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Information acquisition processes and their continuity: Transforming uncertainty into risk

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ABSTRACT

We propose a formal approach to the problem of transforming uncertainty into risk via information revelation processes. Abstractions and formalizations regarding information acquisition processes are common in different areas of information sciences. We investigate the relationships between the way information is acquired and the continuity properties of revelation processes. A class of revelation processes whose continuity is characterized by how information is transmitted is introduced. This allows us to provide normative results regarding the continuity of the information acquisition processes of decision makers (DMs) and their ability to formulate probabilistic predictions within a given confidence range.

1. Introduction

In the context of decision-making under risk, each outcome occurs with a known probability, that is, the probability function defined on a given set of outcomes S is assumed to be known to the decision maker (DM). In the realm of decision-making under uncertainty, the DM faces a set of possible outcomes, each of them happening with an unknown probability (see, e.g., [21]).

In general, decision theory studies uncertainty and risk separately, e.g., [13]. We study the problem of how an uncertain situation may be transformed into a risky one. Moving from an uncertain situation to a risky one implies that some information must be revealed in the process. However, information theory concentrates mainly on the stochastic properties of information without considering the information revelation/acquisition process as a topological object, e.g., [5].

We propose a novel purely set-theoretical and topological approach to information revelation through time that allows a DM to move from an initial situation of uncertainty to a final situation of risk. In our framework, *revealing (or transmitting) information through time* means providing the DM with the information necessary to determine the true probability (probability

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mass or probability density according to the nature of the set of outcomes) of each outcome. Throughout the paper, the theoretical analysis will concentrate on the *revelation* perspective of the *information* process. However, when presenting the managerial interpretation of the results we will consider the *information acquisition* perspective of the DM. Both approaches are equivalent within the current environment. That is, the object of study is the process of information transmission, which can be analyzed either from the perspective of the transmitter or that of the receiver (the DM). In our setting, the properties of the process studied do not depend on the perspective considered.

Letting $T = [0,1]$ represent the time interval during which information is revealed, we will consider revelation processes with the following characteristics. At time 0, no information is available to the DM, who faces the highest uncertainty: any probability function definable on S could be the true one. As time goes by, the DM receives additional information until a certain time $t \leqslant 1$ when all the information is revealed. Clearly, the more information that is revealed the less uncertainty is faced by the DM. As the amount of information on the true probability function increases, the DM excludes more and more elements from the set of all possible probability functions on S. At time $t \leqslant 1$, when all the information becomes available, the DM is left only with the true probability function. Thus, information being fully revealed is equivalent to uncertainty being completely transformed into risk.

To formalize the proposed problem we define information revelation processes by means of multifunctions from the time interval T to the set PR of all the probability functions defined on S. A revelation process is continuous if it satisfies both lower and upper semi-continuity as a multifunction. No restriction is imposed initially on when and how information is transmitted and the full information revealed. The main objective of the paper is to investigate the relationship between when and/or how information is transmitted and the continuity properties of revelation processes.

We show that the continuity – both lower and upper semi-continuity – of revelation processes is determined by how information is transmitted through time and not by when it is transmitted. In this regard, we introduce a class of revelation processes whose continuity is characterized by how information is transmitted. These revelation processes present a desirable and natural contraction-like pattern towards the true probability function. Finally, we characterize a subclass of these revelation processes satisfying a useful reflecting property.

The main results are intuitively introduced through several examples explicitly discussing different information transmission situations. These examples demonstrate the applications of our framework to management and information sciences.

1.1. Theoretical motivation

The last decades have witnessed a significant amount of effort on theoretical and practical explorations of intelligent techniques and soft computing that deal with the uncertainty inherent in scientific, engineering and business decision-making processes. Information acquisition has become an important multi-disciplinary domain of information sciences receiving increasing attention (see [6,25] for special issues on this research area and reviews of the corresponding literature).

When dealing with an abstract intangible object such as the flow of information and its acquisition, a formal approach leads naturally to the fields of topology and set theory, see [31,14]. In particular, the formal modelization of uncertain and partial information environments within the information sciences builds on the developments of fuzzy set theory [39,38], possibility theory [9,40], probability and belief structures [30,35,36], and rough set theory [28,37,38]. Generally, the convergence properties of uncertain series are studied from a measure-theoretical perspective, e.g., [20,34]. In the same way, the complex interactions defining the dynamics of information acquisition processes have been analyzed to measure theoretical terms [19]. In this paper, we take the convergence of the process as given and concentrate on its continuity based on the information acquisition behavior of the DMs.

We follow a topological/set-theoretical approach to study the process of information acquisition within an initially uncertain environment. Our approach differs significantly from the standard rough set theoretical and information theoretical ones, summarized by [7,20], and has been chosen for two main reasons. First, it allows us to characterize information as a tractable object whose properties may be defined and analyzed. Second, we are able to derive properties that relate directly to the information acquisition behavior of DMs. In particular, the characterization of the information acquisition process as a topological object allows us to study the effects of its continuity in making reasonably sound decisions. The main result of the current paper, besides the formal characterization of information processes, is how the way DMs acquire information and learn about an initially uncertain variable has important consequences regarding their ability to make accurate probabilistic judgments. That is, if the information acquisition process does not satisfy a key continuity condition [refer to the reflecting Property ($\triangleright\triangleleft$) in Section 6], the ability of DMs to formulate probabilistic predictions within a given confidence range can only be guaranteed in the limit time defining the process. As a result, DMs may continue acquiring information after being able to generate sufficiently accurate judgments, which is suboptimal both in behavioral and information acquisition terms.

Finally, our topological interpretation of an information revelation process as a multifunction H from the time interval T to the set PR of all the probability functions defined on S allows for both: its interpretation in rough set-theoretical terms and potential fuzzy set-theoretical expansions. Refer, in particular, to [38] for a description of the main differences between both theories. More precisely, rough sets could be used to describe the indiscernibility among the probability functions in the image sets of our revelation processes, while the use of fuzziness would constitute a generalization of the dynamical evolution of these image sets. We will further elaborate on these potential extensions in the conclusion section.

1.2. Managerial interpretation

Even though the model introduced in this paper is formal, it provides important intuitive results regarding the continuity of the information acquisition processes of DMs and their ability to formulate probabilistic predictions within a given confidence range. The implications derived from our results in terms of information management are varied and important.

Information is a critical input into any decision process. For example, a positive correlation has been found between business performance and the practice of decision-making, e.g., [22]. Similarly, [11] illustrate how the information feed to project managers significantly influences the strategic value created by megaprojects. In this regard, many project managers presume their decision-making capabilities to be above average, e.g., [23], and consequently do not care about improving their quality, e.g., [15]. This attitude could very well influence the rigor of their approach in seeking information in support of decision-making and can potentially result in wrong judgments that could have been prevented. Consider the case of information technology (IT) professionals, who rely on technological knowledge renewal to stay effective on their jobs. Rong and Grover [29] show that the renewal effectiveness of IT professionals is influenced by their career orientation, perceived IT dynamism, tolerance of ambiguity, and delegation. In other words, the [highly subjective] information acquisition strategies of DMs have a direct and relevant effect on the evolution and performance of the projects they manage.

Information acquisition involves searching both external, e.g., [4], and internal, e.g., [33], environments to identify important events that could affect an organization and its objectives. The quality and quantity of information available to DMs in business organizations has been found to correlate with the quality of their decisions, e.g., [26]. In this sense, top DMs, who usually have access to far more information than they can deal with, see [24], must become selective in favor of the information they consider to be most useful. Davis and Tuttle [8] highlight the fact that individuals change their dynamic information processing strategies to reduce the costs of cognitive effort rather than maximize accuracy. Similarly, Ghani and Lusk [12] measured the impact of presentation format and amount of information on decision performance and found that after an increase in the volume of information the initial performance deteriorated, with or without a change in the presentation format. Thus, even though DMs using more information tend to be more comfortable in dealing with ambiguity and uncertainty, e.g., [32], the acquisition process presents highly subjective biases.

The setting and results introduced in this paper may easily relate to the design of expert systems. This branch of the literature has readily emphasized the importance of the quality of the interactions between the DM and the computer information system, e.g., [18], and their positive effects in the form of persistently better and faster decision-making, e.g., [27]. In this regard, given the dynamic nature defining information acquisition, the design of information systems must account for time pressure and its effects on the information acquisition process of DMs. For example, information timeliness has been shown to condition the benefits derived from implementing decisions based upon the information acquired, e.g., [16]. Moreover, when resources are limited, strategies requiring a high degree of analysis may be excluded from consideration. This type of behavior may lead to the selection of suboptimal [underperforming] strategies, e.g., [2] for an early discussion of this problem and [17] for a review of the literature. At the individual level, research has shown that as the time and effort required to complete a task increases, DMs tend to reduce information search at the expense of decision quality, e.g., [1,10]. Thus, environmental characteristics influence the way information is acquired, which has a substantial effect on determining the behavior of DMs and the performance of the projects they manage.

When shifting among different information sources, our continuity condition [reflecting property] implies that DMs must complete the information acquisition process from a given source or project before shifting to another one. The time of conclusion must be clearly established, while the shift to a different project providing additional information should be adaptive, with DMs gradually joining the new flow of information and adapting to its characteristics. Violating this continuity requirement can be interpreted in the exact opposite way, with DMs leaving a project incomplete as they start acquiring information from a new source. Therefore, starting multiple projects or acquiring information from multiple sources without a clear completion target in mind may lead to suboptimal results when making judgments based on the information acquired. This is particularly important when DMs face time-constrained environments where their information acquisition behavior is volatile and subject to environmental influences.

2. Preliminaries and notations

Let S be the set of all possible realizations of a fixed real random variable defined on the σ-algebra of all subsets of a given set of alternatives. Following the standard approach to probability theory, we allow for the random variable to be either absolutely continuous or discrete. As a consequence, S can be identified either with a real subinterval – when the random variable is absolutely continuous – or with a real discrete (finite or countably infinite) subset – if the random variable is discrete. The generic element of S will be denoted by i.

Let PR denote the set of all probability functions defined on S. Depending on the nature of the set of realizations S, the *probability function* may indicate a probability mass function or a probability density function. Thus, PR is the set of all functions $x \in [0,1]^S$, either discrete or piecewise continuous, such that either $\sum_{i \in S} x(i) = 1$ or $\int_S x(i)di = 1$.

As usual, given $\alpha, \beta \in \Re$, with $\alpha < \beta$, the symbols (α, β), $(\alpha, \beta]$, $[\alpha, \beta)$, $[\alpha, \beta]$ will denote the open, the left half-open, the right half-open and the closed interval of end-points α and β, respectively.

The unit interval $[0,1]$ will play two different roles: (1) the range of a probability function x defined on S; and (2) the time interval where the transmission of information happens. To avoid confusion, $[0,1]$ will be denoted by T whenever it represents the time interval and the letters r, s, t will be used to denote time instants. In both cases, the real interval $[0,1]$ will be assumed to be endowed with the induced Euclidean topology. Moreover, PR will be assumed to be endowed with the subspace topology induced by the product topology on $[0,1]^S$.

Finally, we need to recall a few facts about multifunctions. Given two nonempty sets A and B, a *multifunction*, or *set-valued function*, from A to B is a map assigning to each element of A a (possibly empty) subset of B [3, Chapter 6].

A multifunction H from A to B is usually denoted by $H : A \overset{\rightarrow}{\longrightarrow} B$. Also, for every $U \subseteq B$, $H^{-1}(U) = \{a \in A: H(a) \cap U \neq \emptyset\}$ is the *inverse image* of U under H.

A multifunction $H : A \overset{\rightarrow}{\longrightarrow} B$ is said to be *lower semi-continuous* (LSC) if $H^{-1}(U)$ is open in A, whenever U is open in B. H is said to be *upper semi-continuous* (USC) if $\{a \in A: H(a) \subseteq U\}$ is open in A, whenever U is open in B.

The following result is well known (see, e.g., [3, Theorem 6.2.5]).

Proposition 2.1. *Given two Hausdorff spaces A and B and a multifunction $H : A \overset{\rightarrow}{\longrightarrow} B$, the following are equivalent:*

(a) *$H : A \overset{\rightarrow}{\longrightarrow} B$ is USC;*
(b) *$\forall U$ closed in B, $H^{-1}(U)$ is closed in A.*

3. Revelation processes through time

Given the time interval T, we provide a natural description for information revelation processes such that no information is available at time 0 and full information is available at time 1. In the present framework, to reveal full information means to provide the DM with the true probability function, denoted by x^*, on the set of outcomes S. The DM collects an increasing amount of partial information until x^* becomes known. As partial information increases, the set of probability functions that the DM considers as possible candidates for the true one decreases. When full information is finally revealed, there will be only one probability function left, the true one. The following definition provides a natural, but quite accurate formalization of the process just described.

Definition 3.1. Let $x^* \in PR$ and $H : T \overset{\rightarrow}{\longrightarrow} PR$. We say that H is a *revelation process through time for x^** (RP for x^*)} if:

(1) $H(0) = PR$,
(2) $H(1) = \{x^*\}$,
(3) $\forall r, s \in T$, $r > s \Rightarrow H(s) \subseteq H(r)$.

For every $\forall r \in T$, the set $H(r)$ represents the set of the probability functions on S that, at time r and on the base of the available partial information, are still believed to be possible candidates for the true probability function.

Properties (1)–(3) formalize the display of increasing information and the revelation process through time. Property (1) says that at time 0 there is no information available and any $x \in PR$ could be the true probability function. Property (2) indicates that at time 1, when full information is available, there must be no more uncertainty, that is, the true probability function must be known. Finally, Property (3) expresses the fact that the set of possible candidates for the true probability function becomes smaller through time.

Definition 3.1 does not impose any restriction on "when" new information is transmitted. That is, new information does not need to be revealed instant by instant and there are not particular prearranged instants of time at which new information is revealed. Furthermore, between time 0 and time 1, the images $H(r)$ do not need to be different from PR or the singleton x^*. In fact, the "no information" stage could persist from time 0 until any instant $t < 1$, while information could be fully revealed before time 1. Finally, sudden revelations of full information are allowed. This means that there could be a time instant $t \leqslant 1$ at which the DM moves suddenly from the no information stage to the full information one.

Similarly, Definition 3.1 does not impose any restriction on "how" new information is transmitted. That is, the image set of an RP H is not in general constrained to satisfy other properties other than being a decreasing sequence. The shape and the rate of decrease of the image sets are not predetermined.

The following definition provides a classification of revelation processes based on when new information is transmitted.

Definition 3.2. Let $x^* \in PR$ and $H : T \overset{\rightarrow}{\longrightarrow} PR$ be a RP for x^*. We say that the information of the RP H is

- *perpetually transmitted on the time subinterval* $[\alpha, \beta] \subseteq T$ if
$$\forall r, s \in [\alpha, \beta], \quad r \neq s \iff H(r) \neq H(s);$$
- *perpetually transmitted through time* if it is perpetually transmitted on T;
- *sequentially transmitted on the time subinterval* $[\alpha, \beta] \subseteq T$ if either $\exists t_1, \ldots, t_n \in [\alpha, \beta]$, with n a natural number and $\alpha = t_1 < \cdots < t_n = \beta$, such that
$$\forall i = 1, \ldots, n-1, \quad H(t_{i+1}) \subset H(t_i) \wedge \forall s \in (t_i, t_{i+1}), \quad H(s) = H(t_i)$$

or $\exists\{t_n\}_{n\geqslant 0}\subset[\alpha,\beta]$, with $t_0=\alpha$ and $t_n<t_{n+1}$, such that

$$\forall n\geqslant 0,\quad H(t_{n+1})\subset H(t_n)\ \wedge\ \forall s\in(t_n,t_{n+1}),\quad H(s)=H(t_n);$$

- *sequentially transmitted through time* if it is sequentially transmitted on T;
- *suddenly revealed*, if $\exists r\in(0,1]$ such that $H(r)=\{x^*\}\wedge\forall s\in[0,r)$, $H(s)=PR$.

Note that sequential transmissions (either at a finite or at a denumerable number of instants) are cases of non-perpetual transmission. Clearly, a case of sudden revelation can be considered as a particular instance of sequential transmission.

Definition 3.2 does not exhaust all information transmission types. Moreover, combinations and variants of the listed cases are possible and some will be considered in the examples.

Providing a classification of revelation processes based on how new information is transmitted constitutes a much more complicated issue due to the generality of the definition we proposed for revelation processes (Definition 3.1). Therefore, we will concentrate on designing a particular class of revelation processes consistent with the main objectives of the paper.

One question arises immediately: how does the way information is transmitted (when and how) relate to the continuity properties of the revelation process considered as a multifunction?

The following definition introduces lower and the upper semi-continuity in our framework.

Definition 3.3. Let $x^*\in PR$ and $H:T\overset{\rightarrow}{\longrightarrow}PR$ be a RP for x^*. We say that H is a

- *continuous time revelation process for* x^* (CTRP for x^*)} if it is LSC and USC;
- *discontinuous time revelation process for* x^* (DTRP for x^*)} if it is not a CTRP (that is, it is either not LSC or not USC).

Suppose that H is a RP for x^*. Then:

$$\forall\wp\subseteq PR,\quad x^*\in\wp\iff H^{-1}(\wp)=[0,1]. \tag{1}$$

In other words, preimages of subsets of PR containing the true probability function are trivially open and closed. Thus, to determine if H is LSC or USC it suffices to analyze the preimages of open or closed subsets of PR not containing x^*.

Furthermore, given $\wp\subseteq PR$, the set $\{r\colon H(r)\subseteq\wp\}$ is either empty (when $x^*\notin\wp$) or contains the time instant 1 (when $x^*\in\wp$). Therefore, to determine if H is USC it suffices to verify that $\{r\colon H(r)\subseteq\wp\}$ is open for every $\wp$ open subset containing x^*.

These remarks yield the following proposition.

Proposition 3.4. *Let H be a RP for x^*.*

(a) *H is LSC if and only if for every $\wp$ open in PR, either $H^{-1}(\wp)=[0,1]$ (when $x^*\in\wp$) or there exists $s\in(0,1)$ such that $H^{-1}(\wp)=[0,s)$ (when $x^*\notin\wp$).*
(b) *H is USC if and only if for every $\wp$ open in PR, either $\{r\colon H(r)\subseteq\wp\}=\varnothing$ (when $x^*\notin\wp$) or there exists $s\in(0,1)$ such that $\{r\colon H(r)\subseteq\wp\}=(s,1]$ (when $x^*\in\wp$).*

Statement (b) of Proposition 3.4 can also be stated as follows:

(b′) H is USC if and only if for every $\wp$ closed in PR, either $H^{-1}(\wp)=[0,1]$ (when $x^*\in\wp$) or there exists $s\in[0,1)$ such that $H^{-1}(\wp)=[0,s]$ (when $x^*\notin\wp$).

4. Examples of continuous and discontinuous time revelation processes

We analyze several examples of revelation processes with different degrees of continuity.

Example 1. Fix $x^*\in[0,1]^S$. Let $H:T\overset{\rightarrow}{\longrightarrow}PR$ be defined as follows:

$$H(r)=\begin{cases}PR & \text{if } 0\leqslant r<1,\\ x^* & \text{if } r=1.\end{cases} \tag{2}$$

The multifunction H clearly satisfies (1)–(3) of Definition 3.1, hence it is a RP for x^*.

Furthermore, H is LSC but not USC To see this, let $\wp$ be a nonempty subset of PR not containing x^* and note that $H^{-1}(\wp)=[0,1)$. Thus, $H^{-1}(\wp)$ is open whenever $\wp$ is open. Similarly, all nonempty closed subset of PR not containing x^* have an open preimage. Thus, H is a DTRP for x^* (Definition 3.3).

Example 2. Fix $x^*\in[0,1]^S$. Let $H:T\overset{\rightarrow}{\longrightarrow}PR$ be defined as follows:

$$H(r)=\begin{cases}PR & \text{if } r=0,\\ x^* & \text{if } 0<r\leqslant 1.\end{cases} \tag{3}$$

The multifunction H clearly satisfies (1)–(3) of Definition 3.3, hence it is a RP for x^*.

At the same time, H is USC but not LSC. Indeed, for every nonempty subset $\wp$ of PR not containing x^* we have $H^{-1}(\wp) = \{0\}$. Thus, $H^{-1}(\wp)$ is closed whenever $\wp$ is closed, but all nonempty open subset of PR not containing x^* still have a closed preimage. As in Example 1, H is a DTRP for x^* (Definition 3.3).

Both examples above, one being the dual of the other, can be generalized so as to allow for situations where the full revelation happens suddenly at a certain intermediate time instant t.

Example 3. Fix $x^* \in [0,1]^S$. For every $t \in T$, $t \neq 0$, let $H_t : T \xrightarrow{\rightarrow} PR$ be defined as follows:

$$H_t(r) = \begin{cases} PR & \text{if } 0 \leqslant r < t, \\ x^* & \text{if } t \leqslant r \leqslant 1. \end{cases} \tag{4}$$

For every $t \in T$, $t \neq 1$, let $H^t : T \xrightarrow{\rightarrow} PR$ be defined as follows:

$$H^t(r) = \begin{cases} PR & \text{if } 0 \leqslant r \leqslant t, \\ x^* & \text{if } t < r \leqslant 1. \end{cases} \tag{5}$$

Using the same reasoning as in Examples 1 and 2, it is easy to confirm that all the multifunctions of the form H_t are LSC but not USC, while all those of the form H^t are USC but not LSC.

The lack of LSC or USC for the multifunctions of Examples 1–3 is due to the fact that information is transmitted only once through the sudden release of all the information required to reveal x^*.

Note that the fact that S may be the realization set of either an absolutely continuous or a discrete random variable does not play a role in determining whether the RP's in these examples are lower or upper semi-continuous.

Example 4. Let S consist of two elements. In this case, PR is homeomorphic to the anti-diagonal D of the square $[0,1]^2$ endowed with the induced Euclidean topology. Let x^* be the point $\langle 1,0\rangle$, that is, the degenerate probability mass function whose support is one of the elements of S.

Define $H : T \xrightarrow{\rightarrow} PR$ as follows:

$$H(r) = \begin{cases} PR & \text{if } r = 0, \\ \{\langle t, 1-t\rangle : t \geqslant r\} & \text{if } 0 < r < 1, \\ \langle 1,0\rangle & \text{if } r = 1. \end{cases} \tag{6}$$

Clearly, H is an RP for x^* (Definition 3.1). We show now that H is LSC and USC; thus, H is a CTRP.

Let U be an open subset of D. By Proposition 3.4(a), we can assume that $\langle 1,0\rangle \notin U$. Let $s = \sup \pi_1(U)$, where π_1 is the projection map on the first coordinate space. Since $\pi_1(U)$ is an open subset of $[0,1]$, $s \notin \sup \pi_1(U)$. It is easy to check that the preimage of U is open; in fact, $H^{-1}(U) = \{r\text{: } H(r) \cap U \neq \emptyset\} = [0,s)$. Hence, H is LSC.

To show that H is USC, by Proposition 3.4(b), consider an open subset U of D containing $\langle 1,0\rangle$. Let w be the infimum of the t's such that the interior of the line segment joining the points $\langle t,\ 1-t\rangle$ and $\langle 1,0\rangle$ is contained in U. Thus, $\{r\text{: } H(r) \subseteq U\} = (w,1]$ is an open subset of $[0,1]$ and H is USC.

Example 5. Fix $x^* \in [0,1]^S$ and define $H : T \xrightarrow{\rightarrow} PR$ as follows:

$$H(r) = \begin{cases} PR & \text{if } r = 0, \\ \{y \in PR : \quad \forall i \in S, \quad |y(i) - x^*(i)| \leqslant 1 - r\} & \text{if } 0 < r \leqslant 1. \end{cases} \tag{7}$$

Clearly, H is a RP for x^*. We claim that H is LSC and USC.

To prove that H is LSC, fix a basic open subset $\wp$ of PR not containing x^* (see Proposition 3.4(a)) and find $H^{-1}(\wp)$.

Recall that a basic open subset $\wp$ of PR is of the form $\left(\prod_{i\in F} V_i \times \prod_{i\notin F} [0,1]\right) \cap PR$, where either F is a finite subset of S if S is infinite or $F = S$ if S is finite, and each V_i is an open subset of the ith factor of the Cartesian product $[0,1]^S$.

Since $x^* \notin \wp$, there exist some i's in F such that $x^*(i) \notin \pi_i(\wp)$, where π_i is the projection map on the i-th coordinate space. For each of such i's, let $s_i = \sup\{t : \pi_i(\wp) \cap [x^*(i) - 1 + t, x^*(i) + 1 - t] \neq \emptyset\}$. Finally, let $s = \min\{s_i : i \in F \wedge x^*(i) \notin \pi_i(\wp)\}$; hence, $H^{-1}(\wp) = [0,\ s)$ is an open subset of $[0,1]$ and H is LSC.

To prove that H is USC, fix a basic open subset $\wp$ of PR containing x^* (see Proposition 3.4(b)). Thus, there exists a finite subset F of S such that for all $i \in F$, $x^*(i) \in \pi_i(\wp)$. Let $w_i = \inf\{t : [x^*(i) - 1 + t, x^*(i) + 1 - t] \subseteq \pi_i(\wp)\}$ and $w = \max\{w_i\text{: } i \in F\}$.

Then, the set $\{r\text{: } H(r) \subseteq \wp\} = (w,1]$ is an open subset of $[0,1]$ and H is USC.

An interesting variant of Example 5 is presented in the following example. In this example, the images of H still contract smoothly towards x^*, but these contractions involve only an initial segment of the set of possible coordinates at each instant r.

Example 6. Fix $x^* \in [0,1]^S$ and define $H : T \xrightarrow{\rightarrow} PR$ as follows:

$$H(r) = \begin{cases} PR & \text{if } r = 0, \\ \{y \in PR : \quad \forall i \in S,\ i \leqslant \frac{1}{1-r}, \quad |y(i) - x^*(i)| \leqslant 1 - r\} & \text{if } 0 < r \leqslant 1. \end{cases} \tag{8}$$

Clearly, H is a RP for x^*. We claim that H is LSC and USC.

Consider for instance the case where $S = \aleph$. The images of H contract toward x^* in a smooth way through time, but not uniformly through the coordinates since the restriction $|y(i) - x^*(i)| \leqslant 1 - r$ affects only a finite number of coordinates at each time instant r.

For LSC, fix a basic open subset $\wp = \left(\prod_{i\in F} V_i \times \prod_{i\notin F}[0,1]\right) \cap PR$ of PR not containing x^* (see Proposition 3.4(a)). For every $i \in F$ such that $x^*(i) \notin \pi_i(\wp)$, let $s_i = \sup\{r : \pi_i(\wp) \cap [x^*(i) - 1 + r, x^*(i) + 1 - r] \neq \emptyset\}$ and let $s = \min\{s_i : i \in F \wedge x^*(i) \notin \pi_i(\wp)\}$.

Note that $\pi_i(\wp) \cap [x^*(i) - 1 + r, x^*(i) + 1 - r] = \emptyset$ necessarily implies $i \leqslant \frac{1}{1-r}$. Hence, $H^{-1}(\wp) = [0, s)$.

To prove that H is USC, fix a basic open subset $\wp$ of PR containing x^* (see Proposition 3.4(b)). Thus, there exists a finite subset F of S such that for all $i \in F$, $x^*(i) \in \pi_i(\wp)$. Let $w_i = \inf\{t : [x^*(i) - 1 + t, x^*(i) + 1 - t] \subseteq \pi_i(\wp)\}$ and $w = \max\{w_i: i \in F\}$.

Note that $\pi_i(\wp) \subseteq [x^*(i) - 1 + r, x^*(i) + 1 - r]$ necessarily implies $i \leqslant \frac{1}{1-r}$. Therefore, $\{r: H(r) \subseteq \wp\} = (w, 1]$.

Note that both the lower and the upper semi-continuity of the RP's of Examples 5 and 6 hold independently of the properties of the set S, that is, S can be freely assumed to be either a real subinterval or a discrete (finite or countably infinite) real subset.

Examples 4–6 may be used to model situations where the information is perpetually transmitted through time (see Definition 3.2). At the same time, they present RP's which are defined by coordinate smooth contractions towards x^*, that is, for every $i \in S$, a closed neighborhood of the ith coordinate $x^*(i)$ of x^* contracting towards $x^*(i)$ is defined.

These two characteristics together seem to be relevant in order to guarantee the continuity of the processes in the sense of Definition 3.3. However, the following Examples 7–10 illustrate RPs that, exploiting the presence of sudden coordinatewise contractions towards x^*, are shown to miss either LSC, USC or both.

Regarding "when information is transmitted", while in Examples 7–9 the information is sequentially transmitted on the base of a fixed sequence of time instants, in Example 10 it is perpetually transmitted. Finally, Example 11, shows a RP where, despite the fact that information is sequentially transmitted, the images of the multifunction do contract smoothly and, in fact, the RP can be proved to be a CTRP.

Thus, we conjecture that for a RP H to be continuous, it suffices to assume that the images of H satisfy a coordinatewise smooth contraction-like property. In other words, the conditions to guarantee the continuity of a RP H must rely on the shape that the image sets of H take, hence on "how" rather than on "when" information is transmitted.

Example 7. Fix $x^* \in [0,1]^S$ and define $H : T \xrightarrow{\rightarrow} PR$ as follows:

$$H(r) = \begin{cases} PR & \text{if } r = 0, \\ \left\{y \in PR : \quad \forall i \in S,\ i \leqslant j, \quad |y(i) - x^*(i)| \leqslant \frac{1}{j}\right\} & \text{if } t_j \leqslant r < t_{j+1}, \\ \{x^*\} & \text{if } r = 1, \end{cases} \tag{9}$$

where for all $j \in \aleph, t_j = \frac{j-1}{j}$.

Clearly, H is a RP for x^* where new information is transmitted at a countably infinite set of time instants, namely, at each $t_j = \frac{j-1}{j}$. We show that H is LSC but not USC, hence a DTRP, independently of the nature of S.

To make a parallel with Example 6, consider the case where $S = \aleph$. The images of H still contract towards x^* involving only an initial segment of the set of possible coordinates at each instant r, but not in a smooth way. In order to see this, consider a few image sets:

$$\begin{aligned} & H(0) = \{y \in PR : \forall i = 1, |y(i) - x^*(i)| \leqslant 1\} = PR, \\ & 0 \leqslant r < \frac{1}{2} \quad \Rightarrow \quad H(r) = PR, \\ & H\left(\frac{1}{2}\right) = \left\{y \in PR : \forall i = 1,2, |y(i) - x^*(i)| \leqslant \frac{1}{2}\right\}, \\ & \frac{1}{2} \leqslant r < \frac{2}{3} \quad \Rightarrow \quad H(r) = H\left(\frac{1}{2}\right), \\ & H\left(\frac{2}{3}\right) = \left\{y \in PR : \forall i = 1,2,3, |y(i) - x^*(i)| \leqslant \frac{1}{3}\right\}, \\ & \frac{2}{3} \leqslant r < \frac{3}{4} \quad \Rightarrow \quad H(r) = H\left(\frac{2}{3}\right), \\ & \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{aligned} \tag{10}$$

To see that H is LSC, fix a basic open subset $\wp$ of PR not containing x^*. Given the images of H, there exists a $\hat{j}$ such that $H(t_{\hat{j}}) \cap \wp = \emptyset$. Thus, for all $i \geqslant \hat{j}, H(t_i) \cap \wp = \emptyset$. Hence, by the definition of H, $H^{-1}(\wp) = [0, t_{\hat{j}})$. This shows that H is LSC.

Finally, let us check that H is not USC. Consider the following open subset of PR containing x^*:

$$\wp = \left(\left(\left(x^*(1) - \frac{1}{2}, x^*(1) + \frac{1}{2}\right) \cap [0,1]\right) \times \prod_{i\neq 1}[0,1]\right) \cap PR. \tag{11}$$

Given the image sets listed above, it is easy to check that

$$\{r : H(r) \subseteq \wp\} = \left\{r : H(r) \subseteq H\left(\frac{2}{3}\right)\right\} = \left[\frac{2}{3}, 1\right], \tag{12}$$

hence not an open subset of $[0,1]$.

Example 7 admits a dual formulation, where the discontinuity of the RP is due to the lack of lower semi-continuity instead of upper semi-continuity.

Example 8. Fix $x^* \in [0,1]^S$ and define $H : T \xrightarrow{\rightarrow} PR$ as follows:

$$H(r) = \begin{cases} PR & \text{if } r = 0, \\ \left\{y \in PR : \quad \forall i \in S,\ i \leqslant j, \quad |y(i) - x^*(i)| \leqslant \frac{1}{j}\right\} & \text{if } t_j < r \leqslant t_{j+1}, \\ \{x^*\} & \text{if } r = 1, \end{cases} \tag{13}$$

where for all $j \in \aleph, t_j = \frac{j-1}{j}$.

H is a DTRP for x^*, where new information is transmitted *immediately after* each $t_j = \frac{j-1}{j}$. We leave it to the reader to check that H is USC but not LSC, independently of the nature of S.

Combining the information revelation structures of the last two examples it is possible to define a DTRP lacking both lower and upper semi-continuity.

Example 9. Fix $x^* \in [0,1]^S$ and define $H : T \xrightarrow{\rightarrow} PR$ as follows:

$$H(r) = \begin{cases} PR & \text{if } 0 \leqslant r < \frac{1}{2}, \\ \{y \in PR : \quad \forall i \in S, \quad |y(i) - x^*(i)| \leqslant \frac{1}{3}\} & \text{if } \frac{1}{2} \leqslant r \leqslant \frac{2}{3}, \\ \{y \in PR : \quad \forall i \in S, \quad |y(i) - x^*(i)| \leqslant \frac{1}{4}\} & \text{if } \frac{2}{3} < r < \frac{3}{4}, \\ \{x^*\} & \text{if } \frac{3}{4} \leqslant r \leqslant 1. \end{cases} \tag{14}$$

H is a DTRP for x^* that is neither LSC nor USC. Indeed, consider the following open subset of PR not containing x^*:

$$\wp = \left(\left(x^*(1) + \frac{2}{3}, 1\right] \times \prod_{i \neq 1} [0,1]\right) \cap PR. \tag{15}$$

Since $H^{-1}(\wp) = [0, \frac{2}{3}]$, H is not LSC. At the same time, given the following open subset of PR containing x^*:

$$\wp = \left(\left(\left(x^*(1) - \frac{1}{2}, x^*(1) + \frac{1}{2}\right) \cap [0,1]\right) \times \prod_{i \neq 1} [0,1]\right) \cap PR, \tag{16}$$

we have $\{r : H(r) \subseteq \wp\} = [\frac{1}{2}, 1]$. Thus, H is not USC.

Example 10. Fix $x^* \in [0,1]^S$ and define $H : T \xrightarrow{\rightarrow} PR$ as follows:

$$H(r) = \begin{cases} PR & \text{if } r = 0, \\ \left\{y \in PR : \quad \forall i \in S, i \leqslant j, \quad |y(i) - x^*(i)| \leqslant 1 - r - \frac{1}{j+1}\right\} & \text{if } t_j \leqslant r < t_{j+1}, \\ \{x^*\} & \text{if } r = 1, \end{cases} \tag{17}$$

where for all $j \in \aleph, t_j = \frac{j-1}{j}$.

Unlike Example 7, H is a RP for x^* where new information is transmitted perpetually through time, since $H(r) \neq H(s)$ for all $r \neq s$. However, similar to Example 7, it can be shown that H is LSC but not USC, hence a DTRP, independently of the nature of S.

Example 11. Fix $x^* \in [0,1]^S$ and define $H : T \xrightarrow{\rightarrow} PR$ as follows:

$$H(r) = \begin{cases} PR & \text{if } r = 0, \\ \{y \in PR : \quad \forall i \in S, \quad |y(i) - x^*(i)| \leqslant 1 - r\} & \text{if } 0 < r \leqslant \frac{1}{2}, \\ \{y \in PR : \quad \forall i \in S, \quad |y(i) - x^*(i)| \leqslant \frac{1}{2}\} & \text{if } \frac{1}{2} < r \leqslant \frac{3}{4}, \\ \{y \in PR : \quad \forall i \in S, \quad |y(i) - x^*(i)| \leqslant 2 - 2r\} & \text{if } \frac{3}{4} < r \leqslant 1. \end{cases} \tag{18}$$

Using the same reasoning as in Example 5, we can see that H is a CTRP for x^*. However, in contrast to Example 5, information is not being transmitted perpetually through time (indeed, no new information is given between time $\frac{1}{2}$ and $\frac{3}{4}$).

5. Obtaining continuous and discontinuous time revelation processes

Based upon the examples provided in the previous section, the continuity of the revelation process seems to be determined by the way the image sets of H *contract* towards the true probability function x^*. In this sense, the *tubular shape* of the images of the multifunctions defined in the examples plays an essential role. All the remaining properties studied so far for H, that is:

- when new information is transmitted,
- the support S of the probability functions being a real interval or a discrete set,
- the fact that partial information may not allow to approximate simultaneously all the values that x^* takes,

do not affect the lower or upper semi-continuity of the information revelation process.

These ideas lead to the following definitions and results.

Definition 5.1. A multifunction $C : T \xrightarrow{\to} S$ will be called an *index revelation function through time (IRF)* if:

(1) $0 \leqslant r < s \Rightarrow C(r) \subseteq C(s)$;
(2) $\bigcup_{r<1} C(r) = C(1) = S$.

Definition 5.2. A function g: $T \to [0,1]$ will be called an *approximation function through time (AF)* if it is a decreasing function such that $dom(g) = T$, $g(0) = 1$ and $g(1) = 0$.

Definitions 5.1 and 5.2 express two different aspects relating to how information can be transmitted in the assigned time interval. In the following definition, we introduce a class of revelation processes that combines both aspects. The main result of this section shows that the continuity of this kind of revelation processes is equivalent to the continuity of the corresponding approximation functions.

Definition 5.3. Let $x^* \in PR$ and H be a RP for x^*. Let $C : T \xrightarrow{\to} S$ be an IRF and g: $T \to [0,1]$ be an AF. We say that H has *coordinatewise closed balls subject to C and g*, in short $(C, g) - \mathrm{CCB}$, if

$$\forall r \in T, \quad H(r) = \left(\prod_{i \in C(r)} ([x^*(i) - g(r), x^*(i) + g(r)] \cap [0,1]) \times \prod_{i \notin C(r)} [0,1]\right) \cap PR.$$

Proposition 5.4. *Let $x^* \in PR$ and H be a $(C, g) - \mathrm{CCBRP}$ for x^*. If g is continuous, then H is a CTRP.*

Proof. First, let $\wp$ be a basic open subset of PR not containing x^* (see Proposition 3.4(a)). Thus, $\wp = \left(\prod_{i \in F} V_i \times \prod_{i \notin F} [0,1]\right) \cap PR$, where either F is a finite subset of S if S is infinite or $F = S$ if S is finite, and each V_i is an open subset of the ith factor of the Cartesian product $[0,1]^S$.

For every $i \in F$ such that $x^*(i) \notin \pi_i(\wp)$, let $s_i = \sup\{r\colon\ i \in C(r) \wedge \pi_i\ (\wp) \cap [x^*(i) - g(r),\ x^*(i) + g(r)] \neq \emptyset\}$ and let $s = \min\{s_i\colon i \in F \wedge x^*(i) \notin \pi_i(\wp)\}$. By Definition 5.3, it is easy to check that, $H^{-1}(\wp) = [0,s)$. Given the arbitrary choice of $\wp$, H is LSC.

Now, fix a basic open subset $\wp$ of PR containing x^* (see Proposition 3.4(b)). Then, there exists a finite subset F of S such that for all $i \in F$, $x^*(i) \in \pi_i(\wp)$. Letting $w_i = \inf\{r\colon i \in C(r) \wedge [x^*(i) - g(r), x^*(i) + g(r)] \subseteq \pi_i(\wp)\}$ and $w = \max\{w_i\colon i \in F\}$, by Definition 5.3, we have $\{r\colon H(r) \subseteq \wp\} = (w,1]$. By the arbitrary choice of $\wp$, H is USC. □

Proposition 5.5. *Let $x^* \in PR$ and H be a $(C, g) - \mathrm{CCBRP}$ for x^*. If g is discontinuous, then H is a DTRP.*

Proof. Let $\hat{r}$ be a discontinuity point for g. Since g is a decreasing function and $dom(g) = T$, there must exist $a, b \in [0,1]$, with $a < b$, such that one of the following holds.

Case 1: $g(\hat{r}) = g(\hat{r}^-) = b$ and $g(\hat{r}^+) = a$,

Case 2: $g(\hat{r}^-) = b$ and $g(\hat{r}) = g(\hat{r}^+) = a$,

where $g(\hat{r}^-) = \lim_{r \to \hat{r}^-} g(r)$ and $g(\hat{r}^+) = \lim_{r \to \hat{r}^+} g(r)$.

Suppose that Case 1 holds. Then, there are open subsets of PR whose preimages are not open. Consider, for instance, the following basic open subset $\wp$ not containing x^*,

$$\wp = \left(\left(x^*(i) + \frac{a+b}{2}, 1\right] \times \prod_{j \neq i} [0,1]\right) \cap PR, \tag{19}$$

where $i \in C(\hat{r})$. We have $H^{-1}(\wp) = [0, \hat{r}]$. Therefore, H is not LSC.

Suppose now that Case 2 holds. Consider the following basic open subset $\wp$ containing x^*,

$$\wp = \left(\left(\left(x^*(i) - \frac{a+b}{2}, x^*(i) + \frac{a+b}{2}\right) \cap [0,1]\right) \times \prod_{j \neq i}[0,1]\right) \cap PR, \tag{20}$$

where $i \in C(\hat{r})$. We have $\{r : H(r) \subseteq \wp\} = [\hat{r}, 1]$. Thus, H cannot be USC. □

Propositions 5.4 and 5.5 yield the following.

Theorem 5.6. *Let $x^* \in PR$ and H be a (C, g) – CCBRP for x^*. The following are equivalent:*

(*a*) *g is continuous;*
(*b*) *H is a CTRP.*

We end the section with a result that, extending Proposition 5.5, clarifies how the degree of continuity of revelation processes depends on the degree of continuity assigned to the AF g.

According to Proposition 5.5, given a (C, g) – CCBRP, the discontinuity of g implies that H lacks at least one of the two semi-continuity states, either lower or upper, but it does not suffice to establish what happens to the other one.

The following definition introduces a class of revelation processes that are shown to be either LSC but not USC, or USC but not LSC.

Definition 5.7. Let $x^* \in PR$ and H be a (C, g) – CCBRP for x^*. H is called *lower step-revealing* [resp. *upper step-revealing*] if g is a right-continuous function [resp. left-continuous function].

More precisely, H is a lower step-revealing RP if it is a (C, g) – CCBRP and there exists a countable set of time instants where right jump discontinuities occur for g. Hence, either there exist a finite increasing sequence of time instants $t_0 = 0 < t_1 < \cdots < t_m < t_{m+1} = 1$ and finitely many continuous functions g_0: $[t_0, t_1) \to [0,1]$, g_1: $[t_1, t_2) \to [0,1], \ldots, g_m$: $[t_m, t_{m+1}] \to [0,1]$ such that $\forall n < m, g_n(t^-_{n+1}) \neq g_{n+1}(t_{n+1})$ and their union is g, or there exist an infinite increasing sequence of time instants $\{t_n\}_{n \geqslant 0}$, with $t_0 = 0$, and a family of continuous functions $\{g_n$: $[t_n, t_{n+1}) \to [0, 1]\}_{n \geqslant 0}$ such that $\forall n \geqslant 0, g_n(t^-_{n+1}) \neq g_{n+1}(t_{n+1})$ and $graph(g) = \bigcup_{n \geqslant 0} graph(g_n) \cup \{\langle 1, 0 \rangle\}$.

Similarly, H is an upper step-revealing RP if it is a (C, g) – CCBRP and either there exist a finite increasing sequence of time instants $t_0 = 0 < t_1 < \cdots < t_m < t_{m+1} = 1$ and finitely many continuous functions g_0: $[t_0, t_1] \to [0,1]$, g_1: $(t_1, t_2] \to [0,1], \ldots, g_m$: $(t_m, t_{m+1}] \to [0,1]$ such that $\forall n < m, (g_n(t_{n+1}) \neq g_{n+1}(t^+_{n+1})$ and their union is g, or there exist an infinite increasing sequence of time instants $\{t_n\}_{n \geqslant 0}$, with $t_0 = 0$, and a family of continuous functions $\{g_n$: $(t_n, t_{n+1}] \to [0,1]\}_{n \geqslant 0}$ such that $\forall n \geqslant 0$, $g_n(t_{n+1}) \neq g_{n+1}(t^+_{n+1})$ and $graph(g) = \bigcup_{n \geqslant 0} graph(g_n) \cup \{\langle 0, 1 \rangle\}$.

Thus, H being lower (resp. upper) step-revealing means that its image sets are defined as follows: $H(0) = PR$, $H(1) = \{x^*\}$, and for all $r \in [t_n, t_{n+1})$ (resp. $r \in (t_n, t_{n+1}]$), $H(r) = \left(\prod_{i \in C(r)}([x^*(i) - g(r), x^*(i) + g(r)] \cap [0,1]) \times \prod_{i \notin C(r)}[0,1]\right) \cap PR$.

Remark. The RP of Example 1 and the H_t of Example 3 show lower step-revealing RPs where the AF's are left-continuous step functions given, respectively, by

$$g(r) = \begin{cases} 1 & \text{if } 0 \leqslant r < 1 \\ 0 & \text{if } r = 1 \end{cases} \quad \text{and} \quad g_t(r) = \begin{cases} 1 & \text{if } 0 \leqslant r < t, \\ 0 & \text{if } t \leqslant r \leqslant 1. \end{cases} \tag{21}$$

Examples 7 and 10 show lower step-revealing RP's where the corresponding g's are left-continuous functions with countably infinite jumps. Examples 2 and 3 (the H^t subcase) and 8 show upper step-revealing RP's associated with right-continuous step functions. The explicit definition of the AF's of Examples 2 and 3 can be easily derived from the corresponding example. Regarding the AF's of Examples 7, 8 and 10, note that they are also sequentially left-continuous at 1 (see Section 6). For instance, the definition of the AF g of Example 7 is the following:

$$g(r) = \begin{cases} \frac{1}{j} & \text{if } t_j \leqslant r < t_{j+1}, \\ 0 & \text{if } r = 1, \end{cases} \tag{22}$$

where for all $j \in \aleph, t_j = \frac{j-1}{j}$. Hence, $t_j \to 1$ as $j \to +\infty$ and $g(t_j) \to g(1) = 0$ as $t_j \to 1$, implying that g is sequentially left-continuous at 1. A similar reasoning applies to the AF's of Examples 8 and 10.

Proposition 5.8.

(*a*) *Every lower step-revealing RP is LSC but not USC.*
(*b*) *Every upper step-revealing RP is USC but not LSC.*

Proof. (a) Let H be a lower step-revealing RP. We will consider only the case where g has finitely many jumps. The case with countably many jumps is similar. Thus, suppose there exist a finite increasing sequence of time instants $t_0 = 0 < t_1 < \cdots < t_m < t_{m+1} = 1$ and finitely many continuous functions g_0: $[t_0, t_1) \to [0,1]$, g_1: $[t_1, t_2) \to [0,1], \ldots, g_m$: $[t_m, t_{m+1}] \to [0,1]$ such that $\forall n < m$, $g_n(t_{n+1}{}^-) \neq g_{n+1}(t_{n+1})$ and their union is g.

Let $\wp$ be a basic open subset of PR not containing x^* (see Proposition 3.4(a)). Thus, $\wp = \left(\prod_{i\in F} V_i \times \prod_{i\notin F} [0,1]\right) \cap PR$, where either F is a finite subset of S if S is infinite or $F = S$ if S is finite, and each V_i is an open subset of the ith factor of the Cartesian product $[0,1]^S$.

Note that for all $n < m$ and all $r \in [t_n, t_{n+1})$,

$$H(r) = \left(\prod_{i\in C(r)} ([x^*(i) - g_n(r), x^*(i) + g_n(r)] \cap [0,1]) \times \prod_{i\notin C(r)} [0,1]\right) \cap PR. \tag{23}$$

If there exists $\hat{n}$ such that $H(t_{\hat{n}}) \cap \wp = \emptyset$, then $H^{-1}(\wp) = [0, t_{\hat{n}})$. Otherwise, for every $i \in F$ such that $x^*(i) \notin \pi_i(\wp)$, let $s_i = \sup\{r : i \in C(r) \ \wedge \ \pi_i(\wp) \cap [x^*(i) - g_n(r), x^*(i) + g_n(r)] \neq \emptyset\}$ and let $s = \min\{s_i : i \in F \wedge x^*(i) \notin \pi_i(\wp)\}$. It is easy to check that, $H^{-1}(\wp) = [0,s)$. This shows that H is LSC.

Consider now the following open subset $\wp$ of PR containing x^*,

$$\wp = \left(\left(\left(x^*(i) - \frac{g_1(t_2^-) + g_2(t_2)}{2}, x^*(i) + \frac{g_1(t_2^-) + g_2(t_2)}{2}\right) \cap [0,1]\right) \times \prod_{j\neq i} [0,1]\right) \cap PR, \tag{24}$$

where $i \in C(r)$ and $r \in [t_1, t_2)$. By the definition of H, we have $\{r: H(r) \subseteq \wp\} = \{r: H(r) \subseteq H(t_2)\} = [t_2, 1]$. Thus, H cannot be USC (see Proposition 3.4(b)).

(b) Let H be an upper step-revealing RP. Again, we will limit the proof to the case where g has finitely many jumps. Hence, there exists a finite increasing sequence of time instants $t_0 = 0 < t_1 < \cdots < t_m < t_{m+1} = 1$ and finitely many continuous functions g_0: $[t_0, t_1] \to [0,1]$, g_1: $(t_1, t_2] \to [0,1]$, ..., g_m: $(t_m, t_{m+1}] \to [0,1]$ such that $\forall\ n < m$, $g_n(t_{n+1}) \neq g_{n+1}(t_{n+1}^+)$ and their union is g.

To prove that H is USC, fix a basic open subset $\wp$ of PR containing x^* (see Proposition 3.4(b)), and note that $\forall n < m$ and $\forall r \in (t_n, t_{n+1}]$,

$$H(r) = \left(\prod_{i\in C(r)} ([x^*(i) - g_n(r), x^*(i) + g_n(r)] \cap [0,1]) \times \prod_{i\notin C(r)} [0,1]\right) \cap PR. \tag{25}$$

If there exists $\hat{n}$ such that $H(t_{\hat{n}}) \subset \wp$, then $\{r : H(r) \subseteq \wp\} = \{r : H(r) \subseteq H(t_{\hat{n}})\} = (t_{\hat{n}}, 1]$. Otherwise, there exists a finite subset F of S such that for all $i \in F$, $x^*(i) \in \pi_i(\wp)$.

Letting $w_i = \inf\{r : i \in C(r) \ \wedge \ [x^*(i) - g_n(r), x^*(i) + g_n(r)] \subseteq \pi_i(\wp)\}$ and $w = \max\{w_i: i \in F\}$, we have $\{r: H(r) \subseteq \wp\} = (w,1]$.

Let us now consider the following open subset of PR not containing x^*,

$$\wp = \left(\left(x^*(i) + \frac{g_1(t_2) + g_2(t_2^+)}{2}, 1\right] \times \prod_{j\neq i} [0,1]\right) \cap PR, \tag{26}$$

where $i \in C(r)$ and $r \in [t_1, t_2)$. Since $H^{-1}(\wp) = [0, t_2]$, H is not LSC (see Proposition 3.4(a)). □

6. A reflecting property of the information process

Recall that given $E \subseteq S$, $x^*(E)$ is the probability of the set E, that is,

$$x^*(E) = \begin{cases} \sum_{i\in E} x^*(i) & \text{if } S \text{ is the realization set of a discrete r.v.} \\ \int_E x^*(i) di & \text{if } S \text{ is the realization set of an abs. cont. r.v.} \end{cases} \tag{27}$$

In this section we introduce the following *reflecting property* for RPs.

Property ($\bowtie$): $\forall E \subset S$, $\forall a, b \in [0,1]$ with $a < b$, if $a < x^*(E) < b$ then $\exists r \in (0,1)$ s.t. $\forall x \in H(r)$, $a < x(E) < b$.

An equivalent formulation of Property ($\bowtie$) is the following:

$\forall E \subset S$, $\forall a, b \in [0,1]$ with $a < b$, if $a < x^*(E) < b$ then $\exists s \in (0,1)$ s.t. $\forall r > s$ and $\forall x \in H(r)$, $a < x(E) < b$.

Property ($\bowtie$) says that, given a range of variation for the true probability weight of any set of realizations, there can be found a time instant <1 at which the probability weight of the same set of realizations measured by any of the probability functions still believed to be possible falls within the given range.

Hence, Property ($\bowtie$) allows for the following interpretation. For every information transmitted about the true probability function x^* there must exist a certain time instant $r < 1$ after which all the probability functions still believed to be possible satisfy the properties defined by the given piece of information. That is, for every approximation constraint satisfied by x^*, there exists an image set of the RP H "reflecting" the same approximation.

A property trivially stronger than Property ($\bowtie$) is the following:

Property strong ($\bowtie$): $\exists t \in (0, 1)$ such that $H(t)=\{x^*\}$.

This property characterizes RPs where the full information is revealed before time 1, with no other restriction being imposed. The following characterization for (C, g) – CCBRPs in terms of their approximation functions is easy to check.

Lemma 6.1. *Let $x^* \in PR$ and H be a (C, g) – CCBRP for x^*. The following are equivalent:*

(*a*) *g is eventually null, that is, $\exists t \in (0,1)$ such that $\forall r > t$, $g(r) = 0$;*
(*b*) *H satisfies Property strong ($\triangleright\triangleleft$).*

The RP's in Example 3, except for H_t with $t = 1$, and those of Examples 9 and 11 satisfy Property strong ($\triangleright\triangleleft$) and, hence, Property ($\triangleright\triangleleft$). It is clear that S being the realization set of a discrete or an absolutely continuous random variable is irrelevant as far as g is eventually null.

Actually, we will show that the nature of S remains irrelevant when characterizing Property ($\triangleright\triangleleft$) for (C, g) – CCBRPs (see Theorem 6.2 below).

The RP's in Examples 4–8 and 10 can all be proved to satisfy Property ($\triangleright\triangleleft$) but not Property strong ($\triangleright\triangleleft$). For the sake of completeness, we illustrate below how this fact holds in one of the mentioned examples.

Example 12. Fix $x^* \in [0,1]^S$ and define H as in Example 6:

$$H(r) = \begin{cases} PR & \text{if } r = 0, \\ \{y \in PR: \quad \forall i \in S,\ i \leqslant \frac{1}{1-r}, \quad |y(i) - x^*(i)| \leqslant 1 - r\} & \text{if } 0 < r \leqslant 1. \end{cases} \tag{28}$$

Suppose that S is the realization set of an absolutely continuous random variable and fix $E \subset S$ and $a, b \in [0,1]$ such that $a < x^*(E) < b$.

Note that for every $r \in [0,1]$ and every $x \in H(r)$, we have:

$$|x(E) - x^*(E)| \leqslant \left| \int_E (x(i) - x^*(i))\ di \right| \leqslant \int_{\{i \in E: i \leqslant \frac{1}{1-r}\}} |x(i) - x^*(i)| di + \int_{\{i \in E: i > \frac{1}{1-r}\}} |x(i) - x^*(i)| di. \tag{29}$$

Let $\eta = \min\{b - x^*(E), x^*(E) - a\}$. Since $(1 - r) \to 0$ as $r \to 1$, there exists $\exists s \in (0,1)$ such that $\forall r > s$ and $\forall x \in H(r)$:

$$\int_{\{i \in E: i \leqslant \frac{1}{1-r}\}} |x(i) - x^*(i)| di \leqslant \int_{\{i \in E: i \leqslant \frac{1}{1-r}\}} (1 - r) di < \frac{\eta}{2}. \tag{30}$$

At the same time, since $\int_{\{i \in E: i > \frac{1}{1-r}\}} |x(i) - x^*(i)| di$ converges to 0 as $r \to 1$, there exists $t \in (0,1)$ such that $\forall r > t$ and $\forall x \in H(r)$:

$$\int_{\{i \in E: i > \frac{1}{1-r}\}} |x(i) - x^*(i)| di < \frac{\eta}{2}. \tag{31}$$

Hence, $\forall r > \max\{s, t\}$ and $\forall x \in H(r)$, we have $|x(E) - x^*(E)| \leqslant \eta$, that is,

$$a < x^*(E) - \eta \leqslant x(E) \leqslant x^*(E) + \eta < b. \tag{32}$$

Thus, H satisfies Property ($\triangleright\triangleleft$).

A similar reasoning applies to the case where S is the realization set of a discrete random variable, by replacing $|\int_E (x(i) - x^*(i)) di|$ with $|\sum_E (x(i) - x^*(i))|$.

Finally, H is a (C, g) – CCBRP where the AF g is defined by $g(r) = 1 - r$. Hence, g is not eventually null.

In all the examples of (C, g) – CCBRP satisfying Property ($\triangleright\triangleleft$) the corresponding AF's are left-continuous at 1. We show that this requirement on g is equivalent to the RP satisfying Property ($\triangleright\triangleleft$).

Since T is assumed to be endowed with the Euclidean metric, the left-continuity at 1 is equivalent to the sequential left-continuity at 1 for g. By the standard definition of a sequentially left-continuous function, an AF g is sequentially left-continuous at 1 if

(#) for every infinite sequence of time instants $\{t_n\}_{n>0}$ converging to 1, $\lim_{n \to +\infty} g(t_n) = g(1) = 0$.

Theorem 6.2. *Let $x^* \in PR$ and H be a (C, g) – CCBRP for x^*. The following are equivalent:*

(*a*) *g is left-continuous at 1;*
(*b*) *H satisfies Property ($\triangleright\triangleleft$).*

Proof. $(a) \Rightarrow (b)$. Fix $E \subset S$ and $a, b \in [0,1]$ such that $a < x^*(E) < b$. Suppose first that S is the realization set of an absolutely continuous random variable.

Note that for every $r \in [0,1]$ and every $x \in H(r)$, we have:

$$|x(E) - x^*(E)| \leqslant \left| \int_E (x(i) - x^*(i)) di \right| \leqslant \int_{\{i \in E: i \in C(r)\}} |x(i) - x^*(i)| di + \int_{\{i \in E: i \notin C(r)\}} |x(i) - x^*(i)| di. \tag{33}$$

Let $\eta = \min\{b - x^*(E), x^*(E) - a\}$ and $\{t_n\}_{n>0}$ be an infinite sequence of time instants converging to 1. By assumption, there exists an N such that $\forall n > N$ and $\forall x \in H(t_n)$:

$$\int_{\{i \in E : i \in C(t_n)\}} |x(i) - x^*(i)| di \leqslant \int_{\{i \in E : i \in C(t_n)\}} g(t_n) di < \frac{\eta}{2}. \tag{34}$$

At the same time, since $\bigcup_{n>0} C(t_n) = S$, we have $\int_{\{i \in E : i \notin C(t_n)\}} |x(i) - x^*(i)| di$ converges to 0 as $t_n \to 1$. Thus, there exists an N' such that $\forall n > N'$ and $\forall x \in H(t_n)$:

$$\int_{\{i \in E : i \notin C(t_n)\}} |x(i) - x^*(i)| di < \frac{\eta}{2}. \tag{35}$$

Hence, $\forall n > \max\{N, N'\}$ and $\forall x \in H(t_n)$, we have $|x(E) - x^*(E)| < \eta$. Note that if $b - x^*(E) < x^*(E) - a$, then $a < 2x^*(E) - b$. Similarly, if $x^*(E) - a < b - x^*(E)$, then $2x^*(E) - a < b$. Hence:

$$a < x^*(E) - \eta < x(E) < x^*(E) + \eta < b. \tag{36}$$

Suppose now that S is the realization set of a discrete random variable. Using the same reasoning as above, with $\left|\sum_E (x(i) - x^*(i))\right|$ in place of $\left|\int_E (x(i) - x^*(i)) di\right|$, it can be shown that there exists $\hat{r} \in (0,1)$ such that $\forall r > \hat{r}$ and $\forall\ x \in H(r)$, $a < x(E) < b$. Therefore, H satisfies Property ($\triangleright\triangleleft$).

$(b) \Rightarrow (a)$. By contradiction, assume (#) does not hold and let $\{t_n\}_{n>0}$ be a sequence of time instants converging to 1 such that $\mu = \lim_{n \to +\infty} g(t_n) > 0$.

By Definition 5.1, $\exists \hat{s} \in T, \hat{s} < 1$, such that $C(\hat{s}) \neq \emptyset$. Fix $j \in C(\hat{s})$. Hence, $\forall r \geqslant \hat{s}, j \in C(r)$. Since $t_n \to 1$, there exists an $\widehat{N}$ such that $\forall n > \widehat{N}$, we have $t_n > \hat{s}$. Hence, $\forall n > \widehat{N}, j \in C(t_n)$.

At the same time, since g is decreasing, $\forall n > \widehat{N}$, $\exists x_n \in H(t_n)$ such that $\forall i \in C(t_n)$, $g(t_n) \geqslant |x_n(i) - x^*(i)| \geqslant g(t_{n+1}) \geqslant \mu$. In particular, $\forall n > \widehat{N}, \exists x_n \in H(t_n)$ such that $|x_n(j) - x^*(j)| \geqslant \mu$.

Now, let $\xi > 0$ be a natural number such that $0 < a = x^*(j) - \frac{\mu}{\xi} < 1$ and $0 < b = x^*(j) + \frac{\mu}{\xi} < 1$. Clearly, $a < x^*(j) < b$. Since H is assumed to satisfy Property ($\triangleright\triangleleft$), this implies that $\exists \hat{\hat{s}} \in (0,1)$ such that $\forall r > \hat{\hat{s}}$ and $\forall x \in H(r)$, $a < x(j) < b$. Since $t_n \to 1$, it follows that there exists $\widehat{\widehat{N}}$ such that $\forall n > \widehat{\widehat{N}}$ and $\forall x \in H(t_n)$, we have $|x(j) - x^*(j)| < \frac{\mu}{\xi}$.

Therefore, $\forall n > \max\{\widehat{N}, \widehat{\widehat{N}}\}$, there must exist $x_n \in H(t_n)$ such that $\mu \leqslant |x_n(j) - x^*(j)| < \frac{\mu}{\xi}$, which is not possible since $\mu \geqslant \frac{\mu}{\xi}$. □

Corollary 6.3. *Let $x^* \in PR$ and H be a (C, g) – CCBRP for x^*. If H is a CTRP, then H satisfies Property ($\triangleright\triangleleft$).*

Proof. Apply Theorems 5.6 and 6.2. □

Corollary 6.4. *Let $x^* \in PR$ and H be a (C, g) – CCBRP for x^*. If g is left-continuous, then H satisfies Property ($\triangleright\triangleleft$).*

Proof. Every left-continuous function is also left-continuous at 1. Apply Theorem 6.2. □

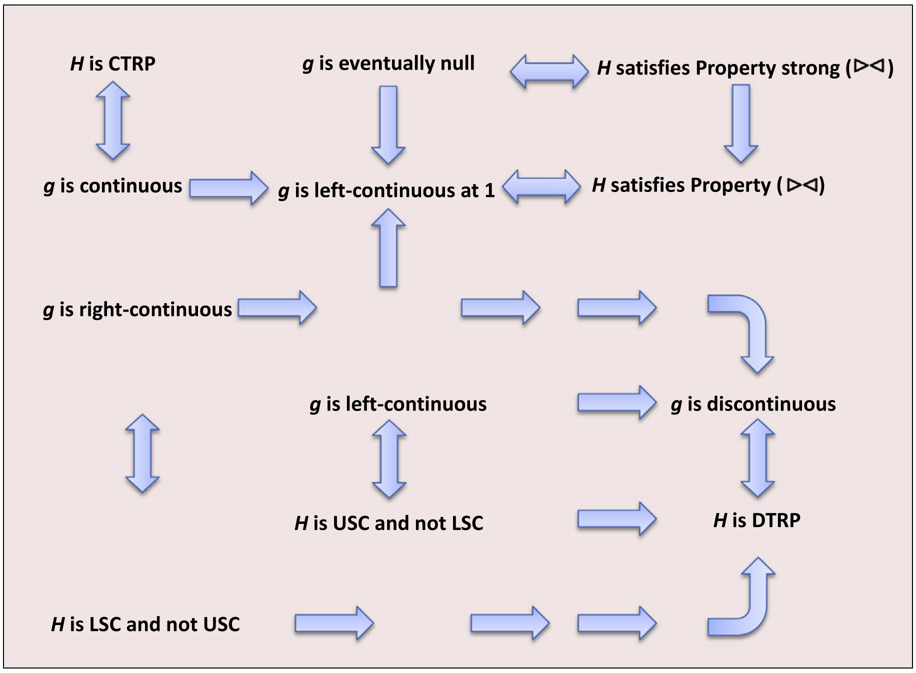

Fig. 1. Results for (C, g) – CCBRPs.

The main results obtained in this and the previous section can be interpreted in the following way. If the DM does not acquire information from different sources but stays always on the same one, he will eventually eliminate all uncertainty when the limit time is reached [Theorem 5.6]. Moreover, he will be able to formulate probabilistic predictions within a given confidence range after a given time period [Property ($\bowtie$) and Theorem 6.2]. If, on the other hand, the DM acquires information from different sources, he may speed up the convergence process and either eliminate all uncertainty before the time limit is reached [Property strong ($\bowtie$) and Lemma 6.1] or formulate probabilistic predictions within a given confidence range before the limit time is reached [Property ($\bowtie$) and Theorem 6.2] or both [Corollaries 6.3 and 6.4]. In all these cases, the information obtained from each source must be exhausted by the time it is abandoned [left-continuity of the AF g and Proposition 5.8]. If the shifting among sources does not satisfy this constraint, then the DM can only be guaranteed to formulate probabilistic predictions within a given confidence range in the limit time.

7. Discussions

This section provides a graphical representation of the main results obtained and further intuition about their managerial significance. We start by describing the main formal findings obtained, whose relationships are summarized in Fig. 1. This figure represents the main implications relating the formal results obtained in Sections 5 and 6. The central part of the diagram is occupied by the left-continuity of g, as it constitutes the weakest condition guaranteeing that Property ($\bowtie$) holds. In order to increase further understanding, Fig. 2 illustrates a family of continuous approximation functions, denoted by g in Definition 5.2. These functions represent the variability of the estimations made by the DMs with respect to the true probability function.

Variability, which is represented in the y-axis of Figs. 2–4, has been normalized within the $[0,1]$ interval and defines the width of the approximation intervals that may be defined by the DM through time relative to the true probability function. That is, assume, for illustrative purposes, that a discrete probability function assigns a value of 0.4 to a given event taking place [note that our analysis allows also for continuous probabilities]. The time zero approximation interval with total uncertainty assigns any probability between zero and one to this event. Figs. 2–4 illustrate how the width of the subsequent approximation intervals approaches zero as the DM acquires information through time, a variable which has also been normalized within $[0,1]$ and accounted for in the x-axis. Note that the approximation intervals generated by a function g do not need to be symmetric through time. Thus, if the approximation function has a value of 0.5, the resulting approximation interval defined relative to the true probability of the event will be $[0,0.9]$, since events cannot have negative probabilities [i.e., in this case -0.1 should define the lower-limit of the symmetric interval].

As stated in the previous section, if the DM stays always on the same information source defining one of the approximation functions, he will eventually eliminate all uncertainty when the limit time is reached. In this sense, different approximation functions may be assumed to be associated with different costs of information, with those functions converging to zero at a faster rate providing more accurate but also more expensive information to the DMs. Thus, the current paper could be extended to formalize the acquisition costs derived from different information sources and to study the optimal behavior of the DMs within the resulting context.

The left-continuity of an approximation function when information is acquired by shifting among different sources or developing different projects is illustrated in Fig. 3. Note that left-continuous means that the time instant when a source is exhausted or a project abandoned is clearly established, while the shifting and adaptation to the new source or project is open [gradual] through time. In contrast, right-continuity is represented in Fig. 4 and consists of exactly the opposite transitional idea. Note that in the right-continuous case, the sources from which information is being acquired remain open

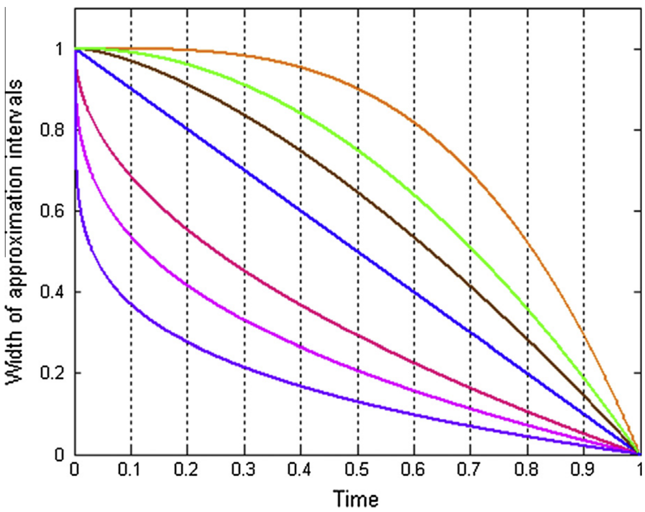

Fig. 2. A family of continuous approximation functions g.

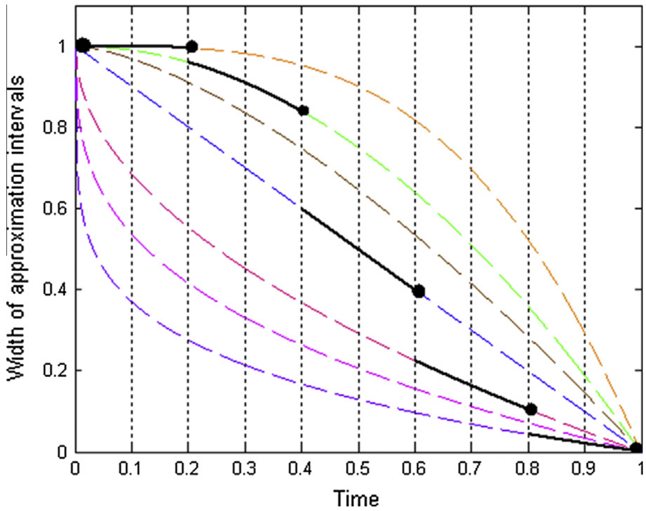

Fig. 3. A left-continuous approximation function has been highlighted in black solid line.

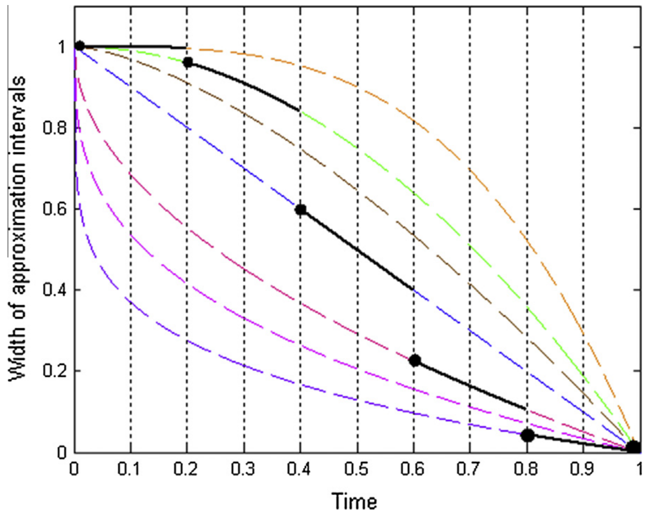

Fig. 4. A right-continuous approximation function has been highlighted in black solid line.

while shifting to new ones at a given point in time. Thus, even though in both cases information is being acquired from the same sources/projects through the same periods of time, the transition process among sources/projects differs significantly. In both cases, however, the starting and finishing time limits of the entire information acquisition process, i.e., time zero and time one, are clearly defined.

As can be observed in Fig. 3, left-continuous functions remain left-continuous at time one, which is described by the upper implication from 'g is left-continuous' to 'g is left-continuous at 1' in Fig. 1. Trivially, continuous functions are both right and left-continuous, though these implications have not been represented in order to simplify Fig. 1, since they do not add any additional information to the results. As shown in Theorem 5.6, continuous time revelation processes are equivalent to approximation functions being continuous. In other words, if DMs do not acquire information continuously through time, discontinuities as the ones illustrated in Figs. 3 and 4 are observed. These figures describe discrete time revelation processes, that is, information acquisition processes that are interrupted through time as DMs search for information among different sources/projects.

Revelation processes may lose part of their continuity, either the upper or the lower one, due to the information search interruptions caused by DMs. In other words, when g is left-continuous, i.e., when the DM completes a project or exhausts an information source before shifting to another one, the corresponding revelation process is upper semi-continuous. This means that the information acquisition process from a source/project is never interrupted until the shift to a different one takes place. Moreover, the shift is gradual and joins an existing source or project through a time period of adaptation, i.e., the shift does not take place immediately but through an open period of time. In contrast, the opposite effects and interpretation follow from a right-continuous process, where an exact time instant determines the immediate beginning of a new project while leaving the previous one unfinished without defining an exact conclusion time.

Finally, consider the upper right part of Fig. 1. The approximation functions illustrated in Fig. 2 are not eventually null, since they reach the value of zero only at the limit time one. Clearly, all these approximation functions conclude at time one and are therefore left-continuous at this time instant. We are however interested in the ability of the DMs to formulate probabilistic predictions within a given confidence range from a given instant before the limit time is reached. If a DM acquires a sufficiently large amount of information from different sources/projects, or remains within a project whose approximation function converges to zero at a very fast rate, then he may be able to eliminate all uncertainty before the limit time is reached, as Property strong ($\triangleright\triangleleft$) states. If this is the case, he will also be able to formulate probabilistic predictions within a given confidence range before the limit time is reached, as Property ($\triangleright\triangleleft$) states. However, and most importantly, in order for DMs to satisfy Property ($\triangleright\triangleleft$), we just need them to follow a left-continuous information acquisition process, that is, to conclude their projects or exhaust their sources before starting acquiring information from new ones. If the shifting process among sources/projects does not satisfy this constraint, then the DM can only be guaranteed to formulate probabilistic predictions within a given confidence range at the limit time one.

8. Conclusions and future research directions

We have considered the problem of information transmission when information is used to transform an uncertain situation into a risky one. That is, we have studied information revelation processes allowing for the true probability function x^* on a set of outcomes S to be revealed through a determined interval of time T. Our analysis of information revelation through time can be applied when considering either the perspective of the information transmitter or that of the receiver (the DM). The properties of the revelation processes we have proposed do not depend on the perspective adopted.

The question of how the way information is transmitted relating to the continuity of revelation processes regarded as multifunctions has been addressed. We have introduced a particular class of revelation processes, namely the class of (C, g) – CCBRPs, and provided necessary and sufficient conditions for their continuity. This is quite a large class of revelation processes satisfying contraction-like patterns towards the true probability function x^*. The diagram in Fig. 1 summarizes the main results obtained for (C, g) – CCBRPs.

The main contribution of our approach consists of endowing the process of information transmission with a topological structure that accounts for its dynamical properties. This formal abstraction leads to immediate behavioral implications regarding the way information should be acquired through time. It should be emphasized that the dynamical aspect is not accounted for in the standard fuzzy and rough set-theoretical models dealing with information transmission.

A particular advantage of our model is its compatibility with both the fuzzy and rough set-theoretical approaches to uncertainty and information transmission. The tubular shape that we assume for the set $H(r)$ of probability functions that the DM considers at a certain time instant r can be extended and interpreted as a rough set. That is, the images of the multifunctions introduced can be considered as equivalence classes with respect to the identification process of x^*. This is due to the fact that the DM cannot distinguish among the probability functions belonging to $H(r)$. At the same time, the boundary of $H(r)$ could be modified to account for a fuzzy characterization of the information transmission process. A fuzzy boundary would lead to a fuzzy set $H(r)$ of probability functions among which to individuate x^*.

The main constraint arising from our model is the fact that it well-behaves when information is revealed on a unique probability function. Therefore, the model can easily be extended to environments with multiple probability functions if information is transmitted on each one of them specifically. However, if the information transmitted on several probability functions is bundled, i.e. not discernible at an individual probability level, the proposed approach would fail. In this case, selection and filtering mechanisms must be introduced in order for the DM to discern what information corresponds to each particular probability function.

Finally, immediate extensions of the current paper may relate directly to the costs and existence of multiple simultaneous information acquisition processes. From a formal perspective, and given the constraint just described, the interactions among multiple complementary or substitutable information acquisition processes could be studied. This would require additional topological structures to be defined, while allowing for interactions and selections among different acquisition processes. The managerial implications following from such extensions should relate to the analysis of priority rules among simultaneous processes. If information acquisition costs are considered as a fundamental decision factor, then differences in process feasibility and accuracy should be accounted for by DMs.

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References

[1] N. Ahituv, M. Igbaria, A. Sella, The effects of time pressure and completeness of information on decision making, J. Manage. Inform. Syst. 15 (2) (1998) 153–172.
[2] L.R. Beach, T.R. Mitchell, A contingency model for the selection of decision strategies, Acad. Manage. Rev. 3 (3) (1978) 439–449.

[3] G. Beer, Topologies on Closed and Closed Convex Sets, Kluwer Academic, Dordrecht, 1993.
[4] M. Coulter, Strategic Management in Action, second ed., Prentice Hall, New York, 2000.
[5] T.M. Cover, J.A. Thomas, Elements of Information Theory, John Wiley and Sons, 2006.
[6] G. Chen, Y. Xu, Special issue on dealing with uncertainty in data mining and information extraction, Inform. Sci. 173 (4) (2005) 277–279.
[7] J. Dai, Q. Xu, Approximations and uncertainty measures in incomplete information systems, Inform. Sci. 198 (2012) 62–80.
[8] J.M. Davis, B.M. Tuttle, A heuristic-systematic model of end-user information processing when encountering IS exceptions, Inform. Manage. 50 (2–3) (2013) 125–133.
[9] D. Dubois, H. Prade, Possibility Theory: An Approach to Computerized Processing of Uncertainty, Plenum Press, New York, 1988.
[10] A. Durrande-Moreau, Waiting for service: ten years of empirical research, Int. J. Serv. Indust. Manage. 10 (2) (1999) 171–189.
[11] J. Eweje, R. Turner, R. Mueller, Maximizing strategic value from megaprojects: the influence of information-feed on decision-making by the project manager, Int. J. Project Manage. 30 (6) (2012) 639–651.
[12] J. Ghani, E.J. Lusk, The impact of a change in information representation and a change in the amount of information on decision performance, Hum. Syst. Manage. 3 (4) (1982) 270–278.
[13] I. Gilboa, Theory of Decision under Uncertainty, Cambridge University Press, 2009.
[14] K.P. Girish, Sunil Jacob John, Multiset topologies induced by multiset relations, Inform. Sci. 188 (2012) 298–313.
[15] P. Goodwin, G. Wright, Decision Analysis for Management Judgment, third ed., Wiley, Chichester, UK, 2004.
[16] W.R. Greer, D.H. Kropp, The incremental benefits of changes in the timeliness of information, Inform. Manage. 6 (6) (1983) 329–336.
[17] M.I. Hwang, Decision making under time pressure: a model for information systems research, Inform. Manage. 27 (4) (1994) 197–203.
[18] V.S. Jacob, J.C. Moore, A.B. Whinston, An analysis of human and computer decision-making capabilities, Inform. Manage. 16 (5) (1989) 247–255.
[19] V.S. Lerner, An observer's information dynamics: acquisition of information and the origin of the cognitive dynamics, Inform. Sci. 184 (1) (2012) 111–139.
[20] B. Liu, Uncertainty Theory, fourth ed., 2013 <http://orsc.edu.cn/liu/ut.pdf>.
[21] R.D. Luce, H. Raiffa, Games and Decisions: Introduction and Critical Survey, John Wiley and Sons, 1957.
[22] S.I. Mackie, S.H. Begg, C. Smith, M.B. Welsh, Decision type-a key to realising the potential of decision making under uncertainty, APPEA J. 47 (1) (2007) 309–319.
[23] C. Massey, D. Robinson, R. Kaniel, Can't wait to look in the mirror: the impact of experience on better-than-average effect, in: Proceedings of the Annual INFORMS Meeting, Pittsburgh, PA, November 5–8, 2006.
[24] H. Mintzberg, Patterns in strategy formulation, Manage. Sci. 24 (9) (1978) 934–948.
[25] J. Montero, D. Ruan, Modelling uncertainty, Inform. Sci. 180 (6) (2010) 799–802.
[26] C.A. O'Reilly, Variations in decision makers' use of information sources: the impact of quality and accessibility of information, Acad. Manage. J. 25 (4) (1982) 756–771.
[27] E. Oz, J. Fedorowicz, T. Stapleton, Improving quality, speed and confidence in decision-making: measuring expert systems benefits, Inform. Manage. 24 (2) (1993) 71–82.
[28] Z. Pawlak, Rough Sets-Theoretical Aspects of Reasoning About Data, Kluwer Academic Publishers, Hingham, MA, 1991.
[29] G. Rong, V. Grover, Keeping up-to-date with information technology: testing a model of technological knowledge renewal effectiveness for IT professionals, Inform. Manage. 46 (7) (2009) 376–387.
[30] G. Shafer, A Mathematical Theory of Evidence, Princeton University Press, Princeton, NJ, 1976.
[31] A. Skowron, On topology in information system, Bull. Polish Acad. Sci. Math. 36 (1988) 477–479.
[32] J.B. Thomas, R.R. McDaniel, Interpreting strategic issues: effects of strategy and the information-processing structure of top management teams, Acad. Manage. J. 33 (2) (1990) 286–306.
[33] J.B. Thomas, M. Shawn, D. Clark, A. Gioia, Strategic sensemaking and organizational performance: linkages among scanning, interpretation, action and outcomes, Acad. Manage. J. 36 (2) (1993) 239–270.
[34] J. Wu, Y. Xia, Relations among convergence concepts of uncertain sequences, Inform. Sci. 198 (2012) 177–185.
[35] R.R. Yager, Modeling uncertainty using partial information, Inform. Sci. 121 (3–4) (1999) 271–294.
[36] R.R. Yager, J. Kacprzyk, M. Fedrizzi, Advances in the Dempster–Shafer Theory of Evidence, Wiley, New York, 1994.
[37] Y. Yao, Three-way decisions with probabilistic rough sets, Inform. Sci. 80 (2010) 341–353.
[38] Y.Y. Yao, A comparative study of fuzzy sets and rough sets, Inform. Sci. 109 (1–4) (1998) 227–242.
[39] L.A. Zadeh, Fuzzy sets, Inform. Control 8 (3) (1965) 338–353.
[40] L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility, Fuzzy Sets Syst. 1 (1978) 3–28.