Journal of the Operational Research Society (2014) 65, 1824–1839

www.palgrave-journals.com/jors/

A new chance-constrained DEA model with birandom input and output data

Madjid Tavana[1,2*], Rashed Khanjani Shiraz[3] and Adel Hatami-Marbini[4]

[1]La Salle University, Philadelphia, USA; [2]University of Paderborn, Paderborn, Germany; [3]Tabriz University, Tabriz, Iran; and [4]Université catholique de Louvain, Louvain-la-Neuve, Belgium

The purpose of conventional Data Envelopment Analysis (DEA) is to evaluate the performance of a set of firms or Decision-Making Units using deterministic input and output data. However, the input and output data in the real-life performance evaluation problems are often stochastic. The stochastic input and output data in DEA can be represented with random variables. Several methods have been proposed to deal with the random input and output data in DEA. In this paper, we propose a new chance-constrained DEA model with birandom input and output data. A super-efficiency model with birandom constraints is formulated and a non-linear deterministic equivalent model is obtained to solve the super-efficiency model. The non-linear model is converted into a model with quadratic constraints to solve the non-linear deterministic model. Furthermore, a sensitivity analysis is performed to assess the robustness of the proposed super-efficiency model. Finally, two numerical examples are presented to demonstrate the applicability of the proposed chance-constrained DEA model and sensitivity analysis.

Journal of the Operational Research Society (2014) **65**(12), 1824–1839. doi:10.1057/jors.2013.157
Published online 27 November 2013

Keywords: Data Envelopment Analysis; chance-constrained programming; birandom variables; quadratic constraints; super-efficiency

1. Introduction

Data Envelopment Analysis (DEA) is a non-parametric approach for frontier estimation, first developed by Charnes, Cooper and Rhodes (CCR) (Charnes *et al*, 1978). On the basis of the original CCR model, Banker, Charnes and Cooper (BCC) (Banker *et al*, 1984) developed a variable returns to scale variation. DEA has been widely used for evaluating the performance and measuring the relative efficiency of a group of firms or Decision-Making Units (DMUs) that uses multiple inputs and multiple outputs. DEA solves a linear programming (LP) model per DMU to determine an efficient frontier using the 100% efficient DMUs. One problem that has been frequently discussed in the DEA literature has been the lack of discrimination in many assessment problems.

Most decision makers are interested in a full ranking method and to go beyond the dichotomized classification (ie, efficient and inefficient DMUs) in order to make a reasonable and insightful analysis. Adler *et al* (2002) reported a comprehensive review on the ranking methods in the DEA context. They divided the ranking methods into six classes involving (1) cross-efficiency ranking methods (eg, see Doyle and Green, 1994; Jahanshahloo *et al*, 2011; Yang *et al*, 2012); (2) super-efficiency ranking techniques (eg, see Andersen and Petersen, 1993; Sueyoshi, 1999; Noura *et al*, 2011); (3) the benchmark ranking method (eg, see Torgersen *et al*, 1996; Lu and Lo, 2009); (4) ranking with multivariate statistics in the DEA context (eg, see Sinuany-Stern *et al*, 1994; Wang *et al*, 2011); (5) the ranking of inefficient DMUs (eg, see Bardhan *et al*, 1996; Jahanshahloo and Afzalinejad, 2006); and (6) DEA and multi-criteria decision-making methods (eg, see Thanassoulis and Dyson, 1992; Halme *et al*, 1999; Yu and Lee, 2013). To improve the discrimination power among the efficient DMUs, a super-efficiency DEA model is a very common technique in which a DMU under evaluation is excluded from the reference set (Andersen and Petersen, 1993; Noguchi *et al*, 2002; Li *et al*, 2007, Noura *et al*, 2011).

The conventional DEA is deterministic, does not require prior weights or explicit specification of the functional relationships between the outputs and inputs and assumes that inputs and outputs are measured precisely. However, the uncertainties inherent in the real-life performance measurement problems inhibit using deterministic DEA models in practice. This limitation of the deterministic DEA models in handling stochastic data has been studied for a long time and a wide range of methods including goal programming (Huang and Li, 1996), fuzzy sets (Kao and Liu, 2000; Lertworasirikul *et al*, 2003), the assurance region concept (Despotis and Smirlis, 2002), chance-constrained programming (Olesen, 2006), joint

**Correspondence: Madjid Tavana, Business Systems and Analytics Department, Lindback Distinguished Chair of Information Systems and Decision Sciences, La Salle University, 1900 West Olney Avenue, Philadelphia, PA 19141, USA.*
E-mail: tavana@lasalle.edu

probabilistic constraints (Bruni *et al*, 2009), local maximum likelihood estimation (Simar and Zelenyuk, 2010) and Monte Carlo simulation (Wong, 2009; Kuah *et al*, 2012) have been proposed to overcome this deficiency.

In addition to the inability of the conventional DEA methods in handling imprecise input and output data in performance measurement problems, DEA is a non-statistical approach that does not allow for hypothesis testing. Form an engineering perspective, the production structure, containing free disposability, economies of scale and convexity, needs to validate the model using statistical inference. Although DEA is not capable of applying statistical inference due to its deterministic nature, there has been some small progress in DEA for testing the model specification. Farrell (1957, pp 269–270) states that '… the introduction of a new factor of production into the analysis cannot lower, and in general, raises the technical efficiency …'. He then suggests 'It is to such frequency distributions [of efficiency scores] that one must look for a measure of the success of the analysis' (Farrell, 1957, p 270). Farrell (1957, p 272) further states 'In going from $x_1x_2x_3$ to $x_1x_2x_3x_4$ [inputs], by introducing capital (another relevant factor) only a little of the apparent variation in efficiency is explained. Thus although in principle $x_1x_2x_3x_4$ is the better analysis, it seems that in fact relatively little difference is made to the efficiency estimates omitting x_4. (This may perhaps be thought of as analogous to a regression analysis where the introduction of a variable does not significantly improve the fit; although here there is no objective measure of significance.)' A statistical testing procedure was later developed by Banker (1993, 1996), Kneip *et al* (1998), Simar and Wilson (1998), Gijbels *et al* (1999) and Tziogkidis (2012) to further improve this testing procedure.

Although some scattered attempts have been made to test the specification of the models in DEA, small samples in DEA normally cannot accommodate sufficient observations to fully represent the production possibilities. As a result, most statistical attempts in DEA produce a small sample error where inefficient DMUs can be imperfectly classified as efficient, or the number of inefficient DMUs can be noticeably underestimated (Banker, 1993; Kneip *et al*, 1998; Gijbels *et al*, 1999). Gijbels *et al* (1999) introduced an analytical method to derive a sampling distribution, while Simar and Wilson (1998) estimated the sampling distribution using bootstrapping techniques. Simar and Wilson (1998) proposed bootstrap DEA to assess the sensitivity and statistical properties of the efficiency scores generated by DEA. They used the percentiles of the bootstrap distribution in their hypothesis testing procedure. The validity and difficulties of the proposed bootstrap DEA by Simar and Wilson (1998) have been a controversial issue among the performance measurement researchers (see, eg, Tziogkidis, 2012). In addition, it is difficult to find an appropriate value of a smoothing parameter in the bootstrap method, and bootstrapping requires a large number of iterations (Hossain *et al*, 2013).

In bootstrapping, efficiency scores are estimated from the original sample data by solving one of several DEA models to form pseudo data where outputs stay the same and inputs are functions of the efficiency scores. A new DEA efficiency score is then calculated for each DMU from these pseudo data. By replicating this process repeatedly, an empirical distribution of efficiency scores can be determined and then used to compute bias-corrected interval estimates (Simar and Wilson, 1998).

In contrast, chance-constrained DEA models are inherently different from the bootstrapping models. In chance-constrained programming (CCP), the input and output data are treated as random variables that replace the deterministic characterizations ('efficient' or 'inefficient') with probabilistic characterizations ('probably efficient' or 'probably inefficient'), and the efficiency measures are calculated based on the probabilistic comparisons with other DMUs. Chance-constraint has a more optimization-based approach compared with bootstrapping (Ceyhan and Benneyan, 2011). In addition, the input and output data collected in the chance-constrained DEA model are treated as random variables, while the input and output data are estimates from the original sample data in bootstrap DEA.

Stochastic input and output variations of DEA have been studied by many researchers including: Land *et al* (1993), Olesen and Petersen (1995), Huang and Li (1996), Li (1998), Morita and Seiford (1999), Cooper *et al* (1996, 1998, 2002, 2004), Bruni *et al* (2009), Khodabakhshi *et al* (2010), Khodabakhshi and Asgharian (2009), Khodabakhshi (2009, 2010), Wu and Lee (2010), Khodabakhshi (2011) and Hosseinzadeh Lotfi *et al* (2012).

Land *et al* (1993) exploited the CCP proposed by Charnes and Cooper (1959) in order to compute efficiency in the face of uncertainty. They developed a chance-constrained model in which inputs were assumed to be deterministic and outputs were jointly normally distributed. Olesen and Petersen (1995) developed a chance-constrained DEA model imposing chance constraints on a DEA formulation in the multiplier form. Joint chance constraints were also the focus in Cooper *et al* (1996) in which the 'satisficing' concepts were introduced in the multiplier form of the DEA model. Li (1998) considered random disturbances in the general input-output data structure. Cooper *et al* (1998) and Huang and Li (2001) suggested the stochastic DEA models with separate chance constraints. These studies defined the efficiency dominance of a DMU using the joint probabilistic comparisons of inputs and outputs with other DMUs.

Morita and Seiford (1999) studied robustness of the efficiency results when input and output data are subject to the stochastic measurement error. Cooper *et al* (2002) provided CCP models that were directed to determining where the efficient and inefficient behaviour occurs with the associated probabilities. Cooper *et al* (2004) extended congestion models in corresponding CCP models. Khodabakhshi and Asgharian (2009) introduced stochastic DEA with an input relaxation. Bruni *et al* (2009) proposed a stochastic model for DEA based on the theory of joint probabilistic constraints. Under the assumption that a robust set of scenarios is available, their models allow for dependencies among the DMUs and among input/output data within the same DMU. Unlike other stochastic DEA models, their deterministic equivalent formulation has a linear integer programming form.

Khodabakhshi (2009) studied the most productive scale size with stochastic data in DEA and Khodabakhshi *et al* (2010) developed an input-oriented super-efficiency measure in stochastic DEA. Khodabakhshi (2011) later proposed the super-efficiency concept based on an input relaxation model in stochastic DEA and Khodabakhshi (2010) used stochastic data with the concept of CCP to develop an output-oriented super-efficiency DEA model. Azadi and Farzipoor Saen (2011) proposed a chance-constrained DEA method to assist the decision makers to identify the most appropriate third-party reverse logistics providers in the presence of both dual-role factors and stochastic data. Hosseinzadeh Lotfi *et al* (2012) recently presented the stochastic centralized resource allocation method for allocating centralized resources in the presence of stochastic inputs and outputs.

There are various types of uncertainties in real-life problems. Random phenomenon is one class of uncertainty that has been studied exhaustively. Another class of uncertainty involves twofold uncertain variables. For example, some input or output variables in a DEA problem could be normally distributed random variables, and their mean values could also be random variables. Thus, the decision maker has to face random values with random parameters to deal with this type of uncertainty. We represent these inputs or outputs in DEA with birandom variables. Birandom variables provide a measurable mapping from a probability space to a collection of random variables for this kind of stochastic phenomenon with incomplete statistical information. Birandom variables were introduced by Liu (2002) and further studied by Yang and Liu (2006) and Peng and Liu (2007). Peng and Zhao (2006) presented the theoretical aspects of the optimistic and pessimistic values in birandom variables. Xu and Zhou (2009) proposed a multi-objective decision-making model with birandom coefficients for the flow shop scheduling problem and transformed the Birandom uncertain model into a deterministic model through an expected value operator. Xu and Ding (2011) developed a chance-constrained multi-objective LP model with birandom coefficients for a vendor selection problem and proposed using a birandom simulation-based genetic algorithm to solve this problem.

In this paper, we propose a chance-constrained DEA model with birandom inputs and outputs for evaluating the efficiency of the DMUs in a stochastic environment. In addition, a super-efficiency approach is developed in which the constraints are treated as birandom events. A deterministic equivalent model is formulated, which is non-linear. We convert the non-linear model into a model with quadratic constraints to solve the non-linear deterministic model. We use a numerical example to show the efficacy of the new model proposed in this study. We use a second example to investigate the stability and robustness of the proposed super-efficiency model with sensitivity analysis. To the best of our knowledge, this is the first study to incorporate birandom variables into a DEA framework.

Liu (2002) showed that the uncertainties inherent in the real-life performance measurement problems are sometimes neither random nor fuzzy. He introduced 'birandom variables' that are measurable mappings from a probability space to a collection of random variables. Yang and Liu (2006) and Peng and Liu (2007) further studied the role of birandom variables in performance management and DEA. The contribution of this paper is sixfold: (1) we address the gap in the DEA literature for problems involving error and random noise in the input and output data; (2) we propose a chance-constrained DEA model with birandom inputs and outputs for evaluating the efficiency of the DMUs in a stochastic environment that is more suitable and less restrictive in some real-life problems; (3) we develop a super-efficiency approach in which the constraints are treated as birandom events; (4) we meticulously examine the stability and the robustness of the proposed super-efficiency model with sensitivity analysis; (5) we present a non-linear deterministic equivalent model that can be converted into a model with quadratic constraints for solving the stochastic optimization problem; and (6) we demonstrate the efficacy and applicability of the proposed models with two numerical examples.

The remainder of the paper is organized as follows. In the next section, we review the primary definitions of birandom variables in the literature followed by a description of the conventional DEA models in Section 3. In Section 4, we propose the stochastic DEA models and in Section 5 we present the Birandom super-efficiency model. In Section 6, we discuss the sensitivity of the Birandom super-efficiency model. We demonstrate the efficacy of the new chance-constrained DEA model and the applicability of our sensitivity analysis in Section 7. Finally, in Section 8, we present our conclusions and future research directions.

2. Preliminaries and definitions

Peng and Liu (2007) recently introduced a new concept of uncertainty, the so-called birandom variables, that is a measurable mapping from a probability space to a collection of random variables. This section presents some basic definitions and properties of birandom variables.

Definition 1 (Peng and Liu, 2007). Let Ω be a non-empty set, A is a σ-algebra of subsets of Ω and Pr is a probability measure. Then, the triplet (Ω, A, Pr) is called a probability space.

Definition 2 (Peng and Liu, 2007). A birandom variable ξ is a mapping from a probability space (Ω, A, Pr) to a collection S random variables such that for any Borel subset B of the real line $\Re$, the induced function $Pr\{\xi(\omega)\in B\}$ is a measurable function with respect to ω.

Definition 3 (Peng and Liu, 2007). An n-dimensional birandom vector ξ is a mapping from the probability space (Ω, A, Pr) to a collection of n-dimensional random vectors such that $Pr\{\xi(\omega)\in B\}$ is a measurable function with respect to ω for any Borel subset B of the real space $\Re^n$.

Theorem 1 (Peng and Liu, 2007). Let $\xi=(\xi_1,\xi_2,\ldots,\xi_n)$ be a birandom vector, and f be a Borel measurable function from R^n to R. Then, $f(\xi)$ is a birandom variable.

Definition 4 (Peng and Liu, 2007). Let $f: R^n \rightarrow \Re$ be a Borel measurable function, and ξ be birandom variables on the probability spaces (Ω_i, A_i, Pr_i), $i=1,\ldots,n$, respectively. Then, the sum of these birandom variables, $\xi=f(\xi_1,\xi_2,\ldots,\xi_n)$, is a birandom variable on $(\Omega_1\times\Omega_2, A_1\times A_2, Pr_1\times Pr_2)$ and is obtained as follows:

$$\xi(\omega_1,\omega_2,\ldots,\omega_n)=f(\xi_1(\omega_1),\xi_2(\omega_2),\ldots,\xi_n(\omega_n))$$

$$\text{for all } (\omega_1,\omega_2,\ldots,\omega_n)\in\Omega_1\times\cdots\times\Omega_n.$$

For example, let $\overline{\mu}\sim\mu(0,1)$ be a uniformly distributed random variable on (Ω, A, Pr). Then $\xi\sim N(\overline{\mu},1)$ is a normally distributed birandom variable.

Definition 5 (Peng and Liu, 2007). Assume that ξ is a birandom variable defined on the probability space (Ω, A, Pr). Then the expected value of the birandom variable ξ is defined as follows:

$$E(\xi)=\int_0^{+\infty} Pr\{\omega\in\Omega \mid E[\xi(\omega)]\geqslant t\}dt - \int_{-\infty}^{0} Pr\{\omega\in\Omega \mid E[\xi(\omega)]\leqslant t\}dt$$

Note that that the above formula involves two integrals and at least one of them is finite.

Theorem 2 (Peng and Liu, 2007). Let $E(\xi)$ and $E[\eta]$ be the finite expected values of the birandom variables ξ and η, respectively. Then we have

$$(1)\ \ E[a\xi+b\eta]=aE[\xi]+bE[\eta]$$

where a and b are arbitrary real numbers.

(2)The variance of the birandom variable ξ is defined as follows:

$$V[\xi]=E\left[(\xi-E[\xi])^2\right]$$

$$V[\xi]=E\left[\xi^2\right]-(E[\xi])^2$$

Definition 6 Let ξ be a normally distributed birandom variable, denoted by $\xi\sim N(\overline{\mu},\overline{\sigma}^2)$, on the probability space (Ω, A, Pr), in which $Pr=\overline{p}(x)$ has the following probability density function:

$$\overline{p}(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\overline{\mu})^2}{2\overline{\sigma}^2}},\quad -\infty<x<+\infty$$

where $\overline{\mu}$ and $\overline{\sigma}^2$ are the mean and variance of ξ, respectively, which are characterized by the random variables on the probability space (Ω^*, A^*, Pr^*).

3. DEA–CCR model

Let us assume that there are n DMUs to be evaluated where every DMU$_j$, $j=1,2,\ldots,n$, produces s outputs, $y_{rj}\,(r=1,\ldots,s)$, using m inputs, $x_{ij}\,(i=1,2,\ldots,m)$. The following problem is posed to calculate the technical input-efficiency of a given DMU$_p$:

$$\begin{aligned}
\min\quad & \theta\\
\text{s.t.}\quad & \\
& \sum_{j=1}^{n} x_{ij}\lambda_j - x_{ip}\theta \leqslant 0,\quad i=1,\ldots,m,\\
& \sum_{j=1}^{n} y_{rj}\lambda_j - y_{rp} \geqslant 0,\quad r=1,\ldots,s,\\
& \lambda_j \geqslant 0,\quad j=1,\ldots,n
\end{aligned}\qquad (1)$$

and the technical output-efficiency of DMU$_p$ is presented as follows:

$$\begin{aligned}
\min\quad & \varphi\\
\text{s.t.}\quad & \\
& \sum_{j=1}^{n} \lambda_j x_{ij} - x_{ip} \leqslant 0,\quad i=1,\ldots,m,\\
& \sum_{j=1}^{n} \lambda_j y_{rj} - \varphi y_{rp} \geqslant 0,\quad r=1,\ldots,s,\\
& \lambda_j \geqslant 0,\quad j=1,\ldots,n
\end{aligned}$$

where λ_j is a non-negative weight associated with each DMU to construct a convex combination of them or a subset of λ_j's. DEA estimates the production possibility set using the efficient DMUs. A DMU is said to be efficient if the optimal value of θ or φ is equal to 1; otherwise, DMU$_p$ is [technically] inefficient.

4. Birandom CCR model

Due to the lack of complete knowledge and information, we cannot explicitly use the conventional deterministic mathematical DEA model to measure the true relative efficiency in complex real-life problems. The statistical information is often used in real-life problems as a bridge between uncertainty and certainty. In this section, we present the mathematical details of the proposed approach for solving the CCR model in which input and output data are assumed to be the birandom variables.

Let us assume that $\tilde{\tilde{X}}_j = \left(\tilde{\tilde{X}}_{1j}, \ldots, \tilde{\tilde{X}}_{mj}\right)^T \in \Re^m$ and $\tilde{\tilde{Y}}_j = \left(\tilde{\tilde{y}}_{1j}, \ldots, \tilde{\tilde{y}}_{sj}\right)^T \in \Re^s$ are the birandom input and output vectors for DMU_j, $j=1, \ldots, n$, and each of them has a normal distribution. Let us also assume, $\overline{X}_j = \left(\overline{x}_{1j}, \ldots, \overline{x}_{mj}\right)^T \in \Re^m$ and $\overline{Y}_j = \left(\overline{y}_{1j}, \ldots, \overline{y}_{sj}\right)^T \in \Re^s$ are the expected vectors of the inputs and outputs of $\tilde{\tilde{X}}_j$ and $\tilde{\tilde{Y}}_j$, respectively. In addition, $X_j=(x_{1j}, \ldots, x_{mj})^T \in \Re^m$ and $Y_j=(y_{1j}, \ldots, y_{sj})^T \in \Re^s$ are the expected vectors of the inputs and outputs of $\overline{X}_j = \left(\overline{x}_{1j}, \ldots, \overline{x}_{mj}\right)^T \in \Re^m$ and $\overline{Y}_j = \left(\overline{y}_{1j}, \ldots, \overline{y}_{sj}\right)^T \in \Re^s$, respectively. The birandom inputs and output accordingly can be summarized as follows:

$$\tilde{\tilde{x}}_{ij} \sim N\left(\overline{x}_{ij}, \overline{\sigma}_{ij}^2\right), \overline{x}_{ij} \sim N\left(x_{ij}, \sigma_{ij}^2\right)$$

$$\tilde{\tilde{y}}_{rj} \sim N\left(\overline{y}_{rj}, \overline{\sigma}_{rj}^2\right), \overline{y}_{rj} \sim N\left(y_{rj}, \sigma_{rj}^2\right)$$

The formulation of the chance-constrained CCR model with the birandom data is represented as follows:

Min θ

s.t.

$$P\left[P\left(\sum_{j=1}^{n} \tilde{\tilde{x}}_{ij}\lambda_j - \tilde{\tilde{x}}_{ip}\theta \leqslant 0\right) \geqslant 1-\alpha\right] \geqslant 1-\beta \quad i = 1, \ldots, m,$$

$$P\left[P\left(\sum_{j=1}^{n} \tilde{\tilde{y}}_{rj}\lambda_j - \tilde{\tilde{y}}_{rp} \geqslant 0\right) \geqslant 1-\alpha\right] \geqslant 1-\beta, \quad r = 1, \ldots, s,$$

$$\lambda_j \geqslant 0, \quad j = 1, \ldots, n. \tag{2}$$

where p denotes 'probability' and $\beta \in (0, 1]$ and $\alpha \in (0, 1]$ are the predetermined thresholds defined by the decision maker for identifying an allowable chance of failing to satisfy the constraints. $P[\cdot]$ in Model (2) denotes the possibility and the probability of $[\cdot]$ event. To solve the probability-probability constrained LP model, we convert the constraints of (2) into the deterministic equivalents using the normal standard distribution.

Let us assume $\tilde{\tilde{h}}_i = \sum_{j=1}^{n} \tilde{\tilde{x}}_{ij}\lambda_j - \tilde{\tilde{x}}_{ip}\theta$. Due to the normal distribution of $\tilde{\tilde{x}}_{ij}, \tilde{\tilde{h}}_i$ also has normal distribution with the following mean:

$$\overline{h}_i = E(\tilde{\tilde{h}}_i) = \sum_{j=1}^{n} \overline{x}_{ij}\lambda_j - \tilde{\tilde{x}}_{ip}\theta$$

For the inner part of the first constraints of Model (2), we have:

$$P\left(\sum_{j=1}^{n} \tilde{\tilde{x}}_{ij}\lambda_j - \tilde{\tilde{x}}_{ip}\theta \leqslant 0\right) \geqslant 1-\alpha$$

$$\Leftrightarrow P\left(\frac{\left(\sum_{j=1}^{n} \tilde{\tilde{x}}_{ij}\lambda_j - \tilde{\tilde{x}}_{ip}\theta\right) - E\left(\tilde{\tilde{h}}_i\right)}{\sqrt{\text{var}\left(\sum_{j=1}^{n} \tilde{\tilde{x}}_{ij}\lambda_j - \tilde{\tilde{x}}_{ip}\theta\right)}} \leqslant \frac{-E\left(\tilde{\tilde{h}}_i\right)}{\sqrt{\text{var}\left(\sum_{j=1}^{n} \tilde{\tilde{x}}_{ij}\lambda_j - \tilde{\tilde{x}}_{ip}\theta\right)}}\right) \geqslant 1-\alpha$$

$$\Leftrightarrow \sum_{j=1}^{n} \overline{x}_{ij}\lambda_j - \overline{x}_{ip}\theta + \sigma_{\tilde{\tilde{h}}_i}^{I}\left(\lambda_j, \theta\right)\Phi^{-1}(1-\alpha) \leqslant 0$$

where

$$\sigma_{\tilde{\tilde{h}}_i}^{I}\left(\lambda_j, \theta\right) = \sqrt{\text{var}\left(\sum_{j=1}^{n} \tilde{\tilde{x}}_{ij}\lambda_j - \tilde{\tilde{x}}_{ip}\theta\right)}$$

$$= \sqrt{\text{var}\left(\sum_{j=1}^{n} \tilde{\tilde{x}}_{ij}\lambda_j\right) + \text{var}(\tilde{\tilde{x}}_{ip}\theta) - 2Cov\left(\sum_{j=1}^{n} \tilde{\tilde{x}}_{ij}\lambda_j, \tilde{\tilde{x}}_{ip}\theta\right)}$$

$$= \sqrt{\sum_{j=1}^{n}\sum_{k=1}^{n} \lambda_j\lambda_k Cov\left(\tilde{\tilde{x}}_{ij}, \tilde{\tilde{x}}_{ik}\right) + \theta^2 \text{var}\left(\tilde{\tilde{x}}_{ip}\right) - 2\theta\sum_{j=1}^{n} \lambda_j Cov\left(\tilde{\tilde{x}}_{ij}, \tilde{\tilde{x}}_{ip}\right)}$$

Therefore, it results in the constraints as follows:

$$P\left[\sum_{j=1}^{n} \overline{x}_{ij}\lambda_j - \overline{x}_{ip}\theta + \sigma_{\tilde{\tilde{h}}_i}^{I}\left(\lambda_j, \theta\right)\Phi^{-1}(1-\alpha) \leqslant 0\right] \geqslant 1-\beta$$

A similar method can be adapted to another constraint of (2) as follows:

$$P\left(\sum_{j=1}^{n} \overline{y}_{rj}\lambda_j - \overline{y}_{rp} - \sigma_{\tilde{\tilde{h}}_r}^{O}\left(\lambda_j\right)\Phi^{-1}(1-\alpha) \geqslant 0\right) \geqslant 1-\beta$$

where

$$\sigma_{\tilde{\tilde{h}}_r}^{O}\left(\lambda_j\right) = \sqrt{\text{Var}\left(\sum_{j=1}^{n} \tilde{\tilde{y}}_{rj}\lambda_j - \tilde{\tilde{y}}_{rp}\right)}$$

$$= \sqrt{\sum_{j=1}^{n}\sum_{k=1}^{n} \lambda_j\lambda_k Cov(\tilde{\tilde{y}}_{rj}, \tilde{\tilde{y}}_{ik}) + \text{Var}(\tilde{\tilde{y}}_{rp}) - 2\sum_{j=1}^{n} \lambda_j Cov(\tilde{\tilde{y}}_{rj}, \tilde{\tilde{y}}_{rp})}$$

Note that the superscripts I and O refer to the input and output, respectively. As a result, Model (2) is changed to the following chance-constrained programming model:

$$
\begin{aligned}
&\min \theta \\
&\text{s.t.} \\
&P\left[\sum_{j=1}^{n}\bar{x}_{ij}\lambda_j-\bar{x}_{ip}\theta+\sigma^{I}_{\tilde{\bar{h}}_i}\left(\lambda_j,\theta\right)\Phi^{-1}(1-\alpha)\leqslant 0\right] \\
&\quad\geqslant 1-\beta, \quad i=1,\dots,m, \\
&P\left[\sum_{j=1}^{n}\bar{y}_{rj}\lambda_j-\bar{y}_{rp}-\sigma^{O}_{\tilde{\bar{h}}_i}\left(\lambda_j\right)\geqslant\Phi^{-1}(1-\alpha)\geqslant 0\right] \\
&\quad\geqslant 1-\beta, \quad r=1,\dots,s, \\
&\sigma^{I}_{\tilde{\bar{h}}_i},\sigma^{O}_{\tilde{\bar{h}}_i},\lambda_j\geqslant 0, \quad j=1,\dots,n.
\end{aligned} \tag{3}
$$

The standardized normal distribution can be applied to transform the above chance-constrained model to a deterministic model. Consequently, let us take the first set of constraints of Model (3) into account as $P\left[\bar{h}_i\leqslant 0\right]\geqslant 1-\beta$ where $\bar{h}_i=\sum_{j=1}^{n}\bar{x}_{ij}\lambda_j-\bar{x}_{ip}\theta+\sigma^{I}_{\tilde{\bar{h}}_i}\left(\lambda_j,\theta\right)\Phi^{-1}(1-\alpha)$. Since $\bar{x}_{ij}$ has the normal distribution, $\bar{h}_i$ also has the normal distribution with the following mean and variance:

$$
\begin{aligned}
h_i=E\left[\bar{h}_i\right]&=E\left[\sum_{j=1}^{n}\bar{x}_{ij}\lambda_j-\bar{x}_{ip}\theta+\sigma^{I}_{\tilde{\bar{h}}_i}(\lambda_j,\theta)\Phi^{-1}(1-\alpha)\right] \\
&=\sum_{j=1}^{n}x_{ij}\lambda_j-x_{ip}\theta+\sigma^{I}_{\tilde{\bar{h}}_i}(\lambda_j,\theta)\Phi^{-1}(1-\alpha)
\end{aligned}
$$

$$
\begin{aligned}
\sigma^{I}_{\bar{h}_i}(\lambda_j,\theta)&=\sqrt{\operatorname{Var}\left(\sum_{j=1}^{n}\bar{x}_{ij}\lambda_j-\bar{x}_{ip}\theta+\sigma^{I}_{\tilde{\bar{h}}_i}(\lambda_j,\theta)\,\Phi^{-1}(1-\alpha)\right)} \\
&=\sqrt{\sum_{j=1}^{n}\sum_{k=1}^{n}\lambda_j\lambda_k Cov\left(\bar{x}_{ij},\bar{x}_{ik}\right)+\theta^2\operatorname{var}(\bar{x}_{ip})-2\theta\sum_{j=1}^{n}\lambda_j Cov\left(\bar{x}_{ij},\bar{x}_{ip}\right)}
\end{aligned}
$$

By standardizing the normal distribution, $Pr\left(\bar{h}_i\leqslant 0\right)\geqslant 1-\beta$ is converted to $Pr\left(z\leqslant -\left(E\left(\bar{h}_i\right)\right)\Big/\sqrt{\operatorname{var}\left(\bar{h}_i\right)}\right)\geqslant 1-\beta$ where $z=\left(h_i-E\left(\bar{h}_i\right)\right)\Big/\sqrt{\operatorname{var}\left(\bar{h}_i\right)}$ is the standard normal random variable with 0 mean and unit variance. The corresponding cumulative distribution function is $\Phi\left(\left(-E\left(\bar{h}_i\right)\right)\Big/\sqrt{\operatorname{Var}\left(\bar{h}_i\right)}\right)\geqslant 1-\beta$ and it is equal to:

$$
\begin{aligned}
&\sum_{j=1}^{n}x_{ij}\lambda_j-x_{ip}\theta+\sigma^{I}_{\tilde{\bar{h}}_i}\left(\lambda_j,\theta\right)\Phi^{-1}(1-\alpha) \\
&\quad+\sigma^{I}_{\bar{h}_i}\left(\lambda_j,\theta\right)\Phi^{-1}(1-\beta)\leqslant 0
\end{aligned}
$$

where $\Phi^{-1}(1-\beta)$ is the inverse of Φ at the level of $1-\beta$.

We can easily mimic the above method on the second set of constraints of (3) to obtain its deterministic form as follows:

$$
\begin{aligned}
&\sum_{j=1}^{n}y_{rj}\lambda_j-y_{rp}-\sigma^{O}_{\tilde{\bar{h}}_r}\left(\lambda_j\right)\Phi^{-1}(1-\alpha) \\
&\quad-\sigma^{O}_{\tilde{\bar{h}}_r}\left(\lambda_j\right)\Phi^{-1}(1-\beta)\geqslant 0
\end{aligned}
$$

$$
\sigma^{O}_{\bar{h}_r}(\lambda_j)=\sqrt{\sum_{j=1}^{n}\sum_{k=1}^{n}\lambda_j\lambda_k Cov(\bar{y}_{rj},\bar{y}_{ik})+\operatorname{Var}(\bar{y}_{rp})-2\sum_{j=1}^{n}\lambda_j Cov(\bar{y}_{rj},\bar{y}_{rp})},
$$

$$r=1,\dots,s.$$

The deterministic equivalent of Model (3) is finally established as follows:

$$
\begin{aligned}
&\min \theta \\
&\text{s.t.} \\
&\sum_{j=1}^{n}x_{ij}\lambda_j-x_{ip}\theta+\sigma^{I}_{\tilde{\bar{h}}_i}\left(\lambda_j,\theta\right)\Phi^{-1}(1-\alpha) \\
&\quad+\sigma^{I}_{\bar{h}_r}\left(\lambda_j,\theta\right)\Phi^{-1}(1-\beta)\leqslant 0, \quad i=1,\dots,m, \\
&\sum_{j=1}^{n}y_{rj}\lambda_j-y_{rp}-\sigma^{O}_{\tilde{\bar{h}}_r}\left(\lambda_j\right)\Phi^{-1}(1-\alpha) \\
&\quad-\sigma^{O}_{\bar{h}_r}\left(\lambda_j\right)\Phi^{-1}(1-\beta)\geqslant 0, \quad r=1,\dots,s, \\
&\sigma^{I}_{\bar{h}_r},\sigma^{O}_{\bar{h}_r},\sigma^{I}_{\tilde{\bar{h}}_i},\sigma^{O}_{\tilde{\bar{h}}_r},\lambda_j\geqslant 0, \quad j=1,\dots,n.
\end{aligned} \tag{4}
$$

It is apparent that the above programme is a non-LP model because of $\sigma^{I}_{\tilde{\bar{h}}_i}\left(\lambda_j,\theta\right),\sigma^{I}_{\bar{h}_i}\left(\lambda_j,\theta\right),\sigma^{o}_{\tilde{\bar{h}}_r}\left(\lambda_j\right)$ and $\sigma^{o}_{\bar{h}_r}\left(\lambda_j\right)$. We demonstrate that this non-linear model can be solved by transforming it into a model with quadratic constraints. Suppose that $\bar{v}_i,v_i,\bar{u}_r$ and u_r are the non-negative variables that are substituted by $\sigma^{I}_{\tilde{\bar{h}}_i}\left(\lambda_j,\theta\right),\sigma^{I}_{\bar{h}_i}\left(\lambda_j,\theta\right),\sigma^{o}_{\tilde{\bar{h}}_r}\left(\lambda_j\right)$ and $\sigma^{o}_{\bar{h}_r}\left(\lambda_j\right)$, respectively. Accordingly, the following model with quadratic constraints is established:

$$
\begin{aligned}
&\min \theta \\
&\text{s.t.} \\
&\sum_{j=1}^{n}x_{ij}\lambda_j-x_{ip}\theta+\bar{v}_i\Phi^{-1}(1-\alpha) \\
&\quad+v_i\Phi^{-1}(1-\beta)\leqslant 0, \quad i=1,\dots,m,
\end{aligned}
$$

$$\sum_{j=1}^{n} y_{rj}\lambda_j - y_{rp} - \bar{u}_r\Phi^{-1}(1-\alpha)$$
$$- u_r\Phi^{-1}(1-\beta) \geqslant 0, \quad r = 1, \ldots, s,$$
$$v_i^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} \lambda_j\lambda_k Cov(\bar{x}_{ij}, \bar{x}_{ik}) + \theta^2 \mathrm{Var}(\bar{x}_{ip})$$
$$- 2\theta \sum_{j=1}^{n} \lambda_j Cov(\bar{x}_{ij}, \bar{x}_{ip}), \quad i = 1, \ldots, m,$$
$$\bar{v}_i^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} \lambda_j\lambda_k Cov(\tilde{\bar{x}}_{ij}, \tilde{\bar{x}}_{ik}) + \theta^2 \mathrm{Var}(\tilde{\bar{x}}_{ip})$$
$$- 2\theta \sum_{j=1}^{n} \lambda_j Cov(\tilde{\bar{x}}_{ij}, \tilde{\bar{x}}_{ip}), \quad i = 1, \ldots, m,$$
$$\bar{u}_r^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} \lambda_j\lambda_k Cov(\tilde{\bar{y}}_{rj}, \tilde{\bar{y}}_{ik}) + \mathrm{Var}(\tilde{\bar{y}}_{rp})$$
$$- 2 \sum_{j=1}^{n} \lambda_j Cov(\tilde{\bar{y}}_{rj}, \tilde{\bar{y}}_{rp}), \quad r = 1, \ldots, s,$$
$$u_r^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} \lambda_j\lambda_k Cov(\bar{y}_{rj}, \bar{y}_{ik}) + \mathrm{Var}(\bar{y}_{rp})$$
$$- 2 \sum_{j=1}^{n} \lambda_j Cov(\bar{y}_{rj}, \bar{y}_{rp}), \quad r = 1, \ldots, s,$$
$$\lambda_j, v_i, \bar{v}_i, u_r, \bar{u}_r \geqslant 0,$$
$$j = 1, \ldots, n; i = 1, \ldots, m; r = 1, \ldots, s. \tag{5}$$

Definition 7 A DMU is said to be probabilistic-probabilistic $\alpha-\beta$ efficient if the optimal value of the objective function of Model (5) is equal to 1 at the probability level α and probability level β; otherwise, it is said to be probabilistic-probabilistic $\alpha-\beta$ inefficient.

Proposition 1 Model (5) for any α and β level is feasible.

Proof Let

$$\lambda_j = \begin{cases} 1 & j = p \\ 0 & j \neq p \end{cases}, \quad j = 1, \ldots, n, \quad \theta = 1.$$

Then $v_i = 0, \bar{v}_i = 0, u_r = 0, \bar{u}_r = 0$. This solution is a feasible solution for Model (5). □

Proposition 2 If $\alpha = 0.5$ and $\beta = 0.5$, then Model (5) changes to the conventional CCR Model (1).

Proof Obviously $\Phi^{-1}(\alpha)$=$\Phi^{-1}(\beta)$=$\Phi^{-1}(0.5)$=0 and the proof is complete. □

Proposition 3 For $\alpha \leqslant 0.5$ and $\beta \leqslant 0.5$, then $0 < \theta^* \leqslant 1$.

Proof Assume the similar feasible solution presented in the Proof of Proposition 1. Because of a minimization model, $\theta^* \leqslant 1$, we only need to prove $\theta^* > 0$. Since $\alpha \leqslant 0.5$ and $\beta \leqslant 0.5$, we have $\Phi^{-1}(1-\alpha) \geqslant 0$, $\Phi^{-1}(1-\beta) \geqslant 0$, $u_r \geqslant 0$, r=1, … , s and $v_i \geqslant 0$, i=1, … , m.
From the first constraints of Model (5), we have the following:

$$\theta \geqslant \frac{\sum_{j=1}^{n} \lambda_j x_{ij}}{x_{ip}}$$

$\theta > 0$ can be derived from the above equation. To demonstrate the non-zero value of θ, let us assume $\theta \leqslant 0$. Thus λ_j=0 and with regard to the second constraints of Model (5) we have $y_{rp} \leqslant 0$, which is a contradiction with $y_{rp} \geqslant 0$ and the proof is complete. □

In order to provide a more comprehensive view in addition to the input-orientation CCR model, we now take the output-oriented CCR model into consideration with the birandom data as follows:

$$\max \quad \varphi$$
$$\text{s.t.}$$
$$\sum_{j=1}^{n} x_{ij}\lambda_j - x_{ip} + \bar{v}_i\Phi^{-1}(1-\alpha)$$
$$+ v_i\Phi^{-1}(1-\beta) \leqslant 0, \quad i = 1, \ldots, m,$$
$$\sum_{j=1}^{n} y_{rj}\lambda_j - \varphi y_{rp} - \bar{u}_r\Phi^{-1}(1-\alpha)$$
$$- u_r\Phi^{-1}(1-\beta) \geqslant 0, \quad r = 1, \ldots, s,$$
$$v_i^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} \lambda_j\lambda_k Cov(\bar{x}_{ij}, \bar{x}_{ik}) + \mathrm{Var}(\bar{x}_{ip})$$
$$- 2 \sum_{j=1}^{n} \lambda_j Cov(\bar{x}_{ij}, \bar{x}_{ip}), \quad i = 1, \ldots, m,$$
$$\bar{v}_i^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} \lambda_j\lambda_k Cov(\tilde{\bar{x}}_{ij}, \tilde{\bar{x}}_{ik}) + \mathrm{Var}(\tilde{\bar{x}}_{ip})$$
$$- 2 \sum_{j=1}^{n} \lambda_j Cov(\tilde{\bar{x}}_{ij}, \tilde{\bar{x}}_{ip}), \quad i = 1, \ldots, m,$$
$$\bar{u}_r^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} \lambda_j\lambda_k Cov(\tilde{\bar{y}}_{rj}, \tilde{\bar{y}}_{ik}) + \varphi^2 \mathrm{Var}(\tilde{\bar{y}}_{rp})$$
$$- 2\varphi \sum_{j=1}^{n} \lambda_j Cov(\tilde{\bar{y}}_{rj}, \tilde{\bar{y}}_{rp}), \quad r = 1, \ldots, s,$$

$$u_r^2=\sum_{j=1}^{n}\sum_{k=1}^{n}\lambda_j\lambda_k Cov(\bar{y}_{rj},\bar{y}_{ik})+\varphi^2\mathrm{Var}(\bar{y}_{rp})$$

$$-2\varphi\sum_{j=1}^{n}\lambda_j Cov(\bar{y}_{rj},\bar{y}_{rp}),\qquad r=1,\ldots,s,$$

$$\lambda_j, v_i, \bar{v}_i, u_r, \bar{u}_r \geqslant 0$$

$$j=1,\ldots,n;\ i=1,\ldots,m;\ r=1,\ldots,s. \qquad (6)$$

Proposition 4 The optimal value of the objection function of Model (6) for any α and β level is larger than or equal to 1, that is, $\varphi^*\geqslant 1$.

Proof Assume the similar feasible solution presented in the Proof of Proposition 1. Model (6) is a maximization model, so we have $\varphi^*\geqslant 1$. □

Proposition 5 Let θ^* and φ^* be the optimal value of Models (5) and (6), respectively. Then we have $\varphi^*=1/\theta^*$.

Proof By substituting$\varphi=1/\theta$ in the constraints of Model (5), we derive the following constraints:

$$\sum_{j=1}^{n}x_{ij}\varphi\lambda_j-x_{ip}+\varphi\bar{v}_i\Phi^{-1}(1-\alpha)$$

$$+\varphi v_i\Phi^{-1}(1-\beta)\leqslant 0,\qquad i=1,\ldots,m,$$

$$\sum_{j=1}^{n}y_{rj}\lambda_j-y_{rp}-\bar{u}_r\Phi^{-1}(1-\alpha)$$

$$-u_r\Phi^{-1}(1-\beta)\geqslant 0,\quad r=1,\ldots,s,$$

$$(\varphi v_i)^2=\sum_{j=1}^{n}\sum_{k=1}^{n}\lambda_j\lambda_k\varphi^2 Cov(\bar{x}_{ij},\bar{x}_{ik})$$

$$+\mathrm{Var}(\bar{x}_{ip})-2\varphi\sum_{j=1}^{n}\lambda_j Cov(\bar{x}_{ij},\bar{x}_{ip}),\qquad i=1,\ldots,m,$$

$$(\varphi\bar{v}_i)^2=\sum_{j=1}^{n}\sum_{k=1}^{n}\lambda_j\lambda_k\varphi^2 Cov(\tilde{\bar{x}}_{ij},\tilde{\bar{x}}_{ik})+\mathrm{Var}(\tilde{\bar{x}}_{ip})$$

$$-2\varphi\sum_{j=1}^{n}\lambda_j\mathrm{cov}(\tilde{\bar{x}}_{ij},\tilde{\bar{x}}_{ip}),\qquad i=1,\ldots,m,$$

$$\bar{u}_r^2=\sum_{j=1}^{n}\sum_{k=1}^{n}\lambda_j\lambda_k Cov(\tilde{\bar{y}}_{rj},\tilde{\bar{y}}_{ik})+\mathrm{Var}(\tilde{\bar{y}}_{rp})$$

$$-2\sum_{j=1}^{n}\lambda_j Cov(\tilde{\bar{y}}_{rj},\tilde{\bar{y}}_{rp}),\qquad r=1,\ldots,s,$$

$$u_r^2=\sum_{j=1}^{n}\sum_{k=1}^{n}\lambda_j\lambda_k Cov(\bar{y}_{rj},\bar{y}_{ik})+\mathrm{Var}(\bar{y}_{rp})$$

$$-2\sum_{j=1}^{n}\lambda_j Cov(\bar{y}_{rj},\bar{y}_{rp}),\qquad r=1,\ldots,s \qquad (7)$$

The constraints in Model (6) are equivalent to the Constraints (7) if we incorporate the non-negative variable substitutions $\bar{\lambda}_j=\varphi\lambda_j, \bar{u}_r=\varphi u_r$ and $\bar{v}_i=\varphi v_i$ into Constraints (7). On the other hand, the minimization of θ is equivalent to the maximization of φ. Then Model (6) will be attained and the proof is complete. □

5. Super-efficiency Birandom DEA model

An acute shortcoming of DEA is that, generally, it specifies more than one efficient DMU, particularly when the number of DMUs is not much bigger than the sum of the number of inputs and outputs. The following super-efficiency model introduced by Andersen and Petersen (1993) is identical to the standard model, except that the unit under evaluation is removed from the reference set in ranking the efficient DMUs:

$$\min\ \theta$$

$$\text{s.t.}$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n}x_{ij}\lambda_j-x_{ip}\,\theta\leqslant 0,\quad i=1,\ldots,m,$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n}y_{rj}\lambda_j-y_{rp}\geqslant 0,\quad r=1,\ldots,s,$$

$$\lambda_j\geqslant 0,\quad j=1,\ldots,n. \qquad (8)$$

Similar to the Birandom CCR model proposed in the previous section, the super-efficiency Birandom model is developed to improve the discrimination power. The generic Birandom super-efficiency model is as follows:

$$\mathrm{Min}\ \theta$$

$$\text{s.t.}$$

$$P\left[P\left(\sum_{\substack{j=1\\ j\neq p}}^{n}\tilde{\bar{x}}_{ij}\lambda_j-\tilde{\bar{x}}_{ip}\theta\leqslant 0\right)\geqslant 1-\alpha\right]$$

$$\geqslant 1-\beta,\quad i=1,\ldots,m,$$

$$P\left[P\left(\sum_{\substack{j=1\\ j\neq p}}^{n}\tilde{\bar{y}}_{rj}\lambda_j-\tilde{\bar{y}}_{rp}\geqslant 0\right)\geqslant 1-\alpha\right]$$

$$\geqslant 1-\beta,\quad r=1,\ldots,s,$$

$$\lambda_j\geqslant 0,\quad j=1,\ldots,n. \qquad (9)$$

Analogously, the above stochastic programme can be converted into the following deterministic equivalent:

$$
\begin{aligned}
&\text{Min}\ \theta^{Super}\\
&\text{s.t.}\\
&\sum_{\substack{j=1\\ j\neq p}}^{n} x_{ij}\lambda_j - x_{ip}\theta - \bar{v}_i\Phi^{-1}(1-\alpha)\\
&\quad - v_i\Phi^{-1}(1-\beta)\leqslant 0, \quad i=1,\dots,m,\\
&\sum_{\substack{j=1\\ j\neq p}}^{n} y_{rj}\lambda_j - y_{rp} + \bar{u}\Phi^{-1}(1-\alpha)_r\\
&\quad + u_r\Phi^{-1}(1-\beta)\geqslant 0, \quad r=1,\dots,s,\\
&v_i^2 = \sum_{\substack{j=1\\ j\neq p}}^{n}\sum_{\substack{k=1\\ j\neq p}}^{n} \lambda_j\lambda_k Cov(\bar{x}_{ij},\bar{x}_{ik}) + \theta^2\text{Var}(\bar{x}_{ip})\\
&\quad -2\theta\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j Cov(\bar{x}_{ij},\bar{x}_{ip}), \quad i=1,\dots,m,\\
&\bar{v}_i^2 = \sum_{\substack{j=1\\ j\neq p}}^{n}\sum_{\substack{k=1\\ j\neq p}}^{n} \lambda_j\lambda_k Cov(\tilde{\bar{x}}_{ij},\tilde{\bar{x}}_{ik}) + \theta^2\text{Var}(\tilde{\bar{x}}_{ip})\\
&\quad -2\theta\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j Cov(\tilde{\bar{x}}_{ij},\tilde{\bar{x}}_{ip}), \quad i=1,\dots,m,\\
&\bar{u}_r^2 = \sum_{\substack{j=1\\ j\neq p}}^{n}\sum_{\substack{k=1\\ j\neq p}}^{n} \lambda_j\lambda_k Cov(\tilde{\bar{y}}_{rj},\tilde{\bar{y}}_{ik}) + \text{Var}(\tilde{\bar{y}}_{rp})\\
&\quad -2\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j Cov(\tilde{\bar{y}}_{rj},\tilde{\bar{y}}_{rp}), \quad r=1,\dots,s,\\
&u_r^2 = \sum_{\substack{j=1\\ j\neq p}}^{n}\sum_{\substack{k=1\\ j\neq p}}^{n} \lambda_j\lambda_k Cov(\bar{y}_{rj},\bar{y}_{ik}) + \text{Var}(\bar{y}_{rp})\\
&\quad -2\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j Cov(\bar{y}_{rj},\bar{y}_{rp}), \quad r=1,\dots,s,\\
&\lambda_j,\ v_i,\ \bar{v}_i,\ u_r,\ \bar{u}_r\geqslant 0,\\
&j=1,\dots,n;\ i=1,\dots,m;\ r=1,\dots,s. \qquad (10)
\end{aligned}
$$

Note that a super-efficiency score less than 1 implies that the DMU is super-inefficient, and scores equal to 1 or greater than 1 imply that the DMU is efficient.

6. Sensitivity analysis

Sensitivity analysis, one of the most important topics in optimization theory, is a systematic study of how sensitive and effective the solutions of the optimization programme are to small changes in the data. In this section, we take into account the data variations in DMU_p to simplify the implementation of sensitivity analysis.

These sensitivity analyses are directed at analysing the allowable limits in data variations for only one DMU at a time and hence contrast with other approaches to sensitivity analysis in DEA that allow all the data for all the DMUs to be varied simultaneously until at least one DMU changes its status from efficient to inefficient, or vice versa (see Cooper *et al*, 2002; Khodabakhshi *et al*, 2010). In other words, random variations of DMU_p in its input and outputs can be shown as follows:

$$
\begin{aligned}
&\sigma^I_{\tilde{\bar{h}}_i}(\lambda_p,\theta)\neq 0, \sigma^I_{\bar{h}_i}(\lambda_p,\theta)\neq 0, \sigma^O_{\tilde{\bar{h}}_r}(\lambda_p)\neq 0,\\
&\sigma^O_{\bar{h}_{rp}}(\lambda_p)\neq 0\\
&\sigma^I_{\tilde{\bar{h}}_i}(\lambda_j,\theta)=0, \sigma^I_{\bar{h}_i}(\lambda_j,\theta)=0, \sigma^O_{\tilde{\bar{h}}_r}(\lambda_j)=0,\\
&\sigma^O_{\bar{h}_r}(\lambda_p)=0\ (j\neq p)
\end{aligned}
$$

for all i and r.

Hence, the deterministic equivalent of the super-efficiency stochastic model can be rewritten as follows:

$$
\begin{aligned}
&\min\ \tilde{\theta}\\
&\text{s.t.}\\
&\sum_{\substack{j=1\\ j\neq p}}^{n} x_{ij}\lambda_j - \hat{x}_{ip}\tilde{\theta}\leqslant 0, \quad i=1,\dots,m,\\
&\sum_{\substack{j=1\\ j\neq p}}^{n} y_{rj}\lambda_j - \hat{y}_{rp}\geqslant 0, \quad r=1,\dots,s,\\
&\lambda_j\geqslant 0, \quad j=1,\dots,n \qquad (11)
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{x}_{ip} &= x_{ip} + \sqrt{\text{Var}(\tilde{\bar{x}}_{ip})}\,\Phi^{-1}(1-\alpha)\\
&\quad + \sqrt{\text{Var}(\bar{x}_{ip})}\,\Phi^{-1}(1-\beta), \quad i=1,\dots,m,\\
\hat{y}_{rp} &= y_{rp} - \sqrt{\text{Var}(\tilde{\bar{y}}_{rp})}\,\Phi^{-1}(1-\alpha)\\
&\quad - \sqrt{\text{Var}(\bar{y}_{rp})}\,\Phi^{-1}(1-\beta)\geqslant 0, \quad r=1,\dots,s
\end{aligned}
$$

Proposition 6 If α=0.5 and β=0.5, then Models (11) and (10) are identical and both models are converted to the conventional super-efficiency Model (8).

Proof Obviously $\Phi^{-1}(\alpha)=\Phi^{-1}(\beta)=\Phi^{-1}(0.5)=0$ and the proof is complete. □

Proposition 7 Let us assume that $0.5<\alpha<1$ and $0.5<\beta<1$.

(I) If $\theta^*>1$ for DMU_p in Model (8), then $\tilde{\theta}^*>1$ in Model (11).
(II) If $\tilde{\theta}^*\leqslant 1$ for DMU_p in Model (11), then $\theta^*\leqslant 1$ in Model (8).

Proof Owing to the fact that $0.5<\alpha<1$ and $0.5<\beta<1$, we have $\Phi^{-1}(1-\alpha)<0$ and $\Phi^{-1}(1-\beta)<0$. Using $\hat{x}_{ip}$ and $\hat{y}_{rp}$ defined in Model (11), we have $\hat{y}_{rp}>y_{rp}$ and $\hat{x}_{ip}<x_{ip}$.

(I) Suppose the contrary, that is $\tilde{\theta}^*\leqslant 1$ in Model (11). Therefore, there exists a solution with $\theta^*=\tilde{\theta}^*\leqslant 1$ for Model (8), when evaluating DMU_p, which is in contrast with $\theta^*>1$. Therefore, this contradiction shows that $\tilde{\theta}^*>1$.
(II) Suppose the contrary, that is $\theta^*>1$ in Model (7). From (I), we must have $\tilde{\theta}^*>1$, which is in contrast with $\tilde{\theta}^*\leqslant 1$. Therefore, this contradiction shows that $\tilde{\theta}^*\leqslant 1$. □

Proposition 8 Let us assume that $0\leqslant\alpha\leqslant 0.5$ and $0\leqslant\beta\leqslant 0.5$.

(I) If $\tilde{\theta}^*>1$ for DMU_p in Model (11), then $\theta^*>1$ in Model (8).
(II) If $\theta^*\leqslant 1$ for DMU_p in Model (8), then $\tilde{\theta}^*\leqslant 1$ Model (11).

Proof Owing to the fact that $0\leqslant\alpha\leqslant 0.5$ and $0\leqslant\beta\leqslant 0.5$, we have $\Phi^{-1}(1-\alpha)>0$ and $\Phi^{-1}(1-\beta)>0$. Using $\hat{x}_{ip}$ and $\hat{y}_{rp}$ defined in Model (11), we have $\hat{y}_{rp}<y_{rp}$ and $\hat{x}_{ip}<x_{ip}$.

(I) Suppose the contrary, that is $\theta^*\leqslant 1$ in Model (7). Therefore, there exists a solution with $\tilde{\theta}^*=\theta^*\leqslant 1$ for Model (8), when evaluating DMU_p, which is in contrast with $\tilde{\theta}^*>1$. Therefore, this contradiction shows that $\theta^*>1$.
(II) The proof is similar to part (II) of Proposition 7. □

In the next two propositions, we discuss the relation between $\tilde{\theta}^*$ (the objective function of Model (11)) and θ^* (the objective function of Model (7)) with additional assumptions.

Proposition 9 Let $0<\alpha<0.5$ and $0<\beta<0.5$ and let $\theta^*>1$ in Model (8). Then, $\tilde{\theta}^*>1$ in Model (11), if

$$\Phi^{-1}(1-\alpha)\left(\sum_{r=1}^{s}\sqrt{\mathrm{Var}(\tilde{\bar{y}}_{rp})}+\sum_{i=1}^{m}\sqrt{\mathrm{Var}(\tilde{\bar{x}}_{ip})}\right)$$
$$+\Phi^{-1}(1-\beta)\left(\sum_{r=1}^{s}\sqrt{\mathrm{Var}(\bar{y}_{rp})}+\sum_{i=1}^{m}\sqrt{\mathrm{Var}(\bar{x}_{ip})}\right)$$
$$<\sum_{r=1}^{s}\beta_r^{+*}+\sum_{i=1}^{m}\beta_i^{-*}$$

where $\sum_{r=1}^{s}\beta_r^{+*}+\sum_{r=1}^{s}\beta_i^{-*}$ is the optimal value of the following model:

$$\min\ \sum_{r=1}^{s}\beta_r^{+}+\sum_{i=1}^{m}\beta_i^{-}$$

s.t.

$$\sum_{\substack{j=1\\ j\neq p}}^{n}x_{ij}\lambda_j-x_{ip}-\beta_i^{-}\leqslant 0,\quad i=1,\ldots,m,$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n}y_{rj}\lambda_j-y_{rp}+\beta_r^{+}\geqslant 0,\quad r=1,\ldots,s,$$

$$\lambda_j,\beta_r^{+},\beta_i^{-}\geqslant 0,\quad j=1,\ldots,n \tag{12}$$

Proof Assume for a contradiction that $\tilde{\theta}^*\leqslant 1$ in Model (11). This implies that there is a $\bar{\lambda}=\left(\bar{\lambda}_1,\ldots,\bar{\lambda}_{p-1},0,\bar{\lambda}_{p+1},\ldots,\bar{\lambda}_n\right)$, $\bar{\lambda}_j\geqslant 0$, $j\neq p$ where $\bar{\lambda}_p=0$ and $\bar{\lambda}_p\geqslant 0(j\neq p)$ such that:

$$\sum_{\substack{j=1\\ j\neq p}}^{n}\hat{x}_{ij}\lambda_j\leqslant\hat{x}_{iP},\quad i=1,\ldots,m,$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n}\hat{y}_{rj}\lambda_j\geqslant\hat{y}_{rP},\quad r=1,\ldots,s$$

Using the constraints of Model (11), we have:

$$\sum_{\substack{j=1\\ j\neq p}}^{n}x_{ij}\bar{\lambda}_j=\sum_{\substack{j=1\\ j\neq p}}^{n}\hat{x}_{ij}\bar{\lambda}_j\leqslant x_{ip}+\sqrt{\mathrm{Var}(\tilde{\bar{x}}_{ip})}\,\Phi^{-1}(1-\alpha)$$
$$+\sqrt{\mathrm{Var}(\bar{x}_{ip})}\,\Phi^{-1}(1-\beta),\quad i=1,\ldots,m,$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n}y_{rj}\bar{\lambda}_j=\sum_{\substack{j=1\\ j\neq p}}^{n}\hat{y}_{ip}\bar{\lambda}_j>y_{rp}-\sqrt{\mathrm{Var}(\tilde{\bar{y}}_{rp})}\,\Phi^{-1}(1-\alpha)$$
$$-\sqrt{\mathrm{Var}(\bar{y}_{rp})}\Phi^{-1}(1-\beta),\quad r=1,\ldots,s,$$

Let

$$\bar{\beta}_i^{-}=\sqrt{\mathrm{Var}(\tilde{\bar{x}}_{ip})}\,\Phi^{-1}(1-\alpha)$$
$$+\sqrt{\mathrm{Var}(\bar{x}_{ip})}\,\Phi^{-1}(1-\beta),\quad i=1,\ldots,m,$$

$$\bar{\beta}_r^{+}=\sqrt{\mathrm{Var}(\tilde{\bar{y}}_{rp})}\,\Phi^{-1}(1-\alpha)$$
$$+\sqrt{\mathrm{Var}(\bar{y}_{rp})}\Phi^{-1}(1-\beta),\quad r=1,\ldots,s$$

We then have a solution $\left(\overline{\beta}_i^-, \overline{\beta}_r^+, \lambda\right)$ that satisfies the constraints in Model (12) with $\sum_{i=1}^{m} \overline{\beta}_i^- + \sum_{r=1}^{s} \overline{\beta}_r^+ < \sum_{i=1}^{m} \overline{\beta}_i^{-*} + \sum_{r=1}^{s} \overline{\beta}_r^{+*}$. This contradiction shows that $\tilde{\theta}^* > 1$ in Model (11). □

Proposition 10 Let $\tilde{\theta}^* > 1$ in Model (11) when $0.5 < \alpha < 1$ and $0.5 < \beta < 1$. Then, $\theta^* > 1$ in Model (8), if

$$-\Phi^{-1}(1-\alpha)\left(\sum_{r=1}^{s}\sqrt{\mathrm{Var}(\tilde{\overline{y}}_{rp})} + \sum_{i=1}^{m}\sqrt{\mathrm{Var}(\tilde{\overline{x}}_{ip})}\right)$$
$$-\Phi^{-1}(1-\beta)\left(\sum_{r=1}^{s}\sqrt{\mathrm{Var}(\overline{y}_{rp})} + \sum_{i=1}^{m}\sqrt{\mathrm{Var}(\overline{x}_{ip})}\right)$$
$$< \sum_{r=1}^{s}\beta_r^{+*} + \sum_{i=1}^{m}\beta_i^{-*}$$

where $\sum_{r=1}^{s}\beta_r^{+*} + \sum_{i=1}^{m}\beta_i^{-*}$ is the optimal value of Model (12).

Proof Assume for a contradiction that $\theta^* \leqslant 1$ in Model (8). This implies that there is a $\overline{\lambda} = (\overline{\lambda}_1, \dots, \overline{\lambda}_{p-1}, 0, \overline{\lambda}_{p+1}, \dots, \overline{\lambda}_n), \overline{\lambda}_j \geqslant 0, j \neq p$ where $\overline{\lambda}_p = 0$ and $\overline{\lambda}_p \geqslant 0 (j \neq p)$ such that:

$$\sum_{\substack{j=1\\ j\neq p}}^{n} x_{ij}\overline{\lambda}_j \leqslant x_{iP}, \quad i = 1, \dots, m,$$
$$\sum_{\substack{j=1\\ j\neq p}}^{n} y_{rj}\overline{\lambda}_j \geqslant y_{rP}, \quad r = 1, \dots, s.$$

Using the constraints of Model (11), we have:

$$\sum_{j=1}^{n} x_{ij}\overline{\lambda}_j = \sum_{\substack{j=1\\ j\neq p}}^{n} x_{ij}\overline{\lambda}_j \leqslant \hat{x}_{ip} - \sqrt{\mathrm{Var}(\tilde{\overline{x}}_{ip})}\,\Phi^{-1}(1-\alpha)$$
$$-\sqrt{\mathrm{Var}(\overline{x}_{ip})}\,\Phi^{-1}(1-\beta), \quad i = 1, \dots, m,$$
$$\sum_{j=1}^{n} y_{rj}\overline{\lambda}_j = \sum_{\substack{j=1\\ j\neq p}}^{n} y_{rj}\overline{\lambda}_j > \hat{y}_{rp} + \sqrt{\mathrm{Var}(\tilde{\overline{y}}_{rp})}\,\Phi^{-1}(1-\alpha)$$
$$+\sqrt{\mathrm{Var}(\overline{y}_{rp})}\Phi^{-1}(1-\beta), \quad r = 1, \dots, s,$$

Let

$$\overline{\beta}_r^- = -\sqrt{\mathrm{Var}(\tilde{\overline{x}}_{ip})}\,\Phi^{-1}(1-\alpha)$$
$$-\sqrt{\mathrm{Var}(\overline{x}_{ip})}\,\Phi^{-1}(1-\beta), \quad i = 1, \dots, m,$$

and

$$\overline{\beta}_r^+ = -\sqrt{\mathrm{Var}(\tilde{\overline{y}}_{rp})}\,\Phi^{-1}(1-\alpha)$$
$$-\sqrt{\mathrm{Var}(\overline{y}_{rp})}\Phi^{-1}(1-\beta), \quad r = 1, \dots, s,$$

We then have a solution $\left(\overline{\beta}_i^-, \overline{\beta}_r^+, \lambda\right)$ that satisfies the constraints in Model (12) with $\sum_{i=1}^{m}\overline{\beta}_i^- + \sum_{r=1}^{s}\overline{\beta}_r^+ < \sum_{i=1}^{m}\overline{\beta}_i^{-*} + \sum_{r=1}^{s}\overline{\beta}_r^{+*}$. This contradiction shows that $\theta^* > 1$ in Model (7). □

7. Numerical examples

In this section, we present two examples. In the first example, we apply birandom programming to the CCR model developed in this study. In the second example, we demonstrate the practical aspects of the sensitivity analysis proposed in this study.

7.1. Example 1

We illustrate the results of the Birandom programming in the DEA formulation through the Birandom CCR and the Birandom super-efficiency solutions of numerical Example 1 presented in Table 1 under the assumption of normality for the inputs and outputs with known mean and variance.

In this example, we evaluate 10 DMUs with two birandom inputs and two birandom outputs that are normally distributed as shown in Table 1. The variances of all birandom input and output data are known and their means are assumed to be a normally distributed random variable with known mean and variance. For instance, the first input for DMU 3 has a normal distribution, its variance is equal to 1, and its mean is a normally distributed random variable with a mean of 2388 and a variance of 9.6. The data in Table 1 can be summarized as follows:

$$\tilde{\overline{x}}_{ij} \sim N\left(\overline{x}_{ij}, \overline{\sigma}_{ij}^2\right) \text{ and } \tilde{\overline{y}}_{rj} \sim N\left(\overline{y}_{rj}, \overline{\sigma}_{rj}^2\right)$$

$$\text{where } \overline{x}_{ij} \sim N\left(x_{ij}, \sigma_{ij}^2\right), \overline{y}_{rj} \sim N\left(y_{rj}, \sigma_{rj}^2\right)$$

It is also assumed that the outputs and inputs of different DMUs are independent. This independence assumption implies that $Cov(\overline{x}_{ij}, \overline{x}_{ik}) = 0$, $Cov(x_{ij}, x_{ik}) = 0$, $Cov(\overline{y}_{rj}, \overline{y}_{rk})$ and $Cov(y_{rj}, y_{rk})=0$. For example, Input 1 for DMUs 1 and 2 are normally and independently distributed as $N(1709, 8.36)$ and $N(1234, 1)$ and the covariance of x_{11} and x_{12} are consequently equal to 0.

The stochastic efficiency and super-efficiency scores are obtained from the implementation of Models (5) and (10) for various combinations of α and β (ie, α=0.5 and β=0.5, α=0.4 and β=0.1, α=0.05 and β=0.01, α=0.05 and β=0.05, α=0.1 and β=0.4, and α=0.01 and β=0.05). The computational results are

Table 1 The birandom inputs and outputs used in Example 1

DMU	*Input* 1	*Input* 2	*Output* 1	*Output* 2
1	N (N(1709, 8.36), 1)	N (N(1234, 1), 6.7)	N (N(2788, 1), 1.95)	N (N(3054, 9.85), 9.33)
2	N (N(1297, 5.94), 1)	N (N(1268, 6.34), 1)	N (N(2456, 1),5.13)	N (N(1940, 9.5), 7.25)
3	N (N(2388, 9.60), 1)	N (N(1381, 1), 6.67))	N (N(2065, 1), 6.19)	N (N(2647, 1), 3.37)
4	N (N(1756, 8.2), 1)	N (N(1469, 4.69), 1)	N (N(3795, 1), 1.79)	N (N(4405, 2.25), 4.4)
5	N (N(2691, 3.82), 1)	N (N(2561, 7.78), 1)	N (N(1211, 1), 1.60)	N (N(2694, 2.6), 2.33)
6	N (N(511, 1.95), 2.44)	N (N(570, 1), 9.9)	N (N(1353, 2.2), 1.05)	N (N(1215, 1), 3.72)
7	N (N(2260, 2.1), 1)	N (N(1881, 1), 8.86)	N (N(7209, 8), 5.73)	N (N(1786, 7.29), 1)
8	N (N(620, 1), 9.8)	N (N(2152, 3.16), 3.5)	N (N(1146, 1), 11.68)	N (N(1093, 1), 9.40)
9	(N(1393, 1), 1)	N (N(1948, 2), 6.36)	N (N(1685, 1), 3.9)	N (N(6866, 5.9), 8.06)
10	N(N(1696, 1), 1)	N (N(1637, 1), 8.5)	N (N(1770, 1), 2.38)	N (N(3899, 1.15), 1)

Table 2 The stochastic efficiency and super-efficiency scores of Example 1 for different α and β values

DMU	*Stochastic efficiency Model* (5)						*Stochastic super-efficiency Model* (10)						*Rank*
	α=0.5, β=0.5	α=0.4, β=0.1	α=0.05, β=0.01	α=0.05, β=0.05	α=0.1, β=0.4	α=0.01, β=0.05	α=0.5, β=0.5	α=0.4, β=0.1	α=0.05, β=0.01	α=0.05, β=0.05	α=0.1, β=0.4	α=0.01, β=0.05	
Column 1	*Column* 2	*Column* 3	*Column* 4	*Column* 5	*Column* 6	*Column* 7	*Column* 8	*Column* 9	*Column* 10	*Column* 11	*Column* 12	*Column* 13	*Column* 14
1	0.8542	0.8609	0.8788	0.8745	0.8632	0.8812	0.8542	0.8609	0.8788	0.8745	0.8632	0.8812	5
2	0.7205	0.7289	0.7478	0.7420	0.7291	0.7474	0.7205	0.7289	0.7478	0.7420	0.7291	0.7474	7
3	0.6266	0.6305	0.6416	0.6393	0.6326	0.6438	0.6266	0.6305	0.6416	0.6393	0.6326	0.6438	9
4	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.1984	1.2073	1.2273	1.2212	1.2073	1.2275	3
5	0.3158	0.3179	0.3221	0.3207	0.3175	0.3218	0.3158	0.3179	0.3221	0.3207	0.3175	0.3218	10
6	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0609	1.0703	1.093	1.0866	1.0714	1.0941	4
7	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.4835	1.494	1.5200	1.5131	1.496	1.5222	2
8	0.7127	0.7225	0.7633	0.7588	0.7382	0.7802	0.7127	0.7225	0.7633	0.7588	0.7382	0.7802	6
9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.9649	1.9805	2.0083	1.9965	1.9739	2.0016	1
10	0.7159	0.7190	0.7299	0.7282	0.7223	0.7333	0.7159	0.7190	0.7299	0.7282	0.7223	0.7333	8

presented in Table 2. Columns 2–7, Columns 8–13 and 14 of Table 2 present the stochastic efficiency scores, the super-efficiency scores and the ranking of the 10 DMUs, respectively.

According to Proposition 3, the stochastic efficiency scores in Model (5) are greater than 0 and less than or equal to 1. Table 2 shows that DMUs 4, 6, 7 and 9 are efficient since their stochastic efficiency scores are equal to 1 for all $\alpha-\beta$ combinations (see Definition 7). In order to rank the efficient DMUs, we implement the super-efficiency model proposed in this study. The results of the stochastic super-efficiency scores are presented in Table 2 as well. For example, as shown in Table 2, when α=0.5 and β=0.5, the stochastic super-efficiency scores of efficient DMUs 4, 6, 7 and 9 are 1.2212, 1.0866, 1.5131 and 1.9965, respectively. Interestingly, this example provides the same ranking order for the 10 DMUs for the $\alpha-\beta$ combinations reported in the last column of Table 2. In addition, as shown in Table 2, the stochastic efficiency scores and the stochastic super-efficiency scores are identical when the efficiency scores are less than 1.

The second column of Table 2, corresponds to α=0.5 and β=0.5 and shows the efficiency scores of the DMUs without the stochastic assumption as stated in Proposition 2. Consequently, the results from Model (5) are identical to the results from the conventional CCR Model (1) when α=0.5 and β=0.5. Similarly, when α=0.5 and β=0.5, the super-efficiency scores of Models (8), (10) and (11) are identical as represented in the eighth column of Table 2 (see Proposition 6).

DMU 4, DMU 6, DMU 7 and DMU 9 are probabilistic-probabilistic $\alpha-\beta$ efficient at different $\alpha-\beta$ levels since their efficiency scores are equal to 1. That is mainly the reason why we use the stochastic super-efficiency Model (10) to rank the probabilistic-probabilistic $\alpha-\beta$ efficient DMUs. Therefore, the corresponding stochastic super-efficiency scores are not the same and can be simply used to provide a complete rank ordering of all DMUs as shown in the last column of Table 2.

Considering the computational results presented in Columns 8–13 of Table 2, DMU 9, with the highest stochastic super-efficiency scores of 1.9649, 1.9805, 2.0083, 1.9965, 1.9739 and 2.0016 corresponding to (α=0.5 and β=0.5), (α=0.4 and β=0.1), (α=0.05 and β=0.01), (α=0.05 and β=0.05), (α=0.1 and β=0.4) and (α=0.01 and β=0.05), respectively, is ranked first by the Birandom super-efficiency model. DMU 7, with the scores of 1.4835, 1.4940, 1.5200, 1.5131, 1.4960 and 1.5222 corresponding to the above $\alpha-\beta$ combinations, is ranked second. Another two top DMUs are DMU 4, with the scores of 1.1984, 1.2073, 1.2273, 1.2073, 1.2212 and 1.2275, and DMU 6, with the scores of 1.0609, 1.0703, 1.0930, 1.0866, 1.0714 and 1.0941, corresponding to the above $\alpha-\beta$ combinations as well. DMU 3, with the scores of 0.6266, 0.6305, 0.6416, 0.6393, 0.6326 and 0.6438, and DMU 5, with the scores of 0.3158, 0.3179, 0.3221, 0.3207, 0.3175 and 0.3218 are ranked at the bottom of the ranking list for (α=0.5 and β=0.5), (α=0.4 and β=0.1), (α=0.05 and β=0.01), (α=0.05 and β=0.05), (α=0.1 and β=0.4) and (α=0.01 and β=0.05), respectively. Therefore, with the exception of DMUs 4, 6, 7 and 9, the rest of the DMUs are probabilistic-probabilistic $\alpha-\beta$ inefficient.

By keeping α constant at the 0.05 level and increasing β, the corresponding efficiency scores of the DMUs are decreased as reported in Columns 2–4 of Table 3. For example, when α=0.05, the efficiency scores of DMU 1 at the β=0.1, β=0.3, β=0.5 and β=0.9 levels are 0.8717, 0.8686, 0.8664 and 0.8612, respectively. On the other hand, when β is kept constant at the 0.05 level and α is increased, the corresponding efficiency scores of the DMUs are decreased as reported in the last four columns of Table 3. For example, as shown in Table 3, the efficiency scores of DMU 1 for α=0.1, 0.3, 0.5 and 0.9 and a constant β of 0.05 are decreased, respectively. Interestingly, DMUs 4, 6, 7 and 9 are still probabilistic-probabilistic efficient for all different $\alpha-\beta$ levels defined in Table 3. We can therefore conclude that the greater levels of α and/or β enable the DMUs to obtain greater relative efficiencies. The reason is because DMUs 4, 6, 7 and 9 remain probabilistic-probabilistic efficient for the larger $\alpha-\beta$ levels.

In summary, we showed the computational results from the super-efficiency Model (10) by keeping α constant and changing β and also by keeping β constant and changing α. When α is kept constant at the 0.05 level and β is increased, the corresponding stochastic super-efficiency scores of the probabilistic-probabilistic efficient DMUs are decreased as represented in Columns 2–4 in Table 4. For example, the super-efficiency scores of DMU 4 at the β=0.1, 0.3, 0.5 and 0.9 levels with a constant α of 0.05 are 1.2173, 1.2129, 1.2098 and 1.2024, respectively. On the other hand, when β is kept constant at the 0.05 level and α is increased, the corresponding stochastic super-efficiency scores of the DMUs are decreased as reported in the last four columns of Table 4. For example, when β=0.05, the stochastic super-efficiency scores of DMU 4 for α=0.1, 0.3, 0.5 and 0.9 levels are decreased, respectively. Note that the stochastic scores of the probabilistic-probabilistic inefficient DMUs remain unchanged in both Tables 3 and 4.

7.2. Example 2

We use an example provided in Table 5 with five DMUs, two inputs and two outputs to demonstrate the practical aspects of the sensitivity analysis proposed in this study. The last two columns of Table 5 present the efficiency and super-efficiency scores of Model (1) and Model (8), respectively.

We perform sensitivity analysis on DMU 2 and DMU 5 separately. When DMU 2 has birandom input and output data, the four remaining DMUs (DMU1, DMU 3, DMU 4 and DMU 5) have crisp input and output data, and when DMU 5 has birandom input and output data the four remaining DMUs (DMU1, DMU 2, DMU 3 and DMU 4) have crisp input and output data. We further assume random variations of inputs $(\overline{\sigma}_{i2} = \sigma_{i2} = 0.4)$ and outputs $(\overline{\sigma}_{r2} = \sigma_{r2} = 0.5)$ for DMU 1, and random variations of inputs $(\overline{\sigma}_{i5} = \sigma_{i5} = 0.4)$ and outputs $(\overline{\sigma}_{r5} = \sigma_{r5} = 0.5)$ associated with DMU 5. To determine the stochastic super-efficiency scores of DMU 2 and DMU 5 for the case (α=0.7, β=0.7), we run Model (11) twice separately for DMU 2 and DMU 5. The obtained stochastic super-efficiency results for DMU 2 and DMU 5 are 3.7331 and 0.9708, respectively. This satisfies Proposition 7 since the super-efficiency scores of DMUs 2 and 5 are 2.5000 (greater than 1) and 0.5000 (less than 1), respectively. According to

Table 3 The stochastic efficiency scores of Example 1 for different α and β values

α	0.05				0.1	0.3	0.5	0.9
β	0.1	0.3	0.5	0.9	0.05			
Column 1	*Column* 2	*Column* 3	*Column* 4	*Column* 5	*Column* 6	*Column* 7	*Column* 8	*Column* 9
1	0.8717	0.8686	0.8664	0.8612	0.8702	0.8654	0.8621	0.8541
2	0.7383	0.7342	0.7314	0.7246	0.7384	0.7341	0.7311	0.7232
3	0.6378	0.6361	0.6349	0.6321	0.6364	0.6331	0.6309	0.6255
4	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
5	0.3197	0.3186	0.3179	0.3161	0.3199	0.3191	0.3185	0.3172
6	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
7	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
8	0.7560	0.7527	0.7505	0.7452	0.7453	0.7305	0.7205	0.6968
9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
10	0.7272	0.7260	0.7252	0.7232	0.7250	0.7214	0.7189	0.7129

Table 4 The stochastic super-efficiency scores of Example 1 for different α and β values

α	0.05				0.1	0.3	0.5	0.9
β	0.1	0.3	0.5	0.9	0.05			
Column 1	*Column* 2	*Column* 3	*Column* 4	*Column* 5	*Column* 6	*Column* 7	*Column* 8	*Column* 9
1	0.8717	0.8686	0.8664	0.8612	0.8702	0.8654	0.8621	0.8541
2	0.7383	0.7342	0.7314	0.7246	0.7384	0.7341	0.7311	0.7232
3	0.6378	0.6361	0.6349	0.6321	0.6364	0.6331	0.6309	0.6255
4	1.2173	1.2129	1.2098	1.2024	1.2172	1.2128	1.2098	1.2025
5	0.3197	0.3186	0.3179	0.3161	0.3199	0.3191	0.3185	0.3172
6	1.0824	1.0778	1.0746	1.0669	1.0817	1.0764	1.0727	1.0638
7	1.5086	1.5036	1.5002	1.4918	1.5072	1.5007	1.4963	1.4868
8	0.7560	0.7527	0.7505	0.7452	0.7453	0.7305	0.7205	0.6968
9	1.9888	1.9802	1.9743	1.9599	1.9932	1.9895	1.9869	1.9807
10	0.7272	0.7260	0.7252	0.7232	0.725	0.7214	0.7189	0.7129

Proposition 6, when α=0.5 and β=0.5 in Model (11), the stochastic super-efficiency Model (11) is converted to the deterministic Model (8) and the super-efficiency scores of DMU 2 and DMU 5 remain unchanged (2.5000 and 0.5000, respectively) as reported in Table 4. In order to examine Propositions 8 and 9 in this example, we assume $0 \leqslant \alpha \leqslant 0.5$ and $0 \leqslant \beta \leqslant 0.5$, and implement a sensitivity analysis for DMU 4 involving random variations of inputs ($\overline{\sigma}_{i4} = \sigma_{i4} = 0.5$) and outputs ($\overline{\sigma}_{r4} = \sigma_{r4} = 0.1$) when ($\alpha$=0.4, β=0.1).

In this case, the stochastic super-efficiency ($\tilde{\theta}^*$) of DMU 4 using Model (11) is equal to 1.2098 and condition (I) of Proposition 8 is satisfied because the super-efficiency (θ*) of DMU 4 using Model (8) equals to 2.2500. As shown in Table 4, the super-efficiency of DMU 4 is 2.2500, and under α=0.4 and β=0.1 we can take into account the following relationship to investigate Proposition 9 for DMU 4:

$$\Phi^{-1}(1-\alpha)\left(\sum_{r=1}^{s}\sqrt{\mathrm{Var}(\tilde{\overline{y}}_{rp})}+\sum_{i=1}^{m}\sqrt{\mathrm{Var}(\tilde{\overline{x}}_{ip})}\right)$$

$$+\Phi^{-1}(1-\beta)\left(\sum_{r=1}^{s}\sqrt{\mathrm{Var}(\overline{y}_{rp})}+\sum_{i=1}^{m}\sqrt{\mathrm{Var}(\overline{x}_{ip})}\right)$$

$$<\sum_{r=1}^{s}\beta_r^{+*}+\sum_{i=1}^{m}\beta_i^{-*}\Rightarrow 0.9180<1.4000$$

where $\sum_{r=1}^{s}\beta_r^{+*}+\sum_{i=1}^{m}\beta_i^{-*}$ is calculated by Model (12). This relationship confirms Proposition 9.

To examine Proposition 10, let us assume (α=0.7, β=0.7) and random variations of inputs ($\overline{\sigma}_{i4} = \sigma_{i4} = 0.5$) and outputs ($\overline{\sigma}_{r4} = \sigma_{r4} = 0.1$) for DMU 2. The stochastic super-efficiency ($\tilde{\theta}^*$) of DMU 2 using Model (11) is equal to 3.7331. We then take into consideration the following relationship:

$$-\Phi^{-1}(1-\alpha)\left(\sum_{r=1}^{s}\sqrt{\mathrm{Var}(\tilde{\overline{y}}_{rp})}+\sum_{i=1}^{m}\sqrt{\mathrm{Var}(\tilde{\overline{x}}_{ip})}\right)$$

$$-\Phi^{-1}(1-\beta)\left(\sum_{r=1}^{s}\sqrt{\mathrm{Var}(\overline{y}_{rp})}+\sum_{i=1}^{m}\sqrt{\mathrm{Var}(\overline{x}_{ip})}\right)$$

$$<\sum_{r=1}^{s}\beta_r^{+*}+\sum_{i=1}^{m}\beta_i^{-*}\Rightarrow 0.9540<3.2190$$

We should note that the optimal solution of the objective function of Model (12), $\sum_{r=1}^{s}\beta_r^{+*}+\sum_{i=1}^{m}\beta_i^{-*}$, is 3.2190. The above relationship satisfies and $\theta^* > 1$ as mentioned in Table 5.

Table 5 The input data, output data, efficiency scores and super-efficiency scores of Example 2

DMU	*Input*		*Output*		*Efficiency score*	*Super-efficiency score*
	1	2	1	2		
1	3	4	3	2	0.7805	0.7805
2	4	2	2	3	1.0000	2.5000
3	3	5	4	3	1.0000	1.1538
4	1	3	3	1	1.0000	2.2500
5	3	2	1	1	0.5000	0.5000

8. Conclusions and future research directions

The conventional DEA is deterministic and assumes that inputs and outputs are measured precisely. However, the uncertainties inherent in the real-life performance measurement problems inhibit using deterministic DEA models in practice. In spite of the criticism that in most real-life applications there is error and random noise in the data, a limited number of methods have been proposed to incorporate the stochastic variations of the data.

In this paper, we proposed a new chance-constrained DEA model with birandom inputs and outputs for evaluating the efficiency of the DMUs in a stochastic environment. In addition, a super-efficiency approach was developed in which the constraints were treated as birandom events. A deterministic equivalent model was formulated, which was non-linear. We converted the non-linear model into a model with quadratic constraints. We demonstrated the efficacy of the proposed model with a numerical example. We also investigated the stability and robustness of the proposed super-efficiency model with sensitivity analysis. To the best of our knowledge, this is the first study to incorporate birandom variables into a DEA framework.

For further research, we plan to extend the proposed approach to other types of DEA models. We are also investigating the possibility of applying the birandom approach to other stochastic LP problems.

Acknowledgements—The authors would like to thank the anonymous reviewers and the editor for their insightful comments and suggestions. Adel Hatami-Marbini also likes to thank the FRS-FNRS for the financial support he received as a chargé de recherches at the Université catholique de Louvain during this research project.

References

Adler N, Friedman L and Sinuany-Stern Z (2002). Review of ranking methods in the data envelopment analysis context. *European Journal of Operational Research* **140**(2): 249–265.

Andersen P and Petersen NC (1993). A procedure for ranking efficient units in data envelopment analysis. *Management Science* **39**(10): 1261–1294.

Azadi M and Farzipoor Saen R (2011). A new chance-constrained data envelopment analysis for selecting third party reverse logistics providers in the existence of dual-role factors. *Expert Systems with Applications* **38**(10): 12231–12236.

Banker RD (1993). Maximum likelihood, consistency and data envelopment analysis: A statistical foundation. *Management Science* **39**(10): 1265–1273.

Banker RD (1996). Hypothesis tests using data envelopment analysis. *Journal of Productivity Analysis* **7**(2–3): 139–159.

Banker RD, Charnes A and Cooper WW (1984). Some method for estimating technical and scale inefficiencies in data envelopment analysis. *Management Science* **30**(9): 1078–1092.

Bardhan I, Bowlin WF, Cooper WW and Sueyoshi T (1996). Models for efficiency dominance in data envelopment analysis. Part I: Additive models and MED measures. *Journal of the Operations Research Society of Japan* **39**(3): 322–332.

Bruni ME, Conforti D, Beraldi P and Tundis E (2009). Probabilistically constrained models for efficiency and dominance in DEA. *International Journal of Production Economics* **117**(1): 219–228.

Ceyhan ME and Benneyan JC (2011). Handling estimated proportions in public sector data envelopment analysis. *Annals of Operations Research* **191**(November): 1–26.

Charnes A and Cooper WW (1959). Chance-constrained programming. *Management Science* **6**(1): 73–79.

Charnes A, Cooper WW and Rhodes E (1978). Measuring the efficiency of decision making units. *European Journal of Operational Research* **2**(6): 429–444.

Cooper WW, Huang Z and Li S (1996). Satisficing DEA models under chance constraints. *Annals of Operations Research* **66**(4): 279–295.

Cooper WW, Huang ZM, Lelas V, Li SX and Olesen OB (1998). Chance constrained programming formulations for stochastic characterizations of efficiency and dominance in DEA. *Journal of Productivity Analysis* **9**(1): 530–579.

Cooper WW, Deng H, Huang Z and Li Susan X (2002). Chance constrained programming approaches to technical efficiencies and inefficiencies in stochastic data envelopment analysis. *Journal of the Operational Research Society* **53**(12): 1347–1356.

Cooper WW, Deng H, Huang ZM and Li SX (2004). Chance constrained programming approaches to congestion in stochastic data envelopment analysis. *European Journal of Operational Research* **155**(2): 487–501.

Despotis DK and Smirlis YG (2002). Data envelopment analysis with imprecise data. *European Journal of Operational Research* **140**(1): 24–36.

Doyle JR and Green R (1994). Efficiency and cross-efficiency in DEA: Derivatives, meanings and uses. *Journal of the Operational Research Society* **45**(5): 567–578.

Farrell MJ (1957). The measurement of productive efficiency. *Journal of the Royal Statistical Society*. Series A **120**(III): 253–281.

Gijbels I, Mammen E, Park BU and Simar L (1999). On estimation of monotone and concave frontier functions. *Journal of the American Statistical Association.* **94**(445): 220–228.

Halme M, Joro T, Korhonen P, Salo S and Wallenius J (1999). A value efficiency approach to incorporating preference information in data envelopment analysis. *Management Science* **45**(1): 103–115.

Hossain MK, Kamil AA, Mustafa A and Baten MA (2013). Efficiency in the worst production situation using data envelopment analysis. *Advances in Decision Sciences*, 2013: article 354509, http://dx.doi.org/10.1155/2013/354509.

Hosseinzadeh Lotfi F, Nematollahi N, Behzadi MH, Mirbolouki M and Moghaddas Z (2012). Centralized resource allocation with stochastic data. *Journal of Computational and Applied Mathematics* **236**(7): 1783–1788.

Huang Z and Li SX (1996). Dominance stochastic models in data envelopment analysis. *European Journal of Operational Research* **95**(2): 390–403.

Huang Z and Li SX (2001). Stochastic DEA models with different types of input-output disturbances. *Journal of Productivity Analysis* **15**(2): 95–113.

Jahanshahloo GR and Afzalinejad M (2006). A ranking method based on a full-inefficient frontier. *Applied Mathematical Modeling* **30**(3): 248–260.

Jahanshahloo GR, Khodabakhshi M, Hosseinzadeh Lotfi F and Moazami Goudarzi MR (2011). A cross-efficiency model based on super-efficiency for ranking units through the TOPSIS approach and its extension to the interval case. *Mathematical and Computer Modelling* **53**(9–10): 1946–1955.

Kao C and Liu ST (2000). Fuzzy efficiency measures in data envelopment analysis. *Fuzzy Sets and Systems* **113**(3): 427–437.

Khodabakhshi M (2009). Estimating most productive scale size with stochastic data in data envelopment analysis. *Economic Modeling* **26**(5): 968–973.

Khodabakhshi M (2010). An output oriented super-efficiency measure in stochastic data envelopment analysis: Considering Iranian electricity distribution companies. *Computers and Industrial Engineering* **58**(4): 663–671.

Khodabakhshi M (2011). Super-efficiency in stochastic data envelopment analysis: An input relaxation approach. *Journal of Computational and Applied Mathematics* **235**(16): 4576–4588.

Khodabakhshi M and Asgharian M (2009). An input relaxation measure of efficiency in stochastic data envelopment analysis. *Applied Mathematical Modeling* **33**(4): 2010–2023.

Khodabakhshi M, Asgharian M and Gregoriou GN (2010). An input-oriented super-efficiency measure in stochastic data envelopment analysis: Evaluating chief executive officers of US public banks and thrifts. *Expert Systems with Applications* **37**(3): 2092–2097.

Kneip A, Park BU and Simar L (1998). A note on the convergence of nonparametric DEA estimators for production efficiency scores. *Econometric Theory* **14**(6): 783–793.

Kuah CT, Wong KY and Wong WP (2012). Monte Carlo data envelopment analysis with genetic algorithm for knowledge management performance measurement. *Expert Systems with Applications* **39**(10): 9348–9358.

Land KC, Lovell CAK and Thore S (1993). Chance constrained data envelopment analysis. *Managerial and Decision Economics* **14**(6): 541–554.

Lertworasirikul S, Fang SC, Joines JA and Nuttle HLW (2003). Fuzzy data envelopment analysis (DEA): A possibility approach. *Fuzzy Sets and Systems* **139**(2): 379–394.

Li SX (1998). Stochastic models and variable returns to scales in data envelopment analysis. *European Journal of Operational Research* **104**(3): 532–548.

Li S, Jahanshahloo GR and Khodabakhshi M (2007). A super-efficiency model for ranking efficient units in data envelopment analysis. *Applied Mathematics and Computation* **184**(2): 638–648.

Liu B (2002). *Theory and Practice of Uncertain Programming*, 1st edn. Physica-Verlag: Heidelberg.

Lu WM and Lo SF (2009). An interactive benchmark model ranking performers: Application to financial holding companies. *Mathematical and Computer Modeling* **49**(1–2): 172–179.

Morita H and Seiford LM (1999). Characteristics on stochastic DEA efficiency—Reliability and probability being efficient. *Journal of Operational Research Society of Japan* **42**(4): 389–404.

Noguchi H, Ogawa M and Ishii H (2002). The appropriate total ranking method using DEA for multiple categorized purposes. *Journal of Computational and Applied Mathematics* **146**(1): 155–166.

Noura AA, Hosseinzadeh Lotfi F, Jahanshahloo GR and Fanati Rashidi S (2011). Super-efficiency in DEA by effectiveness of each unit in society. *Applied Mathematics Letters* **24**(5): 623–626.

Olesen OB (2006). Comparing and combining two approaches for chance constrained DEA. *Journal of Productivity Analysis* **26**(2): 103–119.

Olesen OB and Petersen NC (1995). Chance constrained efficiency evaluation. *Management Science* **41**(3): 442–457.

Peng J and Liu B (2007). Birandom variables and birandom programming. *Computers and Industrial Engineering* **53**(3): 433–453.

Peng J and Zhao X (2006). Some theoretical aspects of the critical values of birandom variable. *Journal of Information and Computer Science* **1**(4): 225–234.

Simar L and Wilson PW (1998). Sensitivity analysis of efficiency scores: How to bootstrap in nonparametric frontier models. *Management Science* **44**(1): 49–61.

Simar L and Zelenyuk V (2010). Stochastic FDH/DEA estimators for frontier analysis. *Journal of Productivity Analysis* **36**(1): 1–20.

Sinuany-Stern Z, Mehrez A and Barboy A (1994). Academic departments efficiency via DEA. *Computers and Operations Research* **21**(5): 543–556.

Sueyoshi T (1999). DEA non-parametric ranking test and index measurement: Slack-adjusted DEA and an application to Japanese agriculture cooperatives. *Omega* **27**(3): 315–326.

Thanassoulis E and Dyson RG (1992). Estimating preferred target input-output levels using data envelopment analysis. *European Journal of Operational Research* **56**(1): 80–97.

Torgersen AM, Førsund FR and Kittelsen SAC (1996). Slack-adjusted efficiency measures and ranking of efficient units. *The Journal of Productivity Analysis* **7**(4): 379–398.

Tziogkidis P (2012). *The Simar and Wilson's Bootstrap DEA approach: A Critique*. Cardiff Economics Working Papers, No. E2012/19.

Wang Y-M, Luo Y and Lan Y-X (2011). Common weights for fully ranking decision making units by regression analysis. *Expert Systems with Applications* **38**(8): 9122–9128.

Wong WP (2009). Performance evaluation of supply chain in stochastic environment: Using a simulation based DEA framework. *International Journal of Business Performance and Supply Chain Modelling* **1**(2–3): 203–228.

Wu D and Lee CG (2010). Stochastic DEA with ordinal data applied to a multi-attribute pricing problem. *European Journal of Operational Research* **207**(3): 1679–1688.

Xu J and Zhou X (2009). A class of multi-objective expected value decision-making model with birandom coefficients and its application to flow shop scheduling problem. *Information Sciences* **179**(17): 2997–3017.

Xu J and Ding C (2011). A class of chance constrained multiobjective linear programming with birandom coefficients and its application to vendors selection. *International Journal of Production Economics* **131**(2): 709–720.

Yang F, Ang S, Xia Q and Yang C (2012). Ranking DMUs by using interval DEA cross efficiency matrix with acceptability analysis. *European Journal of Operational Research* **223**(2): 483–488.

Yang L and Liu B (2006). A sufficient and necessary condition for chance distribution of birandom variable. *Information* **9**(1): 33–36.

Yu P and Lee JH (2013). A hybrid approach using two-level SOM and combined AHP rating and AHP/DEA-AR method for selecting optimal promising emerging technology. *Expert Systems with Applications* **40**(1): 300–314.

Received 14 January 2013;
accepted 9 October 2013