_{i=1}^{n} w_i \frac{max\ f_{ij}\ \times \left(1 + \max_j cv_{ij}\right) - f_{ij} \times \left(1 + cv_{ij}\right)}{max\ f_{ij}\ \times \left(1 + \max_j cv_{ij}\right) - min\ f_{ij}\ \times \left(1 + \min_j cv_{ij}\right)} \tag{29}$$

$$\text{Negative criterion} \qquad \sum_{i=1}^{n} w_i \frac{min\ f_{ij}\ \times \left(1 - \max_j cv_{ij}\right) - f_{ij} \times \left(1 - cv_{ij}\right)}{min\ f_{ij}\ \times \left(1 - \max_j cv_{ij}\right) - max\ f_{ij}\ \times \left(1 - \min_j cv_{ij}\right)} \tag{30}$$

$$\text{Negative criterion} \qquad \sum_{i=1}^{n} w_i \frac{min\ f_{ij}\ \times \left(\sqrt{N-1} - \max_j cv_{ij}\right) - f_{ij} \times \left(\sqrt{N-1} - cv_{ij}\right)}{min\ f_{ij}\ \times \left(\sqrt{N-1} - \max_j cv_{ij}\right) - max\ f_{ij}\ \times \left(\sqrt{N-1} - \min_j cv_{ij}\right)} \tag{30$'$}$$

As already explained, risk seekers consider a high value of the coefficient of variation as a desirable feature when rating alternatives. The $max\ f_{ij}\ \times (1 + \max_j cv_{ij})$ expression for a positive criterion and the $min\ f_{ij}\ \times (1 - \max_j cv_{ij})$ for a negative one illustrate the preference of DMs for volatile alternatives, while in the risk averse case volatility was avoided.

Note that we have provided two definitions of S_j for the negative criterion. Similarly to the risk neutral case, the definition presented in Eq. (30′) has been adapted to account for the maximum potential spread of the coefficient of variation.

Given the modifications introduced in the evaluation process, the remaining VIKOR steps follow from those of Opricovic [14] as described by Tavana et al. [25].

5. Coefficients of variation and expected performance of the alternatives

As a measure of the spread exhibited by the realizations of a given variable, the coefficient of variation should condition the expected performance of a given alternative and, therefore, the rankings defined by the DMs. This section introduces the coefficients of variation in the expectation terms defined by the DMs when evaluating the alternatives through the values of S_j and R_j for each $j = 1, 2, \ldots, m$. It should be emphasized that we will not require any additional information on the side of the DMs. The resulting rankings are determined by the evaluations received and their relative spreads.

5.1. The risk averse setting

We introduce expectations on the definitions of S_j and R_j using a set of Beta density functions computed by the DMs for different realizations of the rating functions and the coefficients of variation per alternative and criterion. The values of S_j, $j = 1, 2, 3, \ldots, m$, for a positive criterion are given by:

$$S_j$$

$$\text{Positive criterion} \qquad \sum_{i=1}^{n} \int_0^1 Beta\left(\overline{f_{ij}} | f_{ij} + \gamma \left(\max_j cv_{ij} - cv_{ij}\right)\right) w_i \left[\frac{max\ f_{ij} - f_{ij}}{max\ f_{ij} - min\ f_{ij}}\right] xdx \tag{31}$$

$$\text{Positive criterion} \qquad \sum_{i=1}^{n} \int_0^1 Beta(cv_{ij} | \max_j cv_{ij}) w_i \left[\frac{max\ f_{ij} - f_{ij}}{max\ f_{ij} - min\ f_{ij}}\right] xdx \tag{32}$$

where

$$Beta(x; cv_{ij}, \max_j cv_{ij}) = \frac{x^{cv_{ij}-1}(1-x)^{\max_j cv_{ij}-1}}{\int_0^1 u^{cv_{ij}-1}(1-u)^{\max_j cv_{ij}-1} du} \tag{33}$$

and a similar definition applies to $Beta(x; \overline{f_{ij}}, f_{ij} + \gamma(\max_j cv_{ij} - cv_{ij}))$. $\overline{f_{ij}} = \frac{\sum_j f_{ij}}{j}$ denotes the average of the ratings observed for a set of alternatives within a given criterion and the terms in the square brackets correspond to the expressions described in Eq. (23) or (23′). Note that we have included a new variable, x, in order to generate an expected value derived from the Beta density. This function has been designed to condition the expected value of S_j on the relative coefficients of variation of the corresponding ratings.

We have introduced the subjective beliefs of the DMs through the Beta densities in two different ways. The first one is defined in Eq. (31) and relies on the average ratings and the relative differences among coefficients of variation. The second one is defined in Eq. (32) and relies on the differences among the coefficients of variation, i.e. it focuses solely on the variability of the alternatives. In the first case, the variable γ has been introduced to represent the aversion intensity of the DMs. This variable takes a value of zero if DMs are risk neutral and a bounded positive one if DMs are risk averse. The higher the value of γ, the stronger the risk aversion exhibited by the DM.

The intuition justifying the choice of a Beta distribution as an approximation to the behavior of the DMs when dealing with risk follows from Fig. 2. Note how the same difference of five units between reference points has a stronger bias

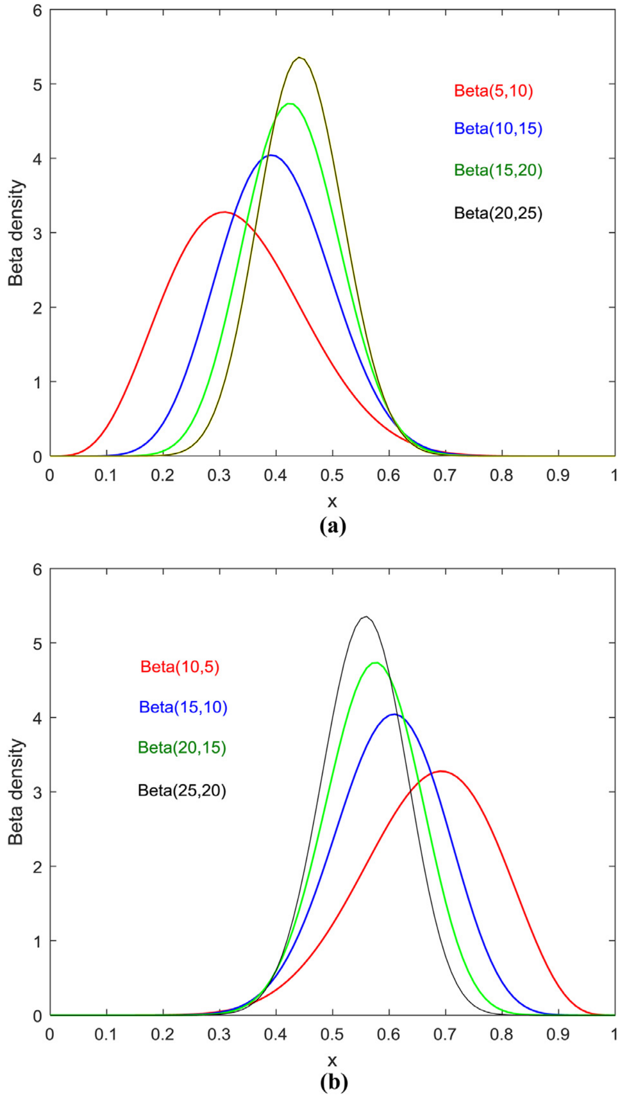

Fig. 2. Relative distance between reference points and resulting Beta densities.

effect when the width of the spread becomes larger relative to value of the reference points. Moreover, the Beta distribution must be skewed towards the lower end of its domain in order to provide a better expected evaluation to the DMs. This requirement follows from the structure of VIKOR, where preferred alternatives exhibit a value of S_j closer to zero.

In this regard, the right hand side of the conditional term defining the Beta density in Eq. (31), i.e. $[f_{ij} + \gamma(\max_j cv_{ij} - cv_{ij})]$, increases as the value of f_{ij} increases and the coefficient of variation of the corresponding alternative decreases. The same type of behavior follows from the $cv_{ij}|\max_j cv_{ij}$ term in Eq. (32), with lower coefficients of variation representing more preferred alternatives and leading to a Beta distribution skewed towards its lower end. Other densities could be considered based on the same type of premises regarding the expectations of the DMs, which would lead to similar results after adjusting the corresponding parameters to the requirements of the current VIKOR environment.

Extending the definition of Eq. (31) using Eq. (23′) for the positive criterion and Eq. (24) for the negative one, we have:

$$S_j$$

$$\text{Positive criterion} \sum_{i=1}^{n} \int_0^1 \left[\frac{\underset{Beta\left(\overline{f_{ij}}\ \mid\ \left[f_{ij}+\gamma\left(\max_j cv_{ij}-cv_{ij}\right)\right]\right) w_i}{max\, f_{ij} \times \left(\sqrt{N-1} - \min_j cv_{ij}\right) - f_{ij} \times \left(\sqrt{N-1} - cv_{ij}\right)}}{max\, f_{ij} \times \left(\sqrt{N-1} - \min_j cv_{ij}\right) - min\, f_{ij} \times \left(\sqrt{N-1} - \max_j cv_{ij}\right)} \right] x dx \tag{34}$$

$$\text{Negative criterion} \sum_{i=1}^{n} \int_0^1 \left[\frac{Beta\left(\overline{f_{ij}} \mid \left[f_{ij} + \gamma\left(\max_j cv_{ij} - cv_{ij}\right)\right]\right) w_i}{\dfrac{\min f_{ij} \times \left(1 + \min_j cv_{ij}\right) - f_{ij} \times \left(1 + cv_{ij}\right)}{\min f_{ij} \times \left(1 + \min_j cv_{ij}\right) - \max f_{ij} \times \left(1 + \max_j cv_{ij}\right)}} \right] xdx \tag{35}$$

5.2. The risk seeker setting

The intuition provided in the previous section applies also when analyzing the risk seeker setting, though in this scenario the DMs prefer alternatives endowed with a higher coefficient of variation. As in the risk averse case, when evaluating the different alternatives their volatility per decision criterion has to be accounted for together with the rating obtained. The corresponding values of S_j, $j = 1, 2, 3, \ldots, m$, for a positive criterion are:

$$S_j$$

$$\text{Positive criterion} \sum_{i=1}^{n} \int_0^1 Beta(\overline{f_{ij}}|f_{ij} + \gamma\left(cv_{ij} - \min_j cv_{ij}\right)) w_i \left[\frac{\max f_{ij} - f_{ij}}{\max f_{ij} - \min f_{ij}} \right] xdx \tag{36}$$

$$\text{Positive criterion} \sum_{i=1}^{n} \int_0^1 Beta(\min_j cv_{ij}|cv_{ij}) w_i \left[\frac{\max f_{ij} - f_{ij}}{\max f_{ij} - \min f_{ij}} \right] xdx \tag{37}$$

where:

$$Beta(x; \min_j cv_{ij}, cv_{ij}) = \frac{x^{\min_j cv_{ij} - 1} (1 - x)^{cv_{ij} - 1}}{\int_0^1 u^{\min_j cv_{ij} - 1} (1 - u)^{cv_{ij} - 1} du} \tag{38}$$

As in the previous setting, a Beta distribution skewed towards its lower end is used to evaluate the more preferred alternatives. Eq. (36) illustrates how such a function requires both a high f_{ij} value and the highest coefficient of variation, with the variable γ reflecting the risk seeking intensity of the DMs. Eq. (37) focuses on the variability of the alternatives to define the corresponding Beta densities, with higher coefficients of variation constituting preferred choices.

Extending the definition of Eq. (36) using Eq. (29) for the positive criterion and Eq. (30′) for the negative one, we have the following expressions determining the expected evaluations obtained by a given alternative when adding over all the criteria:

$$S_j$$

$$\text{Positive criterion} \sum_{i=1}^{n} \int_0^1 \left[\frac{Beta\left(\overline{f_{ij}} \mid \left[f_{ij} + \gamma\left(cv_{ij} - \min_j cv_{ij}\right)\right]\right) w_i}{\dfrac{\max f_{ij} \times \left(1 + \max_j cv_{ij}\right) - f_{ij} \times \left(1 + cv_{ij}\right)}{\max f_{ij} \times \left(1 + \max_j cv_{ij}\right) - \min f_{ij} \times \left(1 + \min_j cv_{ij}\right)}} \right] xdx \tag{39}$$

$$\text{Negative criterion} \sum_{i=1}^{n} \int_0^1 \left[\frac{Beta\left(\overline{f_{ij}} \mid \left[f_{ij} + \gamma\left(cv_{ij} - \min_j cv_{ij}\right)\right]\right) w_i}{\dfrac{\min f_{ij} \times \left(\sqrt{N-1} - \max_j cv_{ij}\right) - f_{ij} \times \left(\sqrt{N-1} - cv_{ij}\right)}{\min f_{ij} \times \left(\sqrt{N-1} - \max_j cv_{ij}\right) - \max f_{ij} \times \left(\sqrt{N-1} - \min_j cv_{ij}\right)}} \right] xdx \tag{40}$$

5.3. Numerical example

In this section, we analyze numerically the different rankings defined by the DMs depending on their attitudes towards risk. In order to do so, we generate ten random alternatives evaluated through five different criteria out of which the first three are positive and the last two negative. Each alternative is assigned ten random realizations per criterion, which are generated using a uniform distribution defined on the following domains: [0,5], [0,10], [0,15], [0,20] and [0,25]. Namely, the domain on which the realizations of the first criterion are defined is [0,5], that of the second criterion is [0,10], and so on. The average rating values and coefficients of variation per alternative and criterion are reported in Table 1.

The lowest and highest values attained by each column variable have been highlighted in blue and red, respectively. We will define six rankings based on the different VIKOR methods designed to account for the risk attitudes of the DMs. The evaluations of the alternatives per VIKOR method are presented in Table 2, whose entries correspond to the value of Q_j when $v = 0.5$. Moreover, throughout the current numerical example we have implicitly assumed an identical weight for each criterion in order to focus on the effect that variability has on the rankings defined by the DMs.

The following notation has been used to represent the different VIKOR models: Tavana refers to the model of Tavana et al. [25] described in Eqs. (10)–(17); RAv corresponds to the risk averse setting presented in Eqs. (19)–(24) with Eq. (23′) used for the positive criteria, while RAvN-1 represents the same risk averse setting but based on $\sqrt{N-1}$ being defined as a

Table 1
$(\bar{f}_{ij}, \boldsymbol{cv})$ of the randomly generated alternatives.

Alternative	C_1		C_2		C_3		C_4		C_5	
	$\bar{X}$	cv	$\bar{X}$	cv	$\bar{X}$	cv	$\bar{X}$	cv	$\bar{X}$	cv
1	2.5	0.805536	1.9	0.762704	8	0.699702	9.8	0.764815	11.3	0.836480
2	3	0.521157	6.3	0.539838	8.3	0.797267	5.9	0.729886	15.7	0.339769
3	3.1	0.635228	5.1	0.802885	7.7	0.721919	8.8	0.631111	14.0	0.518370
4	1.9	1.006290	5	0.632456	10	0.447214	10.7	0.564284	11.7	0.745165
5	1.7	0.736274	3.9	0.787532	6.7	0.656453	10.7	0.703595	12.6	0.574994
6	2.8	0.602339	5.4	0.593364	5.4	0.624647	10	0.516398	11.1	0.680099
7	2.4	0.765780	3.8	0.677202	7.3	0.707542	9.3	0.645261	10.9	0.946976
8	1.8	0.936971	6.3	0.579842	9.6	0.374614	12.6	0.426902	14.1	0.531205
9	2.6	0.633287	3.2	1.119973	5.8	0.824067	10.7	0.586216	10.8	0.837097
10	2.3	0.768253	4.3	0.924075	4.9	0.747990	7.7	0.665175	14.1	0.510824

Table 2
Q_j values of the random alternatives per VIKOR method when $v = 0.5$.

Alternative	Tavana	RAv	RAvN-1	Rsk	RAvW	RSkW
1	0.882105	0.916828	0.879486	0.583503	0.772238	0.344917
2	6.35E-07	2.13E−06	0.126080	0.5	3.63E−17	0.625562
3	0.222481	0.230204	0.200459	0.039085	0.201039	0
4	0.44803	0.696101	0.684285	0.164996	0.760573	0.164891
5	0.782695	0.910764	0.904397	0.854941	0.784207	0.579353
6	0.562774	0.601640	0.58416	0.460293	0.481736	0.313202
7	0.524358	0.472004	0.392066	0.074167	0.451812	0.093582
8	0.572355	0.694363	0.745434	1	0.680657	1
9	0.739957	0.839946	0.808689	0.247435	0.931075	0.129511
10	0.764483	0.921898	0.938602	0.620777	0.932758	0.231728

common reference value for both the positive and negative criteria; RSk is the risk seeker case described in Eqs. (25)–(30); RAvW refers to the risk averse setting defined through the following equations:

$$S_j$$

$$\text{Positive criterion} \sum_{i=1}^{n} \int_0^1 Beta\left(cv_{ij}|\max_j cv_{ij}\right) w_i \left[\frac{max\, f_{ij} \times \left(\sqrt{N-1} - \min_j cv_{ij}\right) - f_{ij} \times \left(\sqrt{N-1} - cv_{ij}\right)}{max\, f_{ij} \times \left(\sqrt{N-1} - \min_j cv_{ij}\right) - min\, f_{ij} \times \left(\sqrt{N-1} - \max_j cv_{ij}\right)}\right] xdx \tag{41}$$

$$\text{Negative criterion} \sum_{i=1}^{n} \int_0^1 Beta\left(cv_{ij}|\max_j cv_{ij}\right) w_i \left[\frac{min\, f_{ij} \times \left(1 + \min_j cv_{ij}\right) - f_{ij} \times \left(1 + cv_{ij}\right)}{min\, f_{ij} \times \left(1 + \min_j cv_{ij}\right) - max\, f_{ij} \times \left(1 + \max_j cv_{ij}\right)}\right] xdx \tag{42}$$

while RSkW corresponds to the risk seeker case determined by:

$$S_j$$

$$\text{Positive criterion} \sum_{i=1}^{n} \int_0^1 Beta\left(\min_j cv_{ij}|cv_{ij}\right) w_i \left[\frac{max\, f_{ij} \times \left(1 + \max_j cv_{ij}\right) - f_{ij} \times \left(1 + cv_{ij}\right)}{max\, f_{ij} \times \left(1 + \max_j cv_{ij}\right) - min\, f_{ij} \times \left(1 + \min_j cv_{ij}\right)}\right] xdx \tag{43}$$

$$\text{Negative criterion} \sum_{i=1}^{n} \int_0^1 Beta\left(\min_j cv_{ij}|cv_{ij}\right) w_i \left[\frac{min\, f_{ij} \times \left(\sqrt{N-1} - \max_j cv_{ij}\right) - f_{ij} \times \left(\sqrt{N-1} - cv_{ij}\right)}{min\, f_{ij} \times \left(\sqrt{N-1} - \max_j cv_{ij}\right) - max\, f_{ij} \times \left(\sqrt{N-1} - \min_j cv_{ij}\right)}\right] xdx \tag{44}$$

In both these latter settings we have used the coefficients of variation presented in Table 1 but multiplied by ten in order to compute the corresponding Beta functions. Note that, otherwise, the low reference values provided by the coefficients of variation would lead to probability densities that differ significantly from the skewed ones described in Fig. 2. The resulting rankings are presented in Table 3.

Regarding the acceptability and stability of the results obtained, it can be easily verified that the rankings defined by Tavana, RAv and RAvW provide an acceptable compromise solution, since they satisfy the *Acceptable advantage* and the

Table 3
Rankings of the random alternatives per VIKOR method.

Alternative	Tavana	RAv	RAvN-1	Rsk	RAvW	RSkW
1	10	9	8	7	7	7
2	1	1	1	6	1	9
3	2	2	2	1	2	1
4	3	6	5	3	6	4
5	9	8	9	9	8	8
6	5	4	4	5	4	6
7	4	3	3	2	3	2
8	6	5	6	10	5	10
9	7	7	7	4	9	3
10	8	10	10	8	10	5

Table 4
Ranking correlations among the different RB-VIKOR methods.

Spearman's rho		Tavana	RAv	RAvW	RSk	RSkW
Tavana	Correlation Coefficient	1.000	.891**	.818**	.612	.261
	Sig. (2-tailed)	.	.001	.004	.060	.467
	N	10	10	10	10	10
RAv	Correlation Coefficient	.891**	1.000	.952**	.515	.127
	Sig. (2-tailed)	.001	.	.000	.128	.726
	N	10	10	10	10	10
RAvW	Correlation Coefficient	.818**	.952**	1.000	.442	.030
	Sig. (2-tailed)	.004	.000	.	.200	.934
	N	10	10	10	10	10
RSk	Correlation Coefficient	.612	.515	.442	1.000	.867**
	Sig. (2-tailed)	.060	.128	.200	.	.001
	N	10	10	10	10	10
RSkW	Correlation Coefficient	.261	.127	.030	.867**	1.000
	Sig. (2-tailed)	.467	.726	.934	.001	.
	N	10	10	10	10	10

**. Correlation is significant at the.01 level (2-tailed).

Acceptable stability in decision making conditions required by Opricovic & Tzeng [16]. That is, $Q_{2nd\ alternative} - Q_{1st\ alternative} \geq \frac{1}{10-1} = 0.11$ and the first alternatives take also the best scores among the S_j and R_j values. On the other hand, the RAvN-1, RSk and RSkW rankings do not satisfy the *Acceptable advantage* condition, as can be observed in Table 2. However, they all satisfy the *Acceptable stability in decision making* condition since their first alternatives take the best scores among either the S_j or R_j values, or both of them.

Note that the risk averse rankings and the one of Tavana select the second alternative as the most preferred one, given its lowest coefficient of variation for the first, second and fifth criterion. However, risk seeker DMs select the third alternative as the most preferred one. Note that this alternative exhibits both relatively high coefficients of variation throughout the criteria and consistently high average realizations. Indeed, as can be observed in Table 3, risk averse DMs rank the third alternative as the second most preferred one. However, the second alternative ranks considerably low among risk seeker DMs due to its low variability and despite the high average realizations achieved.

Table 4 reports the existing correlations among the different rankings. We observe how the risk seeker rankings differ substantially from the risk averse ones and the one of Tavana, which validates the differences observed in Fig. 3 where each alternative is represented on the horizontal axis and its corresponding ranking position on the vertical one. Thus, the risk attitude of the DMs has a direct effect on the resulting ranking of alternatives. Such an effect intensifies if the DMs condition their beliefs on the coefficients of variation when evaluating the corresponding alternatives.

6. Case study

Tavana et al. [25] compared their VIKOR model with the stochastic super-efficiency DEA one of Khodabakhshi et al. [10] by evaluating the performance of several bank branches. In this section, we illustrate how RB-VIKOR behaves when applied to the case study presented in Tavana et al. [25] and analyze the ranking differences arising from modifications in the risk attitudes of the DMs.

6.1. Data acquisition and RB-VIKOR implementation

Tavana et al. [25] selected 22 branches of the Peoples Bank[1] in West Virginia ($A_1, A_2, \ldots, A_{22}$) and chose seven criteria ($C_1, C_2, \ldots, C_7$) to evaluate them: suspicious receivables cost (C_1), personnel cost (C_2), capital cost (C_3), branch equipment

[1] The name was changed to protect the anonymity of the bank.

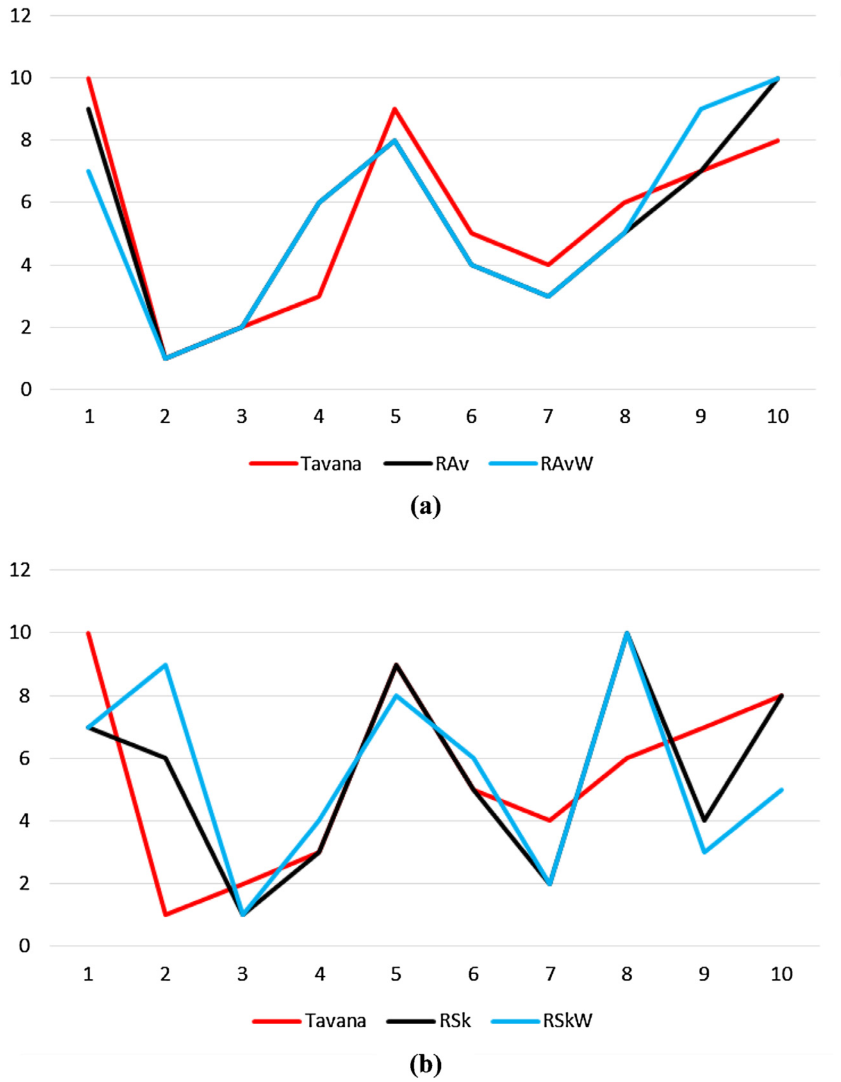

Fig. 3. RB-VIKOR rankings determined by the risk attitudes of the DMs.

cost (C_4), incomes (C_5), deposits (C_6) and banking facilities (C_7). The first four are negative criteria representing input variables while the last three are positive criteria corresponding to output variables. All criteria are measured in million dollar units. The data on the criteria per branch were collected over an eight weeks period.

Tavana et al. [25] verified the normality of the data retrieved and the resulting rating functions and coefficients of variation per branch and criterion are presented in Tables 5 and 6. Note that we are dealing with a risky environment, since the DMs are assumed to know that the data acquired follow a normal distribution. However, RB-VIKOR has been designed to account also for uncertain environments, where a uniform distribution - indicating maximum information entropy - could be assumed on the realizations of the data. A similar formal environment based on such an information structure, where the evaluations of the DMs are determined by the variability and entropy inherent to the data, is provided by Tavana et al. [24].

Tavana et al. [25] computed the weights assigned to each criterion by applying the extent analysis method to a fuzzy AHP framework used to evaluate the linguistic opinions of seven banking experts [23]. The resulting criteria weights are given by

$$W = (0.08671,\ 0.04667,\ 0.25633,\ 0.03724,\ 0.18773,\ 0.22411,\ 0.16121)$$

Given the information retrieved, the corresponding evaluations per branch and VIKOR method are presented in Table 7, whose entries correspond to the value of Q_j when $v = 0.5$. It should be noted that given the low value displayed by the coefficients of variation of the different alternatives, a reference value of one, instead of $\sqrt{N-1}$, has been applied to compute S_j and R_j in both risk settings for the positive and negative criteria.

Table 5
$(\bar{f}_{ij}, \boldsymbol{cv})$ of the negative criteria or inputs per alternative.

Branch Number	C_1		C_2		C_3		C_4	
	$\bar{X}$	cv	$\bar{X}$	cv	$\bar{X}$	cv	$\bar{X}$	cv
$\boldsymbol{A}_1$	0.183092	0.000065903	0.212985	1.24011E−05	0.235626	0.000113243	0.184808	0.000162925
$\boldsymbol{A}_2$	0.208596	7.08299E−05	0.230363	6.08144E−05	0.245014	7.14695E−05	0.241272	2.59317E−05
$\boldsymbol{A}_3$	0.187978	0.000156629	0.252706	4.93194E−06	0.209421	1.76645E−05	0.171436	6.26938E−05
$\boldsymbol{A}_4$	0.225785	3.56953E−05	0.166362	2.14093E−05	0.177091	6.60898E−05	0.234273	8.39646E−05
$\boldsymbol{A}_5$	0.25391	2.98975E−05	0.20528	1.29488E−05	0.223603	6.24711E−05	0.249573	5.52285E−05
$\boldsymbol{A}_6$	0.214122	5.98553E−06	0.177374	6.41937E−05	0.191373	0.000100643	0.164826	6.77854E−05
$\boldsymbol{A}_7$	0.21015	3.18718E−05	0.239554	0.000026188	0.187291	9.16315E−05	0.189364	7.77015E−05
$\boldsymbol{A}_8$	0.243531	0.000105899	0.221798	0.000024126	0.223191	0.000087938	0.234913	9.66622E−05
$\boldsymbol{A}_9$	0.181421	6.17473E−05	0.187905	3.84803E−05	0.214526	3.64076E−05	0.210105	4.21462E−05
$\boldsymbol{A}_{10}$	0.250902	1.82477E−05	0.165296	0.000146387	0.22845	8.40266E−05	0.198068	0.000154162
$\boldsymbol{A}_{11}$	0.156557	3.98902E−05	0.260051	5.74954E−05	0.214174	9.94717E−05	0.25599	9.93986E−06
$\boldsymbol{A}_{12}$	0.190552	3.79299E−05	0.196073	0.00011312	0.231224	0.00009193	0.222687	7.58071E−05
$\boldsymbol{A}_{13}$	0.253144	3.79491E−05	0.206956	8.06985E−05	0.224874	0.000136051	0.167009	8.48621E−05
$\boldsymbol{A}_{14}$	0.243138	5.10019E−05	0.180417	0.000104778	0.192666	0.000162396	0.251582	0.000076942
$\boldsymbol{A}_{15}$	0.202739	4.07969E−05	0.220127	4.42057E−05	0.186853	6.34488E−05	0.186592	3.07689E−05
$\boldsymbol{A}_{16}$	0.238221	2.85497E−05	0.254391	4.22661E−05	0.228408	6.22422E−05	0.19576	7.77746E−05
$\boldsymbol{A}_{17}$	0.192813	4.36255E−05	0.229055	0.000100119	0.175677	6.75783E−05	0.260848	2.45347E−05
$\boldsymbol{A}_{18}$	0.224648	5.34604E−05	0.201761	7.98192E−05	0.17529	7.81931E−05	0.209711	6.92383E−05
$\boldsymbol{A}_{19}$	0.195362	2.14551E−05	0.17204	5.52486E−05	0.213239	0.000088118	0.19405	8.61807E−05
$\boldsymbol{A}_{20}$	0.178847	0.000070373	0.240755	6.96634E−05	0.208716	0.000161973	0.258876	9.15553E−05
$\boldsymbol{A}_{21}$	0.23379	1.51622E-05	0.232072	2.87053E−05	0.251264	6.66647E−05	0.183353	5.70796E−05
$\boldsymbol{A}_{22}$	0.180214	4.28247E−05	0.194373	6.75136E−05	0.226479	7.09521E−05	0.171603	6.16828E−05

Table 6
$(\bar{f}_{ij}, \boldsymbol{cv})$ of the positive criteria or outputs per alternative.

Branch Number	C_5		C_6		C_7	
	$\bar{X}$	cv	$\bar{X}$	cv	$\bar{X}$	cv
$\boldsymbol{A}_1$	0.242661	1.98016E−05	0.17367	4.53261E−05	0.211735	1.34646E−05
$\boldsymbol{A}_2$	0.152329	2.62357E−05	0.220824	9.33601E−06	0.174535	2.03572E−05
$\boldsymbol{A}_3$	0.151927	1.46208E−05	0.166938	4.69472E−05	0.14435	3.64534E−05
$\boldsymbol{A}_4$	0.229268	1.86922E−05	0.29375	2.51355E−05	0.247675	2.12795E−05
$\boldsymbol{A}_5$	0.249618	2.05784E−05	0.205499	3.61937E−05	0.20643	0.000026886
$\boldsymbol{A}_6$	0.184324	1.35987E−05	0.1945	3.31699E−05	0.271635	0.000019659
$\boldsymbol{A}_7$	0.226834	2.49185E−05	0.251782	1.84688E−05	0.212217	1.49143E−05
$\boldsymbol{A}_8$	0.175298	1.50756E−05	0.167357	3.30199E−05	0.254375	1.07385E−05
$\boldsymbol{A}_9$	0.217959	0.000014327	0.15038	3.14595E−05	0.164785	3.47518E−05
$\boldsymbol{A}_{10}$	0.19223	2.53435E−05	0.179498	4.64395E−05	0.160314	3.30879E−05
$\boldsymbol{A}_{11}$	0.252283	1.67157E−05	0.250188	2.16852E−05	0.166095	3.99225E−05
$\boldsymbol{A}_{12}$	0.157571	1.87832E−05	0.149209	4.06469E−05	0.186626	2.92354E-−05
$\boldsymbol{A}_{13}$	0.198878	4.26909E−05	0.164063	6.27114E−05	0.129739	2.42102E−05
$\boldsymbol{A}_{14}$	0.177068	2.27906E−05	0.247196	1.66502E−05	0.221459	2.29682E−05
$\boldsymbol{A}_{15}$	0.240081	1.78092E−05	0.28428	1.01969E−05	0.204324	1.97875E−05
$\boldsymbol{A}_{16}$	0.264702	9.56793E−06	0.277888	1.91376E−05	0.228287	2.20661E−05
$\boldsymbol{A}_{17}$	0.152925	2.63126E−05	0.223313	1.08212E−05	0.279775	0.000009429
$\boldsymbol{A}_{18}$	0.273498	5.52225E−06	0.193881	1.83026E−05	0.264884	0.000028007
$\boldsymbol{A}_{19}$	0.264819	2.62788E−05	0.18166	4.41206E−05	0.189833	4.64215E−05
$\boldsymbol{A}_{20}$	0.198868	3.64654E−05	0.217204	2.84405E−05	0.19879	1.45533E−05
$\boldsymbol{A}_{21}$	0.241816	3.61768E−06	0.184588	8.06913E−06	0.232329	2.02833E−05
$\boldsymbol{A}_{22}$	0.160726	3.17916E−05	0.218817	5.60012E−05	0.251616	3.02976E−05

Regarding the acceptability and stability of the results obtained, it can be easily verified that the rankings defined by Tavana, RAv, Rsk and RSvW provide an acceptable compromise solution, since they satisfy the *Acceptable advantage* and the *Acceptable stability in decision making* conditions. In particular, $Q_{2nd\ alternative} - Q_{1st\ alternative} \geq \frac{1}{22-1} = 0.0476$ and the first alternatives take also the best scores among the S_j and R_j values. On the other hand, the RAvW ranking does not satisfy the *Acceptable advantage* condition. However, it satisfies the *Acceptable stability in decision making* condition since its first alternative takes also the best scores among the S_j and R_j values.

Finally, note that the relatively low values of the coefficients of variation and the ratings of the alternatives have led the rankings of Tavana, RAv and RSk to coincide. This result is modified when introducing the subjective expectations of the DMs in the RAvW and RSkW settings, where the coefficients of variation have been multiplied by 10^5 in order to generate the corresponding Beta functions.

Table 7
Q_j values of the alternatives per VIKOR method when $v = 0.5$.

Branch Number	Tavana	RAv	RAvW	RSk	RSkW
A_1	0.686622	0.686587	0.686109	0.686672	**0.331063**
A_2	0.940992	0.940984	0.638778	0.941031	**0.683534**
A_3	0.80481	0.804802	0.717687	0.804577	**0.740434**
A_4	2.72E-06	−1.4E−06	−9.6E−06	5.49E−07	**1.05E−06**
A_5	0.569357	0.569289	0.389213	0.569435	**0.389848**
A_6	0.418054	0.418042	0.320441	0.4179	**0.309847**
A_7	0.14139	0.141393	0.103318	0.141377	**0.253115**
A_8	0.758847	0.758865	0.606719	0.75866	**0.430667**
A_9	0.780726	0.780734	0.585999	0.780464	**0.542526**
A_{10}	0.748587	0.748519	0.724398	0.748632	**0.386974**
A_{11}	0.378786	0.378679	0.401204	0.378861	**0.243533**
A_{12}	0.914561	0.914575	0.832498	0.914307	**0.455948**
A_{13}	0.846516	0.846525	0.999998	0.846303	**0.452139**
A_{14}	0.448179	0.448185	0.374086	0.448039	**0.224627**
A_{15}	0.091787	0.091794	0.046766	0.091763	**0.180766**
A_{16}	0.475855	0.475814	0.272241	0.475947	**0.333395**
A_{17}	0.491389	0.49139	0.427951	0.491166	**0.409323**
A_{18}	0.291587	0.291581	0.116711	0.291439	**0.323510**
A_{19}	0.514841	0.514847	0.564292	0.514675	**0.221075**
A_{20}	0.427065	0.427055	0.501698	0.426989	**0.324579**
A_{21}	0.888267	0.888297	0.474796	0.888323	**1**
A_{22}	0.578245	0.578265	0.620331	0.578076	**0.278716**

Table 8
Ranking of alternatives: RB-VIKOR versus stochastic super-efficiency DEA.

Branch Number	Tavana	RAv	RAvW	RSk	RSkW	Stochastic DEA
A_1	14	14	18	14	11	**10**
A_2	22	22	17	22	20	**21**
A_3	18	18	19	18	21	**22**
A_4	1	1	1	1	1	**1**
A_5	12	12	8	12	14	**16**
A_6	6	6	6	6	8	**4**
A_7	3	3	3	3	6	**12**
A_8	16	16	15	16	16	**18**
A_9	17	17	14	17	19	**14**
A_{10}	15	15	20	15	13	**19**
A_{11}	5	5	9	5	5	**2**
A_{12}	21	21	21	21	18	**20**
A_{13}	19	19	22	19	17	**15**
A_{14}	8	8	7	8	4	**17**
A_{15}	2	2	2	2	2	**6**
A_{16}	9	9	5	9	12	**9**
A_{17}	10	10	10	10	15	**5**
A_{18}	4	4	4	4	9	**3**
A_{19}	11	11	13	11	3	**7**
A_{20}	7	7	12	7	10	**13**
A_{21}	20	20	11	20	22	**11**
A_{22}	13	13	16	13	7	**8**

6.2. Comparing RB-VIKOR with stochastic DEA

We compare now the rankings obtained using RB-VIKOR and a stochastic version of the super-efficiency DEA model introduced by Andersen and Petersen [1]. Stochastic super-efficiency models are designed to account for the fact that a decision making unit considered to be equally efficient to others may become inefficient when random variations in normally distributed inputs and outputs are introduced; refer to Section 3.2 in Khodabakhshi et al. [10] and Section 6 in Tavana et al. [25] for additional guidelines and intuition regarding these models. The rankings obtained from the different VIKOR methods and the stochastic super-efficiency DEA model of Khodabakhshi et al. [10] are presented in Table 8.

Intuitively, some differences are expected to arise across rankings since, in addition to the criteria weights defined by the experts and the subjective selection of the v parameter, RB-VIKOR accounts for the risk attitudes of the DMs. As Fig. 4 illustrates, these differences are particularly evident when the rankings derived from the RAvW and RSkW settings are considered. This is the case despite the fact that the different RB-VIKOR methods and the stochastic DEA coincide on the first alternative and the lower levels of the corresponding rankings do not differ significantly.

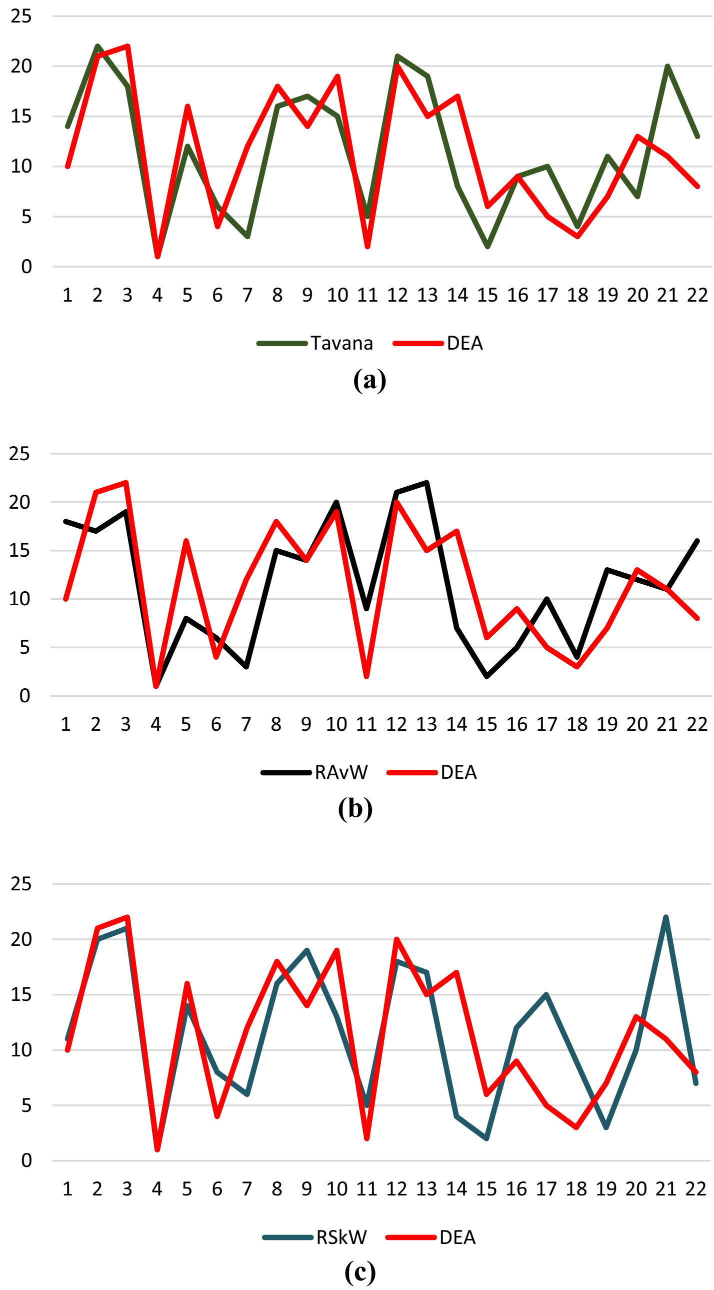

Fig. 4. Pairwise differences between the RB-VIKOR and stochastic DEA rankings.

In other words, the subjective variability introduced in RB-VIKOR distorts what may be considered as the neutral evaluation of DEA, which focuses on the distributional properties of the realizations when computing the efficiency of the alternatives but does not account for subjective judgments or risk attitudes. The existing correlations among the different rankings are presented in Table 9 and confirm the differences between the RB-VIKOR methods, particularly RAvW and RSkW, and the stochastic DEA one. Therefore, if the DMs face uncertainty and do not know the exact distribution from which the observations are being drawn or are not neutral to the inherent risk, their resulting rankings will differ from the one provided by the super-efficiency DEA model.

7. Incorporating the relative importance of the criteria

RB-VIKOR has been defined using the values of the rating functions and their corresponding coefficients of variation. The numerical example introduced in Section 5.3 illustrates the role played by the risk attitudes of DMs in the adjustment of

Table 9
Ranking correlations among the RB-VIKOR methods and stochastic DEA.

Spearman's rho		RAvW	Tavana	RSkW	DEA
RAvW	Correlation Coefficient	1.000	.857**	.633**	.658**
	Sig. (2-tailed)	.	.000	.002	.001
	N	22	22	22	22
Tavana	Correlation Coefficient	.857**	1.000	.860**	.735**
	Sig. (2-tailed)	.000	.	.000	.000
	N	22	22	22	22
RSkW	Correlation Coefficient	.633**	.860**	1.000	.651**
	Sig. (2-tailed)	.002	.000	.	.001
	N	22	22	22	22
DEA	Correlation Coefficient	.658**	.735**	.651**	1.000
	Sig. (2-tailed)	.001	.000	.001	.
	N	22	22	22	22

**. Correlation is significant at the.01 level (2-tailed).

the limit values of the rating functions relative to those defined by Tavana el al. [25]. Risk attitudes were shown to affect the resulting rankings without resorting to modifications in the subjective beliefs of the DMs. At the same time, the Beta densities, built on the ratings received and their dispersion, reinforce the effect of the risk attitudes of the DMs through their beliefs. The case study introduced in Section 6 emphasizes the significant role played by expectations when facing very low coefficients of variation for all the alternatives and criteria.

Note that the case study presented in Section 6 takes the weights of the criteria as given by a group of experts, while Section 5.3 assumes identical weight values for each criterion. However, recent research on decision analysis and operations research has explicitly highlighted the importance of the weights subjectively assigned by the DMs to the different decision criteria [11,19]. Thus, we incorporate the relative importance of the criteria weights when defining the subjective beliefs of the DMs and provide intuition regarding their potential effects on the resulting evaluations and rankings.

7.1. The risk averse setting

We extend the definition of S_j to incorporate the relative importance of the weights of the criteria, $\frac{w_i}{w_{max}}$, with $w_{max} = \max\{w_i,\ i = 1, \ldots, n\}$, when determining the shape of the Beta density. For consistency purposes, it will be assumed that less important criteria lead the DMs to assign a lower weight to the variability of the corresponding alternatives when forming their expectations. The values of S_j, $j = 1, 2, 3, \ldots, m$, for a positive criterion are therefore given by

$$S_j = \sum_{i=1}^{n} \int_0^1 \frac{Beta\left(\bar{f}_{ij}\middle| \left[f_{ij} + \left[1 + \left(\frac{w_i}{w_{max}}\right)\right]\left(max_j\, cv_{ij} - cv_{ij}\right)\right]\right) w_i}{\left[\dfrac{max\, f_{ij} \times \left(\sqrt{N-1} - min_j\, cv_{ij}\right) - f_{ij} \times \left(\sqrt{N-1} - cv_{ij}\right)}{max\, f_{ij} \times \left(\sqrt{N-1} - min_j\, cv_{ij}\right) - min\, f_{ij} \times \left(\sqrt{N-1} - max_j\, cv_{ij}\right)}\right]} xdx \tag{45}$$

$$S_j = \sum_{i=1}^{n} \int_0^1 \frac{Beta\left(cv_{ij}\middle| max_j\, cv_{ij}\left[1 + \left(\frac{w_i}{w_{max}}\right)\right]\right) w_i}{\left[\dfrac{max\, f_{ij} \times \left(\sqrt{N-1} - min_j\, cv_{ij}\right) - f_{ij} \times \left(\sqrt{N-1} - cv_{ij}\right)}{max\, f_{ij} \times \left(\sqrt{N-1} - min_j\, cv_{ij}\right) - min\, f_{ij} \times \left(\sqrt{N-1} - max_j\, cv_{ij}\right)}\right]} xdx \tag{46}$$

It has been assumed that $\gamma = [1 + (\frac{w_i}{w_{max}})]$ so as to focus on the variability of the ratings in both density scenarios, though alternative definitions of beliefs based on $\frac{w_i}{w_{max}}$ could be considered. In this regard, note how $\frac{w_i}{w_{max}}$ determines the relative importance assigned to the variability of the criteria when defining the subjective beliefs of the DM. As a result, the right hand side of the conditional terms defining the Beta densities increases with the relative importance of the criteria. Recall that the Beta functions must be skewed towards the lower end of their domains in order to provide an improved expected evaluation to the DM for the more preferred alternatives.

7.2. The risk seeker setting

The same intuition applies when considering the risk seeker scenario. Thus, if a criterion is relatively important, the DM increases the weight assigned to the variability of the corresponding alternative. The values of S_j, $j = 1, 2, 3, \ldots, m$, for a negative criterion are therefore given by

$$S_j = \sum_{i=1}^{n} \int_0^1 \frac{Beta\left(\bar{f}_{ij} \middle| \left[f_{ij} + \left[1 + \left(\frac{w_i}{w_{max}}\right)\right] \left(cv_{ij} - min_j\, cv_{ij}\right)\right]\right) w_i}{\left[\frac{min\, f_{ij} \times \left(\sqrt{N-1} - max_j\, cv_{ij}\right) - f_{ij} \times \left(\sqrt{N-1} - cv_{ij}\right)}{min\, f_{ij} \times \left(\sqrt{N-1} - max_j\, cv_{ij}\right) - max\, f_{ij} \times \left(\sqrt{N-1} - min_j cv_{ij}\right)}\right] xdx} \tag{47}$$

$$S_j = \sum_{i=1}^{n} \int_0^1 \frac{Beta\left(min_j\, cv_{ij} \middle| cv_{ij} \left[1 + \left(\frac{w_i}{w_{max}}\right)\right]\right) w_i}{\left[\frac{min\, f_{ij} \times \left(\sqrt{N-1} - max_j\, cv_{ij}\right) - f_{ij} \times \left(\sqrt{N-1} - cv_{ij}\right)}{min\, f_{ij} \times \left(\sqrt{N-1} - max_j\, cv_{ij}\right) - max\, f_{ij} \times \left(\sqrt{N-1} - min_j cv_{ij}\right)}\right] xdx} \tag{48}$$

We conclude by noting that there are several sources of subjectivity inherent to the design of the RB-VIKOR model, such as the inclusion of γ and w_i on the beliefs and resulting evaluations of the DMs. As already stated, alternative evaluation scenarios could be defined, with the current one focusing on the effects that the ratings, their variability and the risk attitudes of DMs have on their beliefs and resulting evaluations.

8. Conclusion

We have defined an extended version of the VIKOR method introduced by Tavana et al. [25] that accounts for differences in the risk attitudes of the DMs when ranking stochastic alternatives. In particular, our formal framework allows for modifications in the rating behavior of the DMs that depend on whether they are risk seekers or averters and the coefficient of variation exhibited by each alternative. That is, RB-VIKOR allows the DMs to select the alternative that is more in accordance with their subjective preferences and risk attitudes while accounting for the uncertainty inherent to the real world when evaluating each alternative.

Therefore, if the DMs do not know the exact distribution from which the observations are being drawn or are not neutral to the inherent risk, their resulting rankings will differ from the ones provided by more neutral models such as super-efficiency DEA. Indeed, the proposed method is sufficiently flexible so as to account for both risky and uncertain environments and can be formally extended to other MCDM techniques such as PROMETHEE or TOPSIS.

Potential extensions of the current framework should consider the introduction of expected evaluations whose domain is determined by the entropy inherent to the data, strategic environments whose information structure is conditioned by the credibility of the reporters providing the data, and the emergence of interdependencies among the different decision criteria and their effect on the evaluations of the DMs.

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