J Optim Theory Appl (2016) 170:243–265
DOI 10.1007/s10957-015-0857-y

# Solving Geometric Programming Problems with Normal, Linear and Zigzag Uncertainty Distributions

**Rashed Khanjani Shiraz**[1] · **Madjid Tavana**[2,3] · **Debora Di Caprio**[4,5] · **Hirofumi Fukuyama**[6]

Received: 17 August 2015 / Accepted: 24 November 2015 / Published online: 24 December 2015


**Abstract** Geometric programming is a powerful optimization technique widely used for solving a variety of nonlinear optimization problems and engineering problems. Conventional geometric programming models assume deterministic and precise parameters. However, the values observed for the parameters in real-world geometric programming problems often are imprecise and vague. We use geometric programming within an uncertainty-based framework proposing a chance-constrained geometric programming model whose coefficients are uncertain variables. We assume the uncertain variables to have normal, linear and zigzag uncertainty distributions and show that the corresponding uncertain chance-constrained geometric programming problems can be transformed into conventional geometric programming problems

✉ Madjid Tavana
tavana@lasalle.edu

Rashed Khanjani Shiraz
Rashed.shiraz@gmail.com

Debora Di Caprio
dicaper@mathstat.yorku.ca

Hirofumi Fukuyama
fukuyama@fukuoka-u.ac.jp

1 School of Mathematics Science, University of Tabriz, Tabriz, Iran

2 Distinguished Chair of Business Systems and Analytics, Business Systems and Analytics Department, La Salle University, Philadelphia, PA 19141, USA

3 Business Information Systems Department, Faculty of Business Administration and Economics, University of Paderborn, 33098 Paderborn, Germany

4 Department of Mathematics and Statistics, York University, Toronto M3J 1P3, Canada

5 Polo Tecnologico IISS G. Galilei, Via Cadorna 14, 39100 Bolzano, Italy

6 Faculty of Commerce, Fukuoka University, Fukuoka, Japan

to calculate the objective values. The efficacy of the procedures and algorithms is demonstrated through numerical examples.

**Keywords** Uncertainty theory · Uncertain variable · Linear uncertainty distribution · Normal uncertainty distribution · Zigzag uncertainty distribution

**Mathematics Subject Classification** 90C46 · 65K05 · 28B99 · 90C48 · 49K35

## 1 Introduction

Geometric programming (GP) is a powerful optimization technique developed by the researchers to solve various nonlinear programming problems subject to linear and nonlinear constraints. GP has been applied by many researchers to several engineering problems such as integrated circuit design, engineering design, project management and inventory management.

Chu and Wong in [1] and Hershenson et al. in [2] applied GP to circuit design. Avriel et al. [3], Beightler and Phillips [4] and Choi and Bricker [5] used GP for engineering design. Scott and Jefferson in [6] proposed GP for project management. Cheng in [7], Jung and Klein in [8], Kim and Lee in [9], Lee in [10], Roy and Maiti in [11] and Worrall and Hall in [12] applied GP to inventory management.

Efficient and effective algorithms for solving GP problems with known coefficients have been studied by many scholars (see, for example, [3,4,13–23]).

The conventional GP is based on well-defined, precise and deterministic coefficients. However, the observed values of the parameters in real-world GP problems can be imprecise and vague in nature. Thus, several methods have been proposed to deal with imprecise and ambiguous data in GP models.

Avriel and Wilde [24] proposed stochastic GP where the exponents are deterministic and the coefficients are positive random variables. Dupačová [25] proposed applications of stochastic GP in the context of metal cutting optimization.

Liu [26] applied GP methods to derive the objective value and provide useful information in order to determine the relationship between profit maximization and returns to scale. Liu [27] developed a procedure to obtain the lower and upper bounds of the objective of the posynomial GP problem when the cost and constraint parameters are interval data. Liu [28] developed a procedure to work with GP with fuzzy exponents and fuzzy coefficients. Liu [29] developed a solution procedure for solving GP problems with interval data for the exponents in the objective function, the cost and the constraint coefficients, and the right-hand sides. Liu [30] utilized an extension principle and developed a pair of two-level mathematical programs to calculate the upper and lower bounds of profit values.

Tsai et al. in [31] proposed a method to solve signomial discrete programming problems frequently occurring in engineering design. Li and Tsai [32] proposed a technique to deal with free variables in generalized GP problems providing computationally effective convexification rules for signomial terms with three variables. Tsai et al. in [33] proposed a method for handling non-positive variables with integer powers in generalized GP problems. Tsai [34] proposed a technique for improving the exponential-based methods in a way that they can handle generalized GP problems containing free variables.

Tsai and Lin [35] proposed a generalized technique for treating free variables in signomial discrete programming problems. They converted the problems to the original signomial discrete programming problems and obtained global optimal solutions. Later on, Lin and Tsai [36] integrated the range reduction techniques in a global optimization algorithm for signomial GP to improve computational efficiency.

Uncertainty theory founded by Liu [37] is a new branch of mathematics. In [38], Liu proposed an uncertain stock model and a European option price formula. Following [38], Peng and Yao [39] presented a new uncertain stock model and some option price formulas. Also, Liu [40] and Wang et al. [41,42] applied uncertainty theory to uncertain statistics. In addition, uncertain risk analysis, uncertain reliability analysis and uncertain control were proposed by Liu in [40], Liu in [43] and Zhu in [44], respectively.

Li et al. [45] described the risk as a nonnegative uncertain variable and mainly discussed the premium of uncertain risk within the framework of uncertainty theory. Hang et al. [46] illustrated that uncertainty theory can serve as a powerful tool to deal with the maximum flow in an uncertain network. Ding [47] investigated the uncertain maximum flow problem and formulated the maximum flow and the $\alpha$-maximum flow model in an uncertainty-based framework.

In summary, there already exists an ample literature on posynomial GP, most of which is oriented toward a uncertainty-based approach to GP and its applications. Conventional GP models allow to solve nonlinear optimization problems through the conversion of the primal problem (a nonlinear problem with inequality constraints) into the dual problem (an equivalent linear problem with equality constraints). This fact has been exploited to extend GP modeling so as to solve a myriad of problems, chance-constrained and not, whose coefficients are fuzzy numbers, fuzzy variables or random variables. However, to the best of our knowledge, there is no previous study dealing with the formulation and/or solution of GP problems where the coefficients are given by uncertain variables.

In this paper, we use uncertain variables to account for the unavoidable vagueness of the parameters characterizing real-world GP problems. More precisely, we define three chance-constrained GP models that can be implemented when the coefficients are expressed as uncertain variables with normal, linear or zigzag uncertainty distributions. We show that all the proposed uncertain GPs can be transformed into conventional GPs allowing to calculate the optimal objective using their dual forms.

Our approach to uncertain GP has a clear advantage over those incorporating fuzzy programming and its variants. In standard fuzzy GP, the coefficients are positive interval coefficients, often corresponding to alpha cuts of fuzzy numbers. Thus, in order to solve the optimization problem at hand, it is necessary to define and solve a pair of geometric programs, which allow to identify an upper and lower bound for the optimal objective value at specific alpha level, but not necessarily the optimal objective value. On the other hand, we solve the uncertainty of the problem at hand by constructing an equivalent deterministic model. Thus, our method grants the optimal objective value, not just an approximation of it.

We provide several examples to show the efficacy and applicability of the proposed models. In particular, we develop a general GP model for profit maximization that offers a valid alternative to fuzzy-based approaches.

The paper proceeds as follows. In Sect. 2, we present some basic definitions on uncertainty spaces and uncertain variables. In Sect. 3, we construct a variant of uncertain chance-constrained programming (UCCGP) model and show how it can be converted to a conventional GP in the normal, linear and zigzag uncertainty distribution cases. In Sect. 4, we present the results of two numerical examples illustrating the efficacy of the proposed approach. In Sect. 5, we show an application of the proposed UCCGP model to profit maximization. Finally, in Sect. 6, we present our conclusions.

## 2 Preliminaries and Definitions

In this section, we review some basic concepts of uncertainty theory. All the axioms, definitions and facts included below are taken from [37].

**Definition 2.1** Let $M : \Gamma \to [0,\ 1]$ be a set function defined on a universal set $\Gamma$. $M$ is called an *uncertain measure* iff it satisfies the following axioms.

**Axiom 1** (Normality) $M(\Gamma) = 1$.

**Axiom 2** (Self-Duality) $\forall \Lambda \subseteq \Gamma$, $M(\Lambda) + M(\Lambda^c) = 1$.

**Axiom 3** (Countable subadditivity) $\forall \{\Lambda_i\}_{i=1}^{+\infty}$ countable sequence, $M\left(\bigcup_{i=1}^{\infty} \Lambda_i\right) \leq \sum_{i=1}^{\infty} M(\Lambda_i)$.

Note that Axioms 1–3 also imply monotonicity (i.e., $M(\Lambda_1) \leq M(\Lambda_2)$ whenever $\Lambda_1 \subseteq \Lambda_2$). See Theorem 1.1 in [37].

**Definition 2.2** The triplet $(\Gamma, L, M)$ is called an *uncertainty space* iff $L$ is a $\sigma$-algebra on $\Gamma$ and $M$ is an uncertain measure.

**Definition 2.3** An *uncertain variable* (in short: UV) $\xi$ on the uncertainty space $(\Gamma, L, M)$ is a measurable function from $(\Gamma, L, M)$ to the set IR of real numbers, i.e., for every Borel set $B$, the set $\{\xi \in B\} = \{\Lambda \in \Gamma : \xi(\Lambda) \in B\}$ belongs to $L$.

Note that a crisp number $\varepsilon$ is also an UV, i.e., the constant function $\xi(\Lambda) = \varepsilon \quad \forall \Lambda \in \Gamma$.

**Definition 2.4** An UV $\xi$ is *nonnegative* iff $M\{\xi < 0\} = 0$ and *positive* iff $M\{\xi \leq 0\} = 0$.

**Definition 2.5** Let $\xi_1,\ \xi_2,\ \ldots,\ \xi_n$ be UVs. Then, $\forall \Lambda \in \Gamma$,

$$(\xi_1 +\ \xi_2 +\ \ldots + \xi_n)(\Lambda) := \xi_1(\Lambda) +\ \xi_2(\Lambda) +\ \ldots + \xi_n(\Lambda) \text{ and}$$
$$(\xi_1 \cdot\ \xi_2 \cdots \xi_n)(\Lambda) := \xi_1(\Lambda) \cdot\ \xi_2(\Lambda) \cdots \xi_n(\Lambda).$$

**Proposition 2.6** *If $\xi_1, \xi_2, \ldots, \xi_n$ are UVs and $f$ is a real-valued measurable function, then $f(\xi_1, \xi_2, \ldots, \xi_n)$ is an UV. In particular, sums and products of UVs are UVs.*

**Definition 2.7** Given an UV $\xi$, the function $\Phi_\xi$ :IR→ [0, 1], defined by $\Phi_\xi(x) := M\{\xi \leq x\}$ for every $x \in$IR, is called the *uncertainty distribution* (in short: UD) of $\xi$.

**Definition 2.8** An UV $\xi$ is called *normal*, *linear* or *zigzag* iff it has a normal, linear or zigzag UD, respectively. In symbols:

$\xi$ is *normal* iff:
$$\forall x \in \text{IR},\quad \Phi_\xi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}$$
where $e, \sigma \in$ IR, $\sigma > 0$

$\xi$ is *linear* iff :
$$\Phi_\xi(x) = \begin{cases} 0, & x \leq a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & x \geq b. \end{cases}$$
where $a, b \in$ IR, $a < b$

$\xi$ is *zigzag* iff :
$$\Phi_\xi(x) = \begin{cases} 0, & x \leq a \\ \frac{x-a}{2(b-a)}, & a \leq x \leq b \\ \frac{x+c-2b}{2(c-b)}, & b \leq x \leq c \\ 1, & x \geq c \end{cases}$$
where $a, b, c \in$ IR, $a < b < c$

To indicate that $\xi$ has a normal, linear or zigzag UD we shall write $\xi : N(e, \sigma)$, $\xi : L(a, b)$ and $\xi : Z(a, b, c)$, respectively. For a graphical standard representation of these distributions, see [37].

*Remark 2.1* Clearly, the parameters associated with positive linear and positive zigzag UVs are necessarily positive.

**Definition 2.9** Let $\xi$ be an UV. The *expected value* of $\xi$ is defined by $E[\xi] := \int_0^{+\infty} M\{\xi \geq r\}\,dr - \int_{-\infty}^0 M\{\xi \leq r\}\,dr$,

provided that at least one of the two integrals is finite.

It follows that $E[\xi] = \int_0^{+\infty}(1 - \Phi_\xi(r))dr - \int_{-\infty}^0 \Phi_\xi(r)dr$.

## 3 Uncertain Chance-Constrained Geometric Programming Model

A standard GP allows solving a minimization problem where the objective function is a posynomial whose variables can take only positive values and are constrained by a finite number of inequality constraints. Eq. (1) shows a standard GP problem and its dual form.

GP primal problem
$$\begin{aligned} &\min \sum_{p=1}^{h} \theta_p \prod_{j=1}^{s} x_j^{\alpha_{p,j}} \\ &\text{s.t.} \\ &\sum_{t(i)=1}^{k(i)} \theta_{i,t(i)} \prod_{j=1}^{s} x_j^{\gamma_{i,t(i),j}} \leq \lambda_i,\ i = 1, \ldots, n, \\ &x_j > 0,\ j = 1, \ldots, s \end{aligned}$$

Dual problem
$$\begin{aligned} &\max \prod_{p=1}^{h} \left(\frac{\theta_p}{\beta_p}\right)^{\beta_p} \prod_{i=1}^{n} \prod_{t(i)=1}^{k(i)} \left(\frac{\theta_{i,t(i)}}{\beta_{i,t(i)}\ \lambda_i}\right)^{\beta_{i,t(i)}} \prod_{i=1}^{n} \delta_i^{\delta_i} \\ &\text{s.t.} \\ &\sum_{p=1}^{h} \beta_p = 1\ ,\ \sum_{p=1}^{h} \alpha_{p,j}\ \beta_p + \sum_{i=1}^{n} \sum_{t(i)=1}^{k(i)} \gamma_{i,t(i),j}\ \beta_{i,t(i)} = 0, \\ &\quad j = 1, \ldots, s, \\ &\delta_i = \sum_{t(i)=1}^{k(i)} \beta_{i,t(i)},\ \ i = 1, \ldots, n,\ \ \beta_p > 0,\ \ p = 1, \ldots, h, \\ &\beta_{i,t(i)} \geq 0\ ,\ \ i = 1, \ldots, n\ ,\ \ t(i) = 1, \ldots, k(i) \end{aligned} \tag{1}$$

where $x_j\,(j = 1, \ldots, s)$ are the variables of the problem, $\theta_p\,(p = 1, \ldots, h)$ and $\theta_{i,t(i)}$ $(i = 1, \ldots, n$ and $t(i) = 1, \ldots, k(i))$ are positive constant values, $\alpha_{p,j}\,(p = 1, \ldots, h$

and $j = 1, \ldots, s$) and $\gamma_{i,t(i),j}$ ($i = 1, ..., n \quad t(i) = 1, ..., k(i)$ and $j = 1, .., s$) are arbitrary real numbers, and $\lambda_i$ ($i = 1, ..., n$) are positive constant values.

In this section, we develop an uncertain GP model whose associated chance-constrained version admits an equivalent crisp formulation. First, we transform the conventional GP problem in Eq. (1) in an uncertain GP problem, where $\tilde{\theta}_p$, $\tilde{\theta}_{i,t(i)}$, $\tilde{\lambda}_i$ are UVs.

$$\begin{aligned} &\min \sum_{p=1}^{h} \tilde{\theta}_p \prod_{j=1}^{s} x_j^{\alpha_{p,j}} \\ &\text{s.t.} \\ &\sum_{t(i)=1}^{k(i)} \tilde{\theta}_{i,t(i)} \prod_{j=1}^{s} x_j^{\gamma_{i,t(i),j}} \le \tilde{\lambda}_i\,, \quad i = 1, \ldots, n, \quad x_j > 0\,, \quad j = 1, \ldots, s. \end{aligned} \tag{2}$$

Based on the model in Eq. (2), we can formulate the following generic GP model, which is a variant of uncertain chance-constrained geometric programming (UCCGP) model:

$$\begin{aligned} &\min E\left[\sum_{p=1}^{h} \tilde{\theta}_p \prod_{j=1}^{s} x_j^{\alpha_{p,j}}\right] \\ &\text{s.t.} \\ &M\left(\sum_{t(i)=1}^{k(i)} \tilde{\theta}_{i,t(i)} \prod_{j=1}^{s} x_j^{\gamma_{i,t(i),j}} \le \tilde{\lambda}_i\right) \ge \alpha\,, \quad i=1, \ldots, n, \quad x_j > 0\,, \quad j = 1, \ldots, s. \end{aligned} \tag{3}$$

where $\alpha \in ]0, 1[$, is a pre-specified minimum uncertainty level.

In the following subsections, we propose a solving method for the UCCP problem of Eq. (3) considering three cases: we assume the coefficients to be UVs with normal, linear or zigzag UDs.

*Remark 3.1* Since in the standard GP model in Eq. (1) the coefficients are assumed to be positive, in the following subsections we will restrict the analysis to the case where the coefficients are UVs whose UDs are characterized by positive parameters. Nonetheless, the GP model in Eq. (1) admits a more general formulation that allows to account for situations where the coefficients can also take non-positive values.

| GP primal problem | Dual problem |
|---|---|
| $\min \sum_{p=1}^{h} e_p\, \theta_p \prod_{j=1}^{s} x_j^{\alpha_{p,j}}$<br>s.t.<br>$\sum_{t(i)=1}^{k(i)} e_{i,t(i)}\, \theta_{i,t(i)} \prod_{j=1}^{s} x_j^{\gamma_{i,t(i),j}} \le e_i\, \lambda_i,$<br>$i = 1, \ldots, n,$<br>$x_j > 0, j = 1, \ldots, s,$<br>$e_p,\ e_{i,t(i)},\ e_i = 1 \text{ or } -1.$ | $\max\ e_{oo} \left( \prod_{p=1}^{h} \left(\frac{\theta_p}{\beta_p}\right)^{e_p \beta_p} \prod_{i=1}^{n} \prod_{t(i)=1}^{k(i)} \left(\frac{\theta_{i,t(i)}}{\beta_{i,t(i)}\ \lambda_i}\right)^{e_{i,t(i)}\beta_{i,t(i)}} \prod_{i=1}^{n} \delta_i^{\delta_i} \right)^{e_{oo}}$<br>s.t.<br>$\sum_{p=1}^{h} e_p \alpha_{p,j}\ \beta_p + \sum_{i=1}^{n} \sum_{t(i)=1}^{k(i)} e_{i,t(i)} \gamma_{i,t(i),j}\ \beta_{i,t(i)} = 0, \quad j = 1, \ldots, s,$<br>$e_{oo} \sum_{p=1}^{h} e_p \beta_p = 1\,, \delta_i = e_i \sum_{t(i)=1}^{k(i)} e_{i,t(i)} \beta_{i,t(i)}\,, \quad i = 1, \ldots, n,$<br>$\beta_p > 0, \quad p = 1, \ldots, h, \quad \beta_{i,t(i)} \ge 0, \quad i = 1, \ldots, n\,,$<br>$t(i) = 1, \ldots, k(i), e_p,\ e_{i,t(i)},\ e_i = 1 \text{ or } -1,$<br>$e_{oo}$ is the sign of the primal objective at the optimum. |

### 3.1 Proposed UCCGP Model with Normal Uncertainty Distributions

Let the coefficients $\tilde{\theta}_p, \tilde{\theta}_{i,t(i)}$ and $\tilde{\lambda}_i$ in Eq. (3) be independent normal UVs whose distributions have positive parameters. That is, $\tilde{\theta}_p : N(\theta_p, \sigma_p)$, $\tilde{\theta}_{i,t(i)} : N(\theta_{i,t(i)}, \sigma_{i,t(i)})$ and $\tilde{\lambda}_i : N(\lambda_i, \sigma_i^{\lambda})$, where $\theta_p, \theta_{i,t(i)}, \lambda_i, \sigma_p, \sigma_{i,t(i)}, \sigma_i^{\lambda}$ are all positive real values.

We will use the following lemmas to convert the UCCGP model in Eq. (3) to a deterministic program and, hence, solve it by passing to the dual form.

**Lemma 3.1** ([37]) *Let $\tilde{a}_i : N(a_i, \sigma_i)$, with $i = 1, \ldots, n$, and $\tilde{b} : N(b, \sigma)$ be independent normal UVs. Let $U_i$, with $i = 1, \ldots, n$, be nonnegative variables. Then, for every $\alpha \in ]0, 1[$,*

$$M\left(\sum_{i=1}^{n} \tilde{a}_i U_i \leq \tilde{b}\right) \geq \alpha \Leftrightarrow \sum_{i=1}^{n} U_i \left(a_i + \frac{\sigma_i \sqrt{3}}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right)\right) \leq b - \frac{\sigma\sqrt{3}}{\pi} \ln\left(\frac{1-\alpha}{\alpha}\right).$$

**Lemma 3.2** ([37]) *The expected value of a normal UV $\xi : N(e, \sigma)$ is $E[\xi] = e$.*

By Lemma 3.1, the chance constraints in Eq. (3) admit a deterministic equivalent, that is, $\forall i = 1, \ldots, n$,

$$M\left(\sum_{t(i)=1}^{k(i)} \tilde{\theta}_{i,t(i)} \prod_{j=1}^{s} x_j^{\gamma_{i,t(i),j}} \leq \tilde{\lambda}_i\right)$$
$$\geq \alpha \Leftrightarrow \sum_{t(i)=1}^{k(i)} \left(\theta_{i,t(i)} + \frac{\sigma_{i,t(i)}\sqrt{3}}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right)\right) \prod_{j=1}^{s} x_j^{\gamma_{i,t(i),j}} \leq \lambda_i - \frac{\sigma_i^{\lambda}\sqrt{3}}{\pi} \ln\left(\frac{1-\alpha}{\alpha}\right).$$

In addition, by Lemma 3.2, the objective of the UCCGP problem of Eq. (3) becomes a crisp value:

$$E\left[\sum_{p=1}^{h} \tilde{\theta}_p \prod_{j=1}^{s} x_j^{\alpha_{p,j}}\right] = \sum_{p=1}^{h} E\left[\tilde{\theta}_p\right] \prod_{j=1}^{s} x_j^{\alpha_{p,j}} = \sum_{p=1}^{h} \theta_p \prod_{j=1}^{s} x_j^{\alpha_{p,j}}.$$

Therefore, the model in Eq. (3) is transformed into the model in Eq. (4) below.

$$\begin{aligned}
&\min \sum_{p=1}^{h} \theta_p \prod_{j=1}^{s} x_j^{\alpha_{p,j}} \\
&\text{s.t.} \\
&\sum_{t(i)=1}^{k(i)} \left(\theta_{i,t(i)} + \frac{\sigma_{i,t(i)}\sqrt{3}}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right)\right) \prod_{j=1}^{s} x_j^{\gamma_{i,t(i),j}} \\
&\leq \lambda_i - \frac{\sigma_i^{\lambda}\sqrt{3}}{\pi} \ln\left(\frac{1-\alpha}{\alpha}\right), i = 1, \ldots, n, x_j > 0, \quad j = 1, \ldots, s. \qquad (4)
\end{aligned}$$

As mentioned above, we solve the GP problem in Eq. (4) using the dual algorithm, that is, solving the equivalent dual program with linear constraints (see, e.g, [4,13,48]). Thus, we consider the following dual GP for the normal UDs. The variables of this problem are $\beta_p$ and $\beta_{i,t(i)}$.

$$\max \quad \prod_{p=1}^{h}\left(\frac{\theta_p}{\beta_p}\right)^{\beta_p}\prod_{i=1}^{n}\prod_{t(i)=1}^{k(i)}\left(\frac{\theta_{i,t(i)}+\frac{\sigma_{i,t(i)}\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right)}{\beta_{i,t(i)}\left(\lambda_i-\frac{\sigma_i^{\lambda}\sqrt{3}}{\pi}\ln\left(\frac{1-\alpha}{\alpha}\right)\right)}\right)^{\beta_{i,t(i)}}\prod_{i=1}^{n}\delta_i^{\delta_i} \tag{5}$$

s.t.
Constraints of Model(1).

### 3.2 Proposed UCCGP Model with Linear Uncertainty Distributions

Let the coefficients $\tilde{\theta}_p$, $\tilde{\theta}_{i,t(i)}$ and $\tilde{\lambda}_i$ in Eq. (3) be independent positive linear UVs. That is, $\tilde{\theta}_p : L(\theta_p^a, \theta_p^b)$, with $0 < \theta_p^a < \theta_p^b$, $\tilde{\theta}_{i,t(i)} : L(\theta_{i,t(i)}^a, \theta_{i,t(i)}^b)$, with $0 < \theta_{i,t(i)}^a < \theta_{i,t(i)}^b$, and $\tilde{\lambda}_i : L(\lambda_i^a, \lambda_i^b)$, with $0 < \lambda_i^a < \lambda_i^b$.

We use Lemmas 3.3 and 3.4 below to obtain the crisp equivalent of the model in Eq. (3) with linear UDs.

**Lemma 3.3** ([37]) *Let $\tilde{\xi}_i (i = 1, \ldots, n)$ and $\tilde{\rho}$ be independent linear UVs, that is, $\tilde{\xi}_i : L(a_i, b_i)$, with $a_i < b_i$, and $\tilde{\rho} : L(a_\rho, b_\rho)$, with $a_\rho < b_\rho$. Let $U_i (i = 1, \ldots, n)$ be nonnegative variables. Then, for every $\alpha \in ]0, 1[$,*

$$M\left(\sum_{i=1}^{n}\tilde{\xi}_i U_i \le \tilde{\rho}\right) \ge \alpha \Leftrightarrow \sum_{i=1}^{n}((1-\alpha)\, a_i + \alpha\, b_i)\, U_i \le \alpha\, a_\rho + (1-\alpha)\, b_\rho \quad .$$

**Lemma 3.4** ([37]) *The expected value of a linear UV $\xi : L(a, b)$ is $E[\xi] = \frac{a+b}{2}$.*

By Lemma 3.4, we obtain a deterministic objective for the proposed UCCGP problem in Eq. (3), that is:

$$E\left[\sum_{p=1}^{h}\tilde{\theta}_p\prod_{j=1}^{s}x_j^{\alpha_{p,j}}\right] = \sum_{p=1}^{h}E\left[\tilde{\theta}_p\right]\prod_{j=1}^{s}x_j^{\alpha_{p,j}} = \sum_{p=1}^{h}\left(\frac{\theta_p^a+\theta_p^b}{2}\right)\prod_{j=1}^{s}x_j^{\alpha_{p,j}}.$$

Moreover, by Lemma 3.3, the constraints in Eq. (3) admit the following deterministic equivalent form:

$$\forall i = 1, \ldots, n, M\left(\sum_{t(i)=1}^{k(i)}\tilde{\theta}_{i,t(i)}\prod_{j=1}^{s}x_j^{\gamma_{i,t(i),j}} \le \tilde{\lambda}_i\right)$$
$$\ge \alpha \Leftrightarrow \sum_{t(i)=1}^{k(i)}\left((1-\alpha)\,\theta_{i,t(i)}^a + \alpha\theta_{i,t(i)}^b\right)\prod_{j=1}^{s}x_j^{\gamma_{i,t(i),j}} \le \alpha\lambda_i^a + (1-\alpha)\,\lambda_i^b.$$

Thus, when the coefficients are UVs endowed with linear distributions, the model in Eq. (3) is equivalent to:

$$\begin{aligned}
&\min \sum_{p=1}^{h}\left(\frac{\theta_p^a+\theta_p^b}{2}\right)\prod_{j=1}^{s} x_j^{\alpha_{p,j}}\\
&\text{s.t.}\\
&\sum_{t(i)=1}^{k(i)}\left((1-\alpha)\,\theta_{i,t(i)}^a+\alpha\theta_{i,t(i)}^b\right)\prod_{j=1}^{s} x_j^{\gamma_{i,t(i),j}}\\
&\quad\le \alpha\lambda_i^a+(1-\alpha)\,\lambda_i^b\,,\quad i=1,\ \ldots,\ n, x_j>0\,,\quad j=1,\ \ldots,\ s.
\end{aligned}\tag{6}$$

The corresponding dual problem is as follows. The variables of this problem are $\beta_p$ and $\beta_{i,t(i)}$.

$$\begin{aligned}
&\max\quad \prod_{p=1}^{h}\left(\frac{\theta_p^a+\theta_p^b}{2\beta_p}\right)^{\beta_p}\prod_{i=1}^{n}\prod_{t(i)=1}^{k(i)}\left(\frac{(1-\alpha)\theta_{i,t(i)}^a+\alpha\theta_{i,t(i)}^b}{\beta_{i,t(i)}\left(\alpha\lambda_i^a+(1-\alpha)\lambda_i^b\right)}\right)^{\beta_{i,t(i)}}\prod_{i=1}^{n}\delta_i^{\delta_i}\\
&\text{s.t.}\\
&\text{Constraints of Model (1).}
\end{aligned}\tag{7}$$

### 3.3 Proposed UCCGP Model with Zigzag Uncertainty Distributions

Let the coefficients $\tilde{\theta}_p$, $\tilde{\theta}_{i,t(i)}$ and $\tilde{\lambda}_i$in Eq. (3) be independent positive zigzag UVs. That is, $\tilde{\theta}_p : Z(\theta_p^a, \theta_p^b, \theta_p^c)$, with $0<\theta_p^a<\theta_p^b<\theta_p^c$, $\tilde{\theta}_{i,t(i)} : Z(\theta_{i,t(i)}^a, \theta_{i,t(i)}^b, \theta_{i,t(i)}^c)$, with $0<\theta_{i,t(i)}^a<\theta_{i,t(i)}^b<\theta_{i,t(i)}^c$ and $\tilde{\lambda}_i : Z(\lambda_i^a, \lambda_i^b, \lambda_i^c)$, with $0<\lambda_i^a<\lambda_i^b<\lambda_i^c$.

As in the previous subsections, in order to solve the UCCGP problem with zigzag UDs, we must convert the constraints of the model into their respective crisp equivalents. To do so, we need the following two lemmas.

**Lemma 3.5** ([37]) *Let $\tilde{\xi}_i\,(i=1,\ \ldots,\ n)$ and $\tilde{\rho}$ be independent zigzag UVs, that is, $\tilde{\xi}_i : Z(a_i, b_i, c_i)$, with $a_i<b_i<c_i$, and $\tilde{\rho} : Z(a_\rho, b_\rho, c_\rho)$, with $a_\rho<b_\rho<c_\rho$. Let $U_i\,(i=1,\ \ldots,\ n)$ be nonnegative variables. Then,*

$$M\left(\sum_{i=1}^{n}\tilde{\xi}_i U_i\le\tilde{\rho}\right)\ge\alpha \Leftrightarrow \begin{cases}\sum\limits_{i=1}^{n}\left((1-2\alpha)\,a_i+2\alpha\,b_i\right)U_i\le 2\alpha\,b_\rho+(1-2\alpha)\,c_\rho, \text{ if } \alpha\in]0,\ 0.5[;\\ \sum\limits_{i=1}^{n}\left((2\alpha-1)\,c_i+2(1-\alpha)\,b_i\right)U_i\le(2\alpha-1)\,a_\rho+2(1-\alpha)\,b_\rho, \text{ if } \alpha\in[0.5,\ 1[.\end{cases}$$

**Lemma 3.6** ([37]) *The expected value of a zigzag UV $\xi : Z(a,b,c)$ is $E[\xi]=\frac{a+2b+c}{4}$.*

By Lemma 3.6, the objective of Eq. (3) with zigzag UDs can be rewritten as follows:

$$E\left[\sum_{p=1}^{h}\tilde{\theta}_p\prod_{j=1}^{s}x_j^{\alpha_{p,j}}\right]=\sum_{p=1}^{h}E\left[\tilde{\theta}_p\right]\prod_{j=1}^{s}x_j^{\alpha_{p,j}}=\sum_{p=1}^{h}\left(\frac{\theta_p^a+2\theta_p^b+\theta_p^c}{4}\right)\prod_{j=1}^{s}x_j^{\alpha_{p,j}}.$$

By Lemma 3.5, the constraints of the UCCGP problem in Eq. (3) with zigzag UDs can be rewritten as follows:

$$\forall i = 1,\ \ldots,\ n,\ M\left(\sum_{t(i)=1}^{k(i)} \tilde{\theta}_{i,t(i)} \prod_{j=1}^{s} x_j^{\gamma_{i,t(i),j}} \le \tilde{\lambda}_i\right) \ge \alpha \Leftrightarrow$$

$$\begin{cases} \sum_{t(i)=1}^{k(i)} \left((1-2\alpha)\theta_{i,t(i)}^{a} + 2\alpha\theta_{i,t(i)}^{b}\right) \prod_{j=1}^{s} x_j^{\gamma_{i,t(i),j}} \le 2\alpha\lambda_i^{b} + (1-2\alpha)\,\lambda_i^{c}, & \text{if } \alpha \in ]0,\ 0.5[; \\ \sum_{t(i)=1}^{k(i)} \left((2\alpha-1)\theta_{i,t(i)}^{c} + 2(1-\alpha)\theta_{i,t(i)}^{b}\right) \prod_{j=1}^{s} x_j^{\gamma_{i,t(i),j}} \le (2\alpha-1)\lambda_i^{a} + (2-2\alpha)\,\lambda_i^{b}, & \text{if } \alpha \in [0.5,\ 1[. \end{cases}$$

Thus, the model in Eq. (3) with zigzag UDs can be transformed into two models, corresponding to the case where $\alpha < 0.5$and $\alpha \ge 0.5$, respectively.

$$\begin{array}{ll} \text{For } \alpha < 0.5, \text{ we have:} & \text{For } \alpha \ge 0.5, \text{ we have:} \\ \min \ \sum_{p=1}^{h} \left(\frac{\theta_p^a + 2\theta_p^b + \theta_p^c}{4}\right) \prod_{j=1}^{s} x_j^{\alpha_{p,j}} & \min \ \sum_{p=1}^{h} \left(\frac{\theta_p^a + 2\theta_p^b + \theta_p^c}{4}\right) \prod_{j=1}^{s} x_j^{\alpha_{p,j}} \\ \text{s.t.} & \text{s.t.} \\ \sum_{t(i)=1}^{k(i)} \left((1-2\alpha)\theta_{i,t(i)}^{a} + 2\alpha\theta_{i,t(i)}^{b}\right) \prod_{j=1}^{s} x_j^{\gamma_{i,t(i),j}} & \sum_{t(i)=1}^{k(i)} \left((2\alpha-1)\theta_{i,t(i)}^{c} + 2(1-\alpha)\theta_{i,t(i)}^{b}\right) \prod_{j=1}^{s} x_j^{\gamma_{i,t(i),j}} \\ \quad \le 2\alpha\lambda_i^{b} + (1-2\alpha)\,\lambda_i^{c}, \quad i = 1,\ \ldots,\ n, & \quad \le (2\alpha-1)\lambda_i^{a} + (2-2\alpha)\,\lambda_i^{b}, \quad i = 1,\ \ldots,\ n, \\ x_j > 0\,, \quad j = 1,\ \ldots,\ s. & x_j > 0\,, \quad j = 1,\ \ldots,\ s. \end{array} \tag{8}$$

The corresponding dual problems are given by the following:

$$\begin{array}{l} \max \begin{cases} \sum_{p=1}^{h} \left(\frac{\theta_p^a + 2\theta_p^b + \theta_p^c}{4\beta_p}\right)^{\beta_p} \prod_{i=1}^{n} \prod_{t(i)=1}^{k(i)} \left(\frac{(1-2\alpha)\theta_{i,t(i)}^{a} + 2\alpha\theta_{i,t(i)}^{b}}{\beta_{i,t(i)}(2\alpha\lambda_i^{b} + (1-2\alpha)\lambda_i^{c})}\right)^{\beta_{i,t(i)}} \prod_{i=1}^{n} \delta_i^{\delta_i}, & \text{if } \alpha < 0.5; \\ \sum_{p=1}^{h} \left(\frac{\theta_p^a + 2\theta_p^b + \theta_p^c}{4\beta_p}\right)^{\beta_p} \prod_{i=1}^{n} \prod_{t(i)=1}^{k(i)} \left(\frac{(2\alpha-1)\theta_{i,t(i)}^{c} + 2(1-\alpha)\theta_{i,t(i)}^{b}}{\beta_{i,t(i)}((2\alpha-1)\lambda_i^{a} + (2-2\alpha)\lambda_i^{b})}\right)^{\beta_{i,t(i)}} \prod_{i=1}^{n} \delta_i^{\delta_i}, & \text{if } \alpha \ge 0.5. \end{cases} \\ \text{s.t.} \\ \text{Constraints of Model (1).} \end{array} \tag{9}$$

## 4 Numerical Examples

We now provide some numerical examples to show the efficacy of the proposed GP models.

*Example 4.1* Let us consider the following instance of uncertain GP problem (see Eq. (2)):

$$\begin{array}{l} \min \tilde{\theta}_1 x_1^{-1} x_2^{-\frac{1}{2}} x_3^{-1} + \tilde{\theta}_2 x_1 x_3 + \tilde{\theta}_3 x_1 x_2 x_3 \\ \text{s.t.} \\ \tilde{\theta}_{1,1} x_1^{-2} x_2^{-2} + \tilde{\theta}_{1,2} x_2^{\frac{1}{2}} x_3^{-1} \le 1, x_1 > 0, x_2 > 0, x_3 > 0. \end{array} \tag{10}$$

where all coefficients are assumed to be UVs.

**Table 1** Solving the GP problem of Example 1 with normal uncertainty distributions

| $\alpha$ | Objective value | Corresponding primal solutions | Corresponding dual solutions |
|---|---|---|---|
| 0.25 | 38.4250 | $x_1^* = 0.5912, x_2^* = 1.5301, x_3^* = 0.7379$ | $\beta_1^* = 0.4823, \beta_2^* = 0.1703, \beta_3^* = 0.3474, \beta_{1,1}^* = 0.0177, \beta_{1,2}^* = 0.0354$ |
| 0.50 | 37.7849 | $x_1^* = 0.5911, x_2^* = 1.6918, x_3^* = 0.6988$ | $\beta_1^* = 0.4926, \beta_2^* = 0.1640, \beta_3^* = 0.3699, \beta_{1,1}^* = 0.0207, \beta_{1,2}^* = 0.0413$ |
| 0.75 | 39.1313 | $x_1^* = 0.5894, x_2^* = 1.8445, x_3^* = 0.6680$ | $\beta_1^* = 0.4779, \beta_2^* = 0.1509, \beta_3^* = 0.3712, \beta_{1,1}^* = 0.0221, \beta_{1,2}^* = 0.0442$ |

*Case 1: Normal Uncertainty Distributions*

$$\tilde{\theta}_1 : N(10,\ 1), \tilde{\theta}_2 : N(15,\ 3), \tilde{\theta}_3 : N(20,\ 5), \tilde{\theta}_{1,1} : N\left(\frac{1}{3},\ 0.1\right), \tilde{\theta}_{1,2} : N\left(\frac{4}{3},\ 0.2\right).$$

Using Eq. (4), the problem of Eq. (10) becomes the following deterministic GP:

$$\begin{aligned}
&\min 10x_1^{-1}x_2^{-\frac{1}{2}}x_3^{-1} + 15x_1x_3 + 20x_1x_2x_3 \\
&\text{s.t.} \\
&\left(\tfrac{1}{3} + \tfrac{0.1\sqrt{3}}{\pi}\ln\left(\tfrac{\alpha}{1-\alpha}\right)\right)x_1^{-2}x_2^{-2} + \left(\tfrac{4}{3} + \tfrac{0.2\sqrt{3}}{\pi}\ln\left(\tfrac{\alpha}{1-\alpha}\right)\right)x_2^{\frac{1}{2}}x_3^{-1} \le 1, \\
&\quad x_1 > 0,\ x_2 > 0,\ x_3 > 0.
\end{aligned} \tag{11}$$

Using Eq. (5), we can transform the problem to its dual form as follows:

$$\begin{aligned}
&\max \left(\frac{10}{\beta_1}\right)^{\beta_1}\left(\frac{15}{\beta_2}\right)^{\beta_2}\left(\frac{20}{\beta_3}\right)^{\beta_3}\left(\frac{\frac{1}{3}+\frac{0.1\sqrt{3}}{\pi}ln\left(\frac{\alpha}{1-\alpha}\right)}{\beta_{1,1}}\right)^{\beta_{1,1}}\left(\frac{\frac{4}{3}+\frac{0.2\sqrt{3}}{\pi}ln\left(\frac{\alpha}{1-\alpha}\right)}{\beta_{1,2}}\right)^{\beta_{1,2}}\rho^{\rho} \\
&\text{s.t.} \\
&\beta_1+\beta_2+\beta_3=1,\ -\beta_1+\beta_2+\beta_3-2\beta_{1,1}=0,\ -\beta_1+\beta_2+\beta_3-2\beta_{1,2}=0, \\
&\rho=\beta_{1,1}+\beta_{1,2}, \quad -\tfrac{\beta_1}{2}+\beta_3-2\beta_{1,1}+\tfrac{\beta_{1,2}}{2}=0, \quad \beta_1>0,\ \beta_2>0,\ \beta_2>0,\ \beta_{1,1}\ge 0,\ \beta_{1,2}\ge 0.
\end{aligned} \tag{12}$$

* Table 1 shows the optimal values obtained for the decision variables and the objective.
*Case 2: Linear Uncertainty Distributions*

$$\tilde{\theta}_1 : L(30,\ 50), \tilde{\theta}_2 : L(25,\ 45), \tilde{\theta}_3 : L(35,\ 65), \tilde{\theta}_{1,1} : L\left(\frac{1}{3},\ \frac{2}{3}\right), \tilde{\theta}_{1,2} : L\left(\frac{2}{3},\ \frac{4}{3}\right).$$

**Table 2** Solving the GP problem of Example 1 with linear uncertainty distributions

| $\alpha$ | Objective value | Corresponding primal solutions | Corresponding dual solutions |
|---|---|---|---|
| 0.25 | 118.6675 | $x_1^* = 0.8461, x_2^* = 1.3214, x_3^* = 0.7159$ | $\beta_1^* = 0.4841, \beta_2^* = 0.1787, \beta_3^* = 0.3372, \beta_{1,1}^* = 0.0159, \beta_{1,2}^* = 0.0318$ |
| 0.50 | 119.7881 | $x_1^* = 0.8295, x_2^* = 1.4765, x_3^* = 0.6881$ | $\beta_1^* = 0.4815, \beta_2^* = 0.1668, \beta_3^* = 0.3518, \beta_{1,1}^* = 0.0185, \beta_{1,2}^* = 0.0371$ |
| 0.75 | 120.8784 | $x_1^* = 0.8153, x_2^* = 1.6225, x_3^* = 0.6648$ | $\beta_1^* = 0.4770, \beta_2^* = 0.1577, \beta_3^* = 0.3657, \beta_{1,1}^* = 0.0231, \beta_{1,2}^* = 0.0462$ |

By Eq. (6), the problem of Eq. (10) becomes the following deterministic GP:

$$\begin{aligned} &\min 40x_1^{-1}x_2^{-\frac{1}{2}}x_3^{-1} + 35x_1x_3 + 50x_1x_2x_3 \\ &\text{s.t.} \\ &\left((1-\alpha)\tfrac{1}{3} + \tfrac{2}{3}\alpha\right)x_1^{-2}x_2^{-2} + \left((1-\alpha)\tfrac{2}{3} + \tfrac{4}{3}\alpha\right)x_2^{\frac{1}{2}}x_3^{-1} \le 1, \\ &x_1 > 0, x_2 > 0, x_3 > 0. \end{aligned} \tag{13}$$

Using Eq. (7), we can move the problem to its dual form as follows:

$$\begin{aligned} &\max \left(\frac{40}{\beta_1}\right)^{\beta_1}\left(\frac{35}{\beta_2}\right)^{\beta_2}\left(\frac{50}{\beta_3}\right)^{\beta_3}\left(\frac{\frac{1}{3}(1-\alpha)+\frac{2}{3}\alpha}{\beta_{1,1}}\right)^{\beta_{1,1}}\left(\frac{\frac{2}{3}(1-\alpha)+\frac{4}{3}\alpha}{\beta_{1,2}}\right)^{\beta_{1,2}}\rho^{\rho} \\ &\text{s.t.} \\ &\text{Constraints of Model (12).} \end{aligned} \tag{14}$$

The optimal values of decision variables and objective are presented in Table 2. In particular, it can be noted that the optimal objective values increase as $\alpha$ increases.

*Case 3: Zigzag Uncertainty Distributions*

$$\begin{aligned} &\tilde{\theta}_1 : Z(10,\ 20,\ 40), \tilde{\theta}_2 : Z(15,\ 25,\ 35), \tilde{\theta}_3 : Z(35,\ 65,\ 75), \\ &\tilde{\theta}_{1,1} : Z\left(\frac{1}{3},\ \frac{2}{3},\ 1\right)\tilde{\theta}_{1,2} : Z\left(\frac{2}{3},\ 1,\ \frac{4}{3}\right). \end{aligned}$$

By Eq. (8), the problem of Eq. (10) becomes one of the two following deterministic GP:

$$\begin{array}{ll} \text{For } \alpha < 0.5, \text{ we have:} & \text{For } \alpha \ge 0.5, \text{ we have:} \\ \min 22.5x_1^{-1}x_2^{-\frac{1}{2}}x_3^{-1} + 25x_1x_3 + 60x_1x_2x_3 & \min 22.5x_1^{-1}x_2^{-\frac{1}{2}}x_3^{-1} + 25x_1x_3 + 60x_1x_2x_3 \\ \text{s.t.} & \text{s.t.} \\ \left((1-2\alpha)\frac{1}{3} + 2\alpha\frac{2}{3}\right)x_1^{-2}x_2^{-2} & \left((2\alpha-1) + 2(1-\alpha)\frac{2}{3}\right)x_1^{-2}x_2^{-2} \\ +\left((1-2\alpha)\frac{2}{3} + 2\alpha\right)x_2^{\frac{1}{2}}x_3^{-1} \le 1, & +\left((2\alpha-1)\frac{4}{3} + 2(1-\alpha)\right)x_2^{\frac{1}{2}}x_3^{-1} \le 1, \\ x_1 > 0, x_2 > 0, x_3 > 0. & x_1 > 0, x_2 > 0, x_3 > 0. \end{array} \tag{15}$$

**Table 3** Solving the GP problem of Example 1 with zigzag uncertainty distributions

| $\alpha$ | Objective value | Corresponding primal solutions | Corresponding dual solutions |
|---|---|---|---|
| 0.25 | 92.4646 | $x_1^* = 0.7941, x_2^* = 1.5423, x_3^* = 0.5255$ | $\beta_1^* = 0.4695, \beta_2^* = 0.1128, \beta_3^* = 0.4176, \beta_{1,1}^* = 0.0305, \beta_{1,2}^* = 0.0609$ |
| 0.50 | 94.3989 | $x_1^* = 0.7999, x_2^* = 1.7681, x_3^* = 0.4798$ | $\beta_1^* = 0.4670, \beta_2^* = 0.1016, \beta_3^* = 0.4313, \beta_{1,1}^* = 0.0329, \beta_{1,2}^* = 0.0659$ |
| 0.75 | 96.1164 | $x_1^* = 0.7996, x_2^* = 1.9773, x_3^* = 0.4476$ | $\beta_1^* = 0.4651, \beta_2^* = 0.0931, \beta_3^* = 0.4418, \beta_{1,1}^* = 0.0349, \beta_{1,2}^* = 0.0697$ |

The dual of the above models is obtained using the corresponding models of Eq. (9):

$$\max \begin{cases} \left(\frac{22.5}{\beta_1}\right)^{\beta_1}\left(\frac{25}{\beta_2}\right)^{\beta_2}\left(\frac{60}{\beta_3}\right)^{\beta_3}\left(\frac{(1-2\alpha)\frac{1}{3}+2\alpha\frac{2}{3}}{\beta_{1,1}}\right)^{\beta_{1,1}}\left(\frac{(1-2\alpha)\frac{2}{3}+2\alpha}{\beta_{1,2}}\right)^{\beta_{1,2}}\rho^{\rho}, & \text{if } \alpha < 0.5; \\ \left(\frac{22.5}{\beta_1}\right)^{\beta_1}\left(\frac{25}{\beta_2}\right)^{\beta_2}\left(\frac{60}{\beta_3}\right)^{\beta_3}\left(\frac{(2\alpha-1)+2(1-\alpha)\frac{2}{3}}{\beta_{1,1}}\right)^{\beta_{1,1}}\left(\frac{(2\alpha-1)\frac{4}{3}+2(1-\alpha)}{\beta_{1,2}}\right)^{\beta_{1,2}}\rho^{\rho}, & \text{if } \alpha \geq 0.5. \end{cases}$$
$$\text{s.t.}$$
$$\text{Constraints of Model (12).} \tag{16}$$

The optimal values of decision variables and objective are presented in Table 3. As in the case of linear UDs, the optimal objective values increase as $\alpha$ increases.

*Example 4.2* We now consider a more general instance of uncertain GP problem (see Eq. (2)): both the coefficients and the limits of the constraints are UVs.

$$\begin{aligned} &\min \tilde{\theta}_1 x_1^{-1.1} x_2^{-1} x_3^{-1} + \tilde{\theta}_2 x_1^2 x_2^1 x_3^{-0.38} \\ &\text{s.t.} \\ &\tilde{\theta}_{1,1} x_1 x_2^{-1} x_3^{-1} \leq \tilde{\lambda}_1, \tilde{\theta}_{2,1} x_2^2 x_3^2 + \tilde{\theta}_{2,2} x_1^{1.5} x_3 \leq \tilde{\lambda}_2, x_1 > 0, x_2 > 0, x_3 > 0. \end{aligned} \tag{17}$$

*Case 1: Normal Uncertainty Distributions*

$$\tilde{\theta}_1 : N(6,\ 1), \tilde{\theta}_2 : N(2,\ 1), \tilde{\theta}_{1,1} : N(4,\ 1), \tilde{\theta}_{2,1} : N(2,\ 0.3), \tilde{\theta}_{2,2} : N(15,\ 1), \tilde{\lambda}_1 : N(1,\ 0.1), \tilde{\lambda}_2 : N(9,\ 1).$$

Using Eq. (4), we obtain the following deterministic GP:

$$\begin{aligned} &\min 9 x_1^{-1.1} x_2^{-1} x_3^{-1} + 10 x_1^2 x_2^1 x_3^{-0.38} \\ &\text{s.t.} \\ &\left(4 + \frac{\sqrt{3}}{\pi} ln\left(\frac{\alpha}{1-\alpha}\right)\right) x_1 x_2^{-1} x_3^{-1} \leq 1 - \frac{0.1\sqrt{3}}{\pi} ln\left(\frac{1-\alpha}{\alpha}\right), \end{aligned}$$

**Table 4** Solving the problem of Example 2 with normal uncertainty distributions

| $\alpha$ | Objective value | Corresponding primal solutions | Corresponding dual solutions |
|---|---|---|---|
| 0.25 | 8.0770 | $x_1^* = 0.5537, x_2^* = 2.3346, x_3^* = 0.7590$ | $\beta_1^* = 0.8032, \beta_2^* = 0.1968, \beta_{1,1}^* = 0.0825, \beta_{2,1}^* = 0.3445, \beta_{2,2}^* = 0.2716$ |
| 0.50 | 9.6460 | $x_1^* = 0.4459, x_2^* = 3.0180, x_3^* = 0.5909$ | $\beta_1^* = 0.8481, \beta_2^* = 0.1519, \beta_{1,1}^* = 0.3147, \beta_{2,1}^* = 0.5054, \beta_{2,2}^* = 0.2096$ |
| 0.75 | 12.4754 | $x_1^* = 0.3492, x_2^* = 3.9483, x_3^* = 0.4336$ | $\beta_1^* = 0.8940, \beta_2^* = 0.1060, \beta_{1,1}^* = 0.5520, \beta_{2,1}^* = 0.6700, \beta_{2,2}^* = 0.1463$ |

$$
\begin{aligned}
&\left(2 + \frac{0.3\sqrt{3}}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right)\right) x_2^2 x_3^2 + \left(15 + \frac{\sqrt{3}}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right)\right) x_1^{1.5} x_3, \\
&\leq 9 - \frac{\sqrt{3}}{\pi} \ln\left(\frac{1-\alpha}{\alpha}\right), \\
&x_1 > 0, x_2 > 0, x_3 > 0.
\end{aligned} \tag{18}
$$

Then, by Eq. (5), we transform the problem of Eq. (18) to its dual as follows:

$$
\begin{aligned}
\max\ & \left(\frac{9}{\beta_1}\right)^{\beta_1} \left(\frac{10}{\beta_2}\right)^{\beta_2} \left(\frac{\left(4+\frac{\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right)\right)}{\left(1-\frac{0.1\sqrt{3}}{\pi}\ln\left(\frac{1-\alpha}{\alpha}\right)\right)\beta_{1,1}}\right)^{\beta_{1,1}} \left(\frac{2+\frac{0.3\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right)}{\left(9-\frac{\sqrt{3}}{\pi}\ln\left(\frac{1-\alpha}{\alpha}\right)\right)\beta_{2,1}}\right)^{\beta_{2,1}} \left(\frac{15+\frac{\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right)}{\left(9-\frac{\sqrt{3}}{\pi}\ln\left(\frac{1-\alpha}{\alpha}\right)\right)\beta_{2,2}}\right)^{\beta_{2,2}} \rho^{\rho} \\
\text{s.t.}\ & \\
& \beta_1 + \beta_2 = 1, \quad -1.1\beta_1 + 2\beta_2 + \beta_{1,1} + 1.5\beta_{2,2} = 0, \ -\beta_1 - 0.38\beta_2 - \beta_{1,1} + 2\beta_{2,1} + \beta_{2,2} = 0, \\
& \rho = \beta_{2,1} + \beta_{2,2}, \ -\beta_1 + \beta_2 - \beta_{1,1} + 2\beta_{2,1} = 0, \quad \beta_1 > 0, \ \beta_2 > 0, \ \beta_{1,1} \geq 0, \beta_{2,1} \geq 0, \ \beta_{2,2} \geq 0.
\end{aligned} \tag{19}
$$

Table 4 shows the optimal values obtained for the decision variables and the objective.

*Case 2: Linear Uncertainty Distributions*

$$
\begin{aligned}
&\tilde{\theta}_1 : L(6,\ 7), \tilde{\theta}_2 : L(2,\ 3), \tilde{\theta}_{1,1} : L\,(3.5,\ 4)\,, \tilde{\theta}_{2,1} : L\,(1,\ 2)\,, \tilde{\theta}_{2,2} : L\,(13,\ 15)\,, \\
&\quad \tilde{\lambda}_1 : L(0.5,\ 1), \tilde{\lambda}_2 : L(8,\ 9).
\end{aligned}
$$

By Eq. (6), the problem of Eq. (17) becomes the following deterministic GP:

$$
\begin{aligned}
&\min 6.5 x_1^{-1.1} x_2^{-1} x_3^{-1} + 2.5 x_1^2 x_2^1 x_3^{-0.38} \\
&\text{s.t.} \\
&(3.5(1-\alpha) + 4\alpha)\, x_1 x_2^{-1} x_3^{-1} \leq 0.5\alpha + (1-\alpha), \\
&((1-\alpha) + 2\alpha)\, x_2^2 x_3^2 + (13(1-\alpha) + 15\alpha)\, x_1^{1.5} x_3 \leq 8\alpha + 9(1-\alpha), \\
&x_1 > 0, x_2 > 0, x_3 > 0.
\end{aligned} \tag{20}
$$

**Table 5** Solving the problem of Example 2 with linear uncertainty distributions

| $\alpha$ | Objective value | Corresponding primal solutions | Corresponding dual solutions |
|---|---|---|---|
| 0.25 | 8.7415 | $x_1^* = 0.4880, x_2^* = 2.5549, x_3^* = 0.7913$ | $\beta_1^* = 0.8098, \beta_2^* = 0.1902, \beta_{1,1}^* = 0.1167, \beta_{1,1}^* = 0.3681, \beta_{1,2}^* = 0.2625$ |
| 0.50 | 10.1768 | $x_1^* = 0.4047, x_2^* = 3.0933, x_3^* = 0.6542$ | $\beta_1^* = 0.8538, \beta_2^* = 0.1462, \beta_{1,1}^* = 0.3441, \beta_{1,1}^* = 0.5259, \beta_{1,2}^* = 0.2018$ |
| 0.75 | 12.7303 | $x_1^* = 0.3201, x_2^* = 3.8367, x_3^* = 0.5172$ | $\beta_1^* = 0.9008, \beta_2^* = 0.0092, \beta_{1,1}^* = 0.9534, \beta_{1,1}^* = 0.9225, \beta_{1,2}^* = 0.0127$ |

We solve this problem passing to its dual form (see Eq. (7)) as follows:

$$\begin{aligned}&\max\ \left(\frac{6.5}{\beta_1}\right)^{\beta_1}\left(\frac{2.5}{\beta_2}\right)^{\beta_2}\left(\frac{(3.5(1-\alpha)+4\alpha)}{(0.5\alpha+(1-\alpha))\beta_{1,1}}\right)^{\beta_{1,1}}\left(\frac{(1-\alpha)+2\alpha}{(8\alpha+9(1-\alpha))\beta_{2,1}}\right)^{\beta_{2,1}}\left(\frac{13(1-\alpha)+15\alpha}{(8\alpha+9(1-\alpha))\beta_{2,2}}\right)^{\beta_{2,2}}\rho^{\rho}\\&\text{s.t.}\\&\text{Constraints of Model (19).}\end{aligned} \tag{21}$$

The optimal values of the decision variables and the objective are shown in Table 5.

*Case 3: Zigzag Uncertainty Distributions*

$$\tilde{\theta}_1 : Z(5,\ 6,\ 9), \tilde{\theta}_2 : Z(10,\ 12,\ 15), \tilde{\theta}_{1,1} : Z(3,\ 4,\ 5), \tilde{\theta}_{2,1} : Z\left(1,\ 1.5,\ 2\right), \tilde{\theta}_{2,2} : Z\left(11,\ 13,\ 15\right), \tilde{\lambda}_1 : Z\left(0.5,\ 0.75,\ 1\right), \tilde{\lambda}_2 : Z\left(6, 8,\ 9\right).$$

Using Eq. (8), the problem of Eq. (17) becomes one of the following deterministic GP:

$$\begin{array}{ll}\text{For } \alpha < 0.5, \text{ we have:} & \text{For } \alpha \geq 0.5, \text{ we have:}\\[4pt] \min \frac{26}{4}x_1^{-1.1}x_2^{-1}x_3^{-1} + \frac{49}{4}x_1^2x_2^1x_3^{-0.38} & \min \frac{26}{4}x_1^{-1.1}x_2^{-1}x_3^{-1} + \frac{49}{4}x_1^2x_2^1x_3^{-0.38}\\ \text{s.t.} & \text{s.t.}\\ (3(1-2\alpha)+8\alpha)\,x_1x_2^{-1}x_3^{-1} & (4(2\alpha-1)+10(1-\alpha)\,)\,x_1x_2^{-1}x_3^{-1}\\ \quad \leq \alpha + 0.75(1-2\alpha), & \quad \leq 0.5(2\alpha-1)+1.5(1-\alpha),\\ ((1-2\alpha)+3\alpha)\ x_2^2x_3^2 & (1.5(2\alpha-1)+4(1-\alpha))\,x_2^2x_3^2\\ \quad +(11(1-2\alpha)+26\alpha)\,x_1^{1.5}x_3 & \quad +(13(2\alpha-1)+30(1-\alpha))\,x_1^{1.5}x_3\\ \quad \leq 12\alpha + 8(1-2\alpha), & \quad \leq 6(2\alpha-1)\ +16(1-\alpha),\\ \ x_1 > 0, x_2 > 0, x_3 > 0. & \ x_1 > 0, x_2 > 0, x_3 > 0.\end{array} \tag{22}$$

The dual of the models in Eq. (22) is obtained using Eq. (9).

**Table 6** Solving the problem of Example 2 with zigzag uncertainty distributions

| $\alpha$ | Objective value | Corresponding primal solutions | Corresponding dual solutions |
|---|---|---|---|
| 0.25 | 13.7387 | $x_1^* = 0.3377, x_2^* = 1.7610, x_3^* = 1.0739$ | $\beta_1^* = 0.8257, \beta_2^* = 0.1743, \beta_{1,1}^* = 0.1989, \beta_{2,1}^* = 0.4251, \beta_{2,2}^* = 0.2405$ |
| 0.50 | 22.8516 | $x_1^* = 0.1896, x_2^* = 2.9121, x_3^* = 0.6512$ | $\beta_1^* = 0.9339, \beta_2^* = 0.0661, \beta_{1,1}^* = 0.7583, \beta_{2,1}^* = 0.8130, \beta_{2,2}^* = 0.0912$ |
| 0.75 | 38.5904 | $x_1^* = 0.1093, x_2^* = 4.4601, x_3^* = 0.4412$ | $\beta_1^* = 0.9769, \beta_2^* = 0.0231, \beta_3^* = 0.9806, \beta_{1,1}^* = 0.9672, \beta_{1,2}^* = 0.0319$ |

$$\begin{aligned}
\max & \begin{cases} \left(\frac{26}{4\beta_1}\right)^{\beta_1}\left(\frac{49}{4\beta_2}\right)^{\beta_2}\left(\frac{3(1-2\alpha)+8\alpha}{(\alpha+0.75(1-2\alpha))\beta_{1,1}}\right)^{\beta_{1,1}}\left(\frac{(1-2\alpha)+3\alpha}{(12\alpha+8(1-2\alpha))\beta_{1,1}}\right)^{\beta_{2,1}} \\ \left(\frac{11(1-2\alpha)+26\alpha}{(12\alpha+8(1-2\alpha))\beta_{1,2}}\right)^{\beta_{2,2}}\rho^{\rho}, \qquad \text{if } \alpha < 0.5; \\ \left(\frac{26}{4\beta_1}\right)^{\beta_1}\left(\frac{49}{4\beta_2}\right)^{\beta_2}\left(\frac{4(2\alpha-1)+10(1-\alpha)}{(0.5(2\alpha-1)+1.5(1-\alpha))\beta_{1,1}}\right)^{\beta_{1,1}}\left(\frac{1.5(2\alpha-1)+4(1-\alpha)}{(6(2\alpha-1)+16(1-\alpha))\beta_{1,1}}\right)^{\beta_{2,1}} \\ \left(\frac{13(2\alpha-1)+30(1-\alpha)}{(6(2\alpha-1)+16(1-\alpha))\beta_{2,2}}\right)^{\beta_{2,2}}\rho^{\rho}, \qquad \text{if } \alpha \geq 0.5. \end{cases} \\
\text{s.t.} & \\
& \text{Constraints of Model (19).}
\end{aligned} \tag{23}$$

The optimal values of decision variables and objective are presented in Table 6.

## 5 Applying the Proposed UCCGP to Profit Maximization

In this section, we consider an application of the proposed UCCGP model to inventory management and, in particular, to economic order quantity (EQQ) under cost minimization and profit maximization. We consider the problem of a profit maximizing firm whose goal is to optimize lot-sizing, pricing, and marketing decisions. Similar problems within a GP framework have been discussed, among others, by Jung and Klein in [8], Liu in [49], Liu in [50], Sadjadi et al. in [51], and Samadi et al. in [52].

### 5.1 Model Specification

Consider a profit maximizing firm which introduces a new product in the market and operates under the following assumptions: (1) Demand of the product is affected by the selling price; (2) replenishment is instantaneous; (3) no shortage is allowed.

Similar to the model of Jung and Klein in [8], we consider two decision variables: $p$, price per unit in dollar, and $q$, order quantity in units. The demand per unit $D(p)$ is a decreasing function of the price per unit, more precisely, $D(p) = Kp^{-\beta}$ where $K > 0$ is a constant and $\beta > 1$ is the price elasticity of demand since $\frac{dD(p)}{dp} \cdot \frac{p}{D(p)} = -\beta$.

Therefore, the total revenue is represented by $pD(p) = Kp^{-\beta+1}$. We assume three types of cost items: ordering cost, inventory holding cost and purchase cost. The ordering cost is represented by $AD(p)/q$ where $A$ is the ordering cost per batch. Let the purchase cost per unit be $C(q) = Rq^{-\delta}$ where $R > 0$ and $1 > \delta > 0$. Note that $\delta$ is the cost elasticity of the product since $\frac{dC(q)}{dq} \cdot \frac{q}{C(q)} = -\delta$. The inventory holding cost is given by $0.5Iq \cdot C(q) = 0.5IRq^{-\delta+1}$ where $I > 0$ is the inventory carrying cost rate (% per unit time). The purchase cost (or variable cost) is represented by $D(p)C(q) = KRp^{-\beta}q^{-\delta}$.

The maximum profit $\pi(p, q)$ is obtained by solving the following problem:

$$\begin{aligned}\pi(p,q) &= \max\ (\text{Revenue} - \text{Ordering Cost} - \text{Inventory Holding Cost}\\ &\qquad -\text{Purchase Cost})\\ &= \max\ pD(p) - \frac{AD(p)}{q} - \frac{IqC(q)}{2} - C(q)D(p)\\ &= \max\ Kp^{-\beta+1} - AKp^{-\beta}q^{-1} - 0.5IRq^{-\delta+1} - KRp^{-\beta}q^{-\delta}, \quad (24)\end{aligned}$$

where the decision variables are $p$ and $q$. Eq. (24) can be transformed into the following:

$$\begin{aligned}&\max\quad T\\ &\text{s.t.}\\ &Kp^{-\beta+1} - AKp^{-\beta}q^{-1} - 0.5IRq^{-\delta+1} - KRp^{-\beta}q^{-\delta} \geq T,\end{aligned} \quad (25)$$

which is equivalent to the following GP problem:

$$\begin{aligned}&\min\quad T^{-1}\\ &\text{s.t.}\\ &(K)^{-1}p^{\beta-1}T + Ap^{-1}q^{-1} + 0.5IR(K)^{-1}p^{\beta-1}q^{-\delta+1} + Rp^{-1}q^{-\delta} \leq 1.\end{aligned} \quad (26)$$

Assuming that $K$, $A$, $R$ and $I$ are UVs, the problem of Eq. (26) becomes:

$$\begin{aligned}&\min\quad T^{-1}\\ &\text{s.t.}\\ &\left(\tilde{K}\right)^{-1}p^{\beta-1}T + \tilde{A}p^{-1}q^{-1} + 0.5\tilde{I}\tilde{R}\left(\tilde{K}\right)^{-1}p^{\beta-1}q^{-\delta+1} + \tilde{R}p^{-1}q^{-\delta} \leq 1.\end{aligned} \quad (27)$$

As in Sect. 3, we will solve three instances of this problem, that is, we will assume the coefficients $K$, $A$, $R$ and $I$ to be UVs with normal, linear or zigzag UDs. To convert the problem to its determinist equivalent, we will use the following results from [37].

**Lemma 5.1** *Let $\xi$ be a positive UV with regular UD $\Phi_\xi$. Then, the inverse uncertainty distribution of $1/\xi$ is $\Psi^{-1}_{1/\xi}(\alpha) = \frac{1}{\Phi^{-1}_{\xi}(1-\alpha)}$, where $\alpha \in ]0,\ 1[$.*

**Lemma 5.2** *Let $\xi_1,\ \xi_2,\ \ldots,\ \xi_n$ be positive UVs with regular UDs $\Phi_{\xi_1},\ \Phi_{\xi_2},\ \ldots,\ \Phi_{\xi_n}$, respectively. Then, the inverse UD of the product $\xi = \xi_1 \cdot \xi_2 \cdots \xi_n$ is $\Psi^{-1}_{\xi}(\alpha) = \Phi^{-1}_{\xi_1}(\alpha) \cdot \Phi^{-1}_{\xi_2}(\alpha) \cdots \Phi^{-1}_{\xi_n}(\alpha)$, where $\alpha \in ]0,\ 1[$.*

**Lemma 5.3** *Let $\xi_1,\ \xi_2,\ \ldots,\ \xi_n$ be independent UVs with UDs $\Phi_{\xi_1},\ \Phi_{\xi_2},\ \ldots,\ \Phi_{\xi_n}$, respectively, and $g(x,\xi_1,\ \xi_2,\ \ldots,\ \xi_n)$ be a constraint function strictly increasing with respect to $\xi_1,\ \xi_2,\ \ldots,\ \xi_k$ and strictly decreasing with respect to $\xi_{k+1},\ \xi_{k+2},\ \ldots,\ \xi_n$. Then, for every $\alpha \in ]0,\ 1[$,*

$$M[g(x,\xi_1,\ \xi_2,\ \ldots,\ \xi_n)\le 0]\ge\alpha \Leftrightarrow g(x,\ \Phi_{\xi_1}^{-1}(\alpha),\ \Phi_{\xi_2}^{-1}(\alpha),\ \ldots,\ \Phi_{\xi_k}^{-1}(\alpha),\ \Phi_{\xi_{k+1}}^{-1}(1-\alpha),\ \ldots,\ \Phi_{\xi_n}^{-1}(1-\alpha))\le 0.$$

We also need to recall the algebraic form of the inverse UDs of normal, linear and zigzag UVs.

| Inverse UD of $\xi : N(e,\sigma)$ | Inverse UD of $\xi : L(a,b)$ | Inverse UD of $\xi : Z(a,b,c)$ |
|---|---|---|
| $\Phi^{-1}(\alpha)=e+\frac{\sigma\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right)$ where $\alpha\in]0,\ 1[$ | $\Phi^{-1}(\alpha)=(1-\alpha)a+\alpha b$ where $\alpha\in]0,\ 1[$ | $\Phi^{-1}(\alpha)=\begin{cases}(1-2\alpha)a+2\alpha b, & \text{if } 0<\alpha<0.5;\\ (2-2\alpha)b+(2\alpha-1)c, & \text{if } 0.5\le\alpha<1.\end{cases}$ |

### 5.2 Profit Maximization with Normal Uncertainty Distributions

Let $\tilde{K}: N(K,\sigma_k)$, $\tilde{A}: N(A,\sigma_A)$, $\tilde{R}: N(R,\sigma_R)$, $\tilde{I}: N(I,\sigma_I)$ and all the parameters be positive. Using the inverse of the normal UD, the uncertain GP problem of Eq. (27) takes the following form:

$$\begin{aligned}&\min\quad T^{-1}\\ &\text{s.t.}\\ &\frac{p^{\beta-1}T}{K+\frac{\sigma_K\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right)}+\left(A+\frac{\sigma_A\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right)\right)p^{-1}q^{-1}+\frac{0.5\left(I+\frac{\sigma_I\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right)\right)}{K+\frac{\sigma_K\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right)}\\ &\times\left(R+\frac{\sigma_R\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right)\right)p^{\beta-1}q^{-\delta+1}+\left(R+\frac{\sigma_R\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right)\right)p^{-1}q^{-\delta}\le 1,\end{aligned}\tag{28}$$

whose dual form is the following:

$$\begin{aligned}\max\quad&\left(\frac{1}{K+\frac{\sigma_K\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right)}\right)\left(\frac{A+\frac{\sigma_A\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right)}{\beta_{1,2}}\right)^{\beta_{1,2}}\\ &\left(\frac{0.5\left(I+\frac{\sigma_I\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right)\right)\left(R+\frac{\sigma_R\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right)\right)}{\left(K+\frac{\sigma_K\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right)\right)\beta_{1,3}}\right)^{\beta_{1,3}}\left(\frac{\left(R+\frac{\sigma_R\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right)\right)}{\beta_{1,4}}\right)^{\beta_{1,4}}\rho^{\rho}\\ \text{s.t.}&\\ &\beta_1=1,\quad(\beta-1)\,\beta_{1,1}-\beta_{1,2}+(\beta-1)\,\beta_{1,3}-\beta_{1,4}=0,\ \rho=\beta_{1,1}+\beta_{1,2}+\beta_{1,3}+\beta_{1,4},\\ &-\beta_1+\beta_{1,1}=0,\ -\beta_{1,2}+(-\delta+1)\,\beta_{1,3}-\delta\beta_{1,4}=0,\beta_1>0,\quad\beta_{1,1},\beta_{1,2},\beta_{1,3},\beta_{1,4}\ge 0.\end{aligned}\tag{29}$$

### 5.3 Profit Maximization with Linear Uncertainty Distributions

Let $\tilde{K}: L(K^a,K^b)$, $\tilde{A}: L(A^a,A^b)$, $\tilde{R}: L(R^a,R^b)$, $\tilde{I}: L(I^a,I^b)$ and all the parameters be positive. Using the inverse of the linear UD, the uncertain GP problem of Eq. (27) takes the following form:

$$\begin{array}{l} \min \quad T^{-1} \\ \text{s.t.} \\ \frac{p^{\beta-1}T}{(1-\alpha)K^a+\alpha K^b} + \left((1-\alpha)\, A^a + \alpha A^b\right) p^{-1}q^{-1} \\ +\frac{0.5\left((1-\alpha)I^a+\alpha I^b\right)\left((1-\alpha)R^a+\alpha R^b\right)}{(1-\alpha)K^a+\alpha K^b} p^{\beta-1}q^{-\delta+1} \\ +\left((1-\alpha)\, R^a + \alpha R^b\right) p^{-1}q^{-\delta} \le 1,\ p, q, T > 0, \end{array} \tag{30}$$

which has the following dual form:

$$\begin{array}{l} \max \quad \left(\frac{1}{(1-\alpha)K^a+\alpha K^b}\right) \left(\frac{\left((1-\alpha)A^a+\alpha A^b\right)}{\beta_{1,2}}\right)^{\beta_{1,2}} \\ \times\left(\frac{0.5\left((1-\alpha)I^a+\alpha I^b\right)\left((1-\alpha)R^a+\alpha R^b\right)}{\left((1-\alpha)K^a+\alpha K^b\right)\beta_{1,3}}\right)^{\beta_{1,3}} \times \left(\frac{\left((1-\alpha)R^a+\alpha R^b\right)}{\beta_{1,4}}\right)^{\beta_{1,4}} \rho^{\rho} \\ \text{s.t.} \\ \text{Constraints of Model (29).} \end{array} \tag{31}$$

## 5.4 Profit Maximization with Zigzag Uncertainty Distributions

Let $\tilde{K} : Z(K^a, K^b, K^c)$, $\tilde{A} : Z(A^a, A^b, A^c)$, $\tilde{R} : Z(R^a, R^b, R^c)$, $\tilde{I} : Z(I^a, I^b, I^c)$ and all the parameters be positive. Using the inverse of the zigzag UD, the uncertain GP problem of Eq. (27) takes the following form:

$$\begin{array}{ll} \text{For } \alpha < 0.5, \text{ we have:} & \text{For } \alpha \ge 0.5, \text{ we have :} \\ \min \quad T^{-1}\text{s.t.} & \min \quad T^{-1}\text{s.t.} \\ \frac{p^{\beta-1}T}{(1-2\alpha)K^a+2\alpha K^b} & \frac{p^{\beta-1}T}{(2-2\alpha)K^b+(2\alpha-1)K^c} \\ +\left((1-2\alpha)\, A^a + 2\alpha A^b\right) p^{-1}q^{-1} & +\left((2-2\alpha)\, A^b + (2\alpha-1)\, A^c\right) p^{-1}q^{-1} \\ +\frac{0.5\left((1-2\alpha)I^a+2\alpha I^b\right)}{(1-2\alpha)K^a+2\alpha K^b} & +\frac{0.5\left((2-2\alpha)I^b+(2\alpha-1)I^c\right)}{(2-2\alpha)K^b+(2\alpha-1)K^c} \\ \times\left((1-2\alpha)\, R^a + 2\alpha R^b\right) p^{\beta-1}q^{-\delta+1} & \times\left((2-2\alpha)\, R^b + (2\alpha-1)\, R^c\right) p^{\beta-1}q^{-\delta+1} \\ +\left((1-2\alpha)\, R^a + 2\alpha R^b\right) p^{-1}q^{-\delta} \le 1. & +\left((2-2\alpha)\, R^b+(2\alpha-1)\, R^c\right) p^{-1}q^{-\delta} \le 1. \end{array} \tag{32}$$

whose dual form is given by the following:

$$\max \begin{cases} \left(\frac{1}{(1-2\alpha)K^a+2\alpha K^b}\right) \left(\frac{\left((1-2\alpha)A^a+2\alpha A^b\right)}{\beta_{1,2}}\right)^{\beta_{1,2}} \\ \times\left(\frac{0.5\left((1-2\alpha)I^a+2\alpha I^b\right)\left((1-2\alpha)R^a+2\alpha R^b\right)}{\left((1-2\alpha)K^a+2\alpha K^b\right)\beta_{1,3}}\right)^{\beta_{1,3}} \left(\frac{\left((1-2\alpha)R^a+2\alpha R^b\right)}{\beta_{1,4}}\right)^{\beta_{1,4}} \rho^{\rho}, \text{ if } \alpha < 0.5; \\ \\ \left(\frac{1}{(2-2\alpha)K^b+(2\alpha-1)K^c}\right) \left(\frac{\left((2-2\alpha)A^b+(2\alpha-1)A^c\right)}{\beta_{1,2}}\right)^{\beta_{1,2}} \\ \times\left(\frac{0.5\left((2-2\alpha)I^b+(2\alpha-1)I^c\right)\left((2-2\alpha)R^b+(2\alpha-1)R^c\right)}{\left((2-2\alpha)K^b+(2\alpha-1)K^c\right)\beta_{1,3}}\right)^{\beta_{1,3}} \left(\frac{\left((2-2\alpha)R^b+(2\alpha-1)R^c\right)}{\beta_{1,4}}\right)^{\beta_{1,4}} \rho^{\rho}, \\ \quad \text{if } \alpha \ge 0.5. \end{cases}$$

$$\begin{array}{l} \text{s.t.} \\ \text{Constraints of Model (29).} \end{array} \tag{33}$$

**Table 7** Profit maximization results for specific uncertainty distributions

| A | Objective value and primal solutions (Linear) | Objective value and primal solutions (Normal) | Objective value and primal solutions (Zigzag) |
|---|---|---|---|
| 0.25 | $T^* = 6510.4$ $p^* = 31.3755,$ $q^* = 224.6459$ | $T^* = 7547.7$ $p^* = 27.4083,$ $q^* = 181.5307$ | $T^* = 6819.9$ $p^* = 40.6192,$ $q^* = 121.8140$ |
| 0.50 | $T^* = 6745.8$ $p^* = 33.0269,$ $q^* = 206.8560$ | $T^* = 7798.5$ $p^* = 31.2534,$ $q^* = 163.9387$ | $T^* = 1597.9$ $p^* = 30967,$ $q^* = 0.0146$ |
| 0.75 | $T^* = 6978.9$ $p^* = 34.6860,$ $q^* = 192.0087$ | $T^* = 8054.8$ $p^* = 35.1079,$ $q^* = 150.1615$ | $T^* = 1961.6$ $p^* = 16268,$ $q^* = 0.0296$ |

### 5.5 Results under Various Specifications on Uncertainty Distributions

We end this section showing the results obtained by assigning some specific values to the problem coefficients. Let $\beta = 1.2$, $\delta = 0.01$ and the coefficients have uncertainty distributions specified as follows:

- Case 1 (normal UDs). $\tilde{K} : N(19000, 1800)$, $\tilde{A} : N(75, 5)$, $\tilde{R} : N(5, 1)$, $\tilde{I} : N(0.4, 0.01)$.
- Case 2 (linear UDs). $\tilde{K} : L(15000, 18000)$, $\tilde{A} : L(55, 65)$, $\tilde{R} : L(5, 6)$, $\tilde{I} : L(0.2, 0.3)$.
- Case 3 (zigzag UDs). $\tilde{K} : Z(16000, 19000, 22000)$, $\tilde{A} : Z(65, 75, 85)$, $\tilde{R} : Z(6, 7, 8)$, $\tilde{I} : Z(0.3, 0.4, 0.5)$.

Table 7 shows the profit values for these three cases at $\alpha = 0.25, \alpha = 0.50$ and $\alpha = 0.75$. For each value of $\alpha$, the profit values differ substantially across the specifications assumed for the uncertainty distributions. While the profit values based on the normal and linear uncertainty distributions increase as $\alpha$ increases, with zigzag uncertainty distributions the profits fluctuate. This indicates that practicing managers in charge of introducing a new product to the market need to use the most appropriate uncertainty distributions for $K$, $A$, $R$ and $I$ in order to reflect the real situations and take reliable decisions. In other words, once practicing managers have collected the correct information on the distributions of the problem parameters, they should be able to accurately predict profits.

## 6 Conclusions

Geometric programming (GP) is a powerful optimization technique widely employed for solving a variety of nonlinear optimization problems and engineering problems.

The conventional GP models assume deterministic and precise parameters. However, the parameters in the real-life GP problems are often imprecise and ambiguous. We have approached the problem of formalizing and implementing imprecise and non-deterministic parameters through uncertainty theory.

There exists an ample literature on uncertain GP and its applications to problems (chance-constrained and not) whose coefficients are fuzzy numbers, fuzzy variables or random variables. However, to the best of our knowledge, there is no previous study dealing with the formulation and/or solution of GP problems where the coefficients are given by uncertain variables.

In this paper, we introduced an uncertain chance-constrained GP (UCCGP) model and proposed a solving method that applies to three of the most used uncertainty distribution cases: We assumed the coefficients to be uncertain variables with normal, linear or zigzag uncertainty distributions. We proved that the corresponding uncertain GP models can be transformed into conventional geometric programs with crisp coefficients and, hence, an optimal solution found by the duality algorithm. The capacity of our UCCGP model to provide an optimal objective value (not just an approximation of it) constitutes a clear advantage of the model over other UCCP and fuzzy CCP models in the literature.

We have shown the efficacy of the procedures and the algorithms through three numerical examples. In particular, we have discussed an inventory management model of economic order quantity (EQQ) under cost minimization and profit maximization and shown that our UCCGP method can be used by practicing managers to make accurate predictions in terms of profits and, more in general, to collect important information for planning purposes.

We believe that the framework proposed in this paper contributes to shed light on the applications of GP to concrete problems opening the way to further research in engineering and managerial problems.

**Acknowledgments** The authors would like to thank the anonymous reviewers and the editor for their insightful comments and suggestions.

## References

1. Chu, C., Wong, D.F.: VLSI circuit performance optimization by geometric programming. Ann. Oper. Res. **105**, 37–60 (2001)
2. Hershenson, M.D., Boyd, S.P., Lee, T.H.: Optimal design of a CMOS op-amp via geometric programming. IEEE Trans. Comput. Aid. Design. **20**, 1–21 (2001)
3. Avriel, M., Dembo, R., Passy, U.: Solution of generalized geometric programs. Int. J. Numer. Method Eng. **9**, 149–168 (1975)
4. Beightler, C.S., Philips, D.T.: Applied Geometric Programming. Wiley, New York (1976)
5. Choi, J.C., Bricker, D.L.: Effectiveness of a geometric programming algorithm for optimization of machining economics models. Comput. Oper. Res. **10**, 957–961 (1996)
6. Scott, C.H., Jefferson, T.R.: Allocation of resources in project management. Int. J. Syst. Sci. **26**, 413–420 (1995)
7. Cheng, T.C.E.: An economic order quantity model with demand-dependent unit production cost and imperfect production process. IIE Trans. **23**, 23–28 (1991)
8. Jung, H., Klein, C.M.: Optimal inventory policies under decreasing cost functions via geometric programming. Eur. J. Oper. Res. **132**, 628–642 (2001)
9. Kim, D., Lee, W.J.: Optimal joint pricing and lot sizing with fixed and variable capacity. Eur. J. Oper. Res. **109**, 212–227 (1998)
10. Lee, W.J.: Determining order quantity and selling price by geometric programming. Optimal solution, bounds, and sensitivity. Decis. Sci. **24**, 76–87 (1993)
11. Roy, T.K., Maiti, M.: A fuzzy EOQ model with demand-dependent unit cost under limited storage capacity. Eur. J. Oper. Res. **99**, 425–432 (1997)

12. Worrall, B.M., Hall, M.A.: The analysis of an inventory control model using posynomial geometric programming. Int. J. Prod. Res. **20**, 657–667 (1982)
13. Duffin, R.J., Peterson, E.L., Zener, C.: Geometric Programming Theory and Applications. Wiley, New York (1967)
14. Duffin, R.J., Peterson, E.L.: Geometric programming with signomials. J. Optim. Theory Appl. **11**, 3–35 (1973)
15. Fang, S.C., Peterson, E.L., Rajasekera, J.R.: Controlled dual perturbations for posynomial programs. Eur. J. Oper. Res. **35**, 111–117 (1988)
16. Kortanek, K.O., No, H.: A second order affine scaling algorithm for the geometric programming dual with logarithmic barrier. Optimization **23**, 303–322 (1992)
17. Kortanek, K.O., Xu, X., Ye, Y.: An infeasible interior-point algorithm for solving primal and dual geometric programs. Math. Program. **76**, 155–181 (1997)
18. Maranas, C.D., Floudas, C.A.: Global optimization in generalized geometric programming. Comput. Chem. Eng. **21**, 351–369 (1997)
19. Rajgopal, J.: An alternative approach to the refined duality theory of geometric programming. J. Math. Anal. Appl. **167**, 266–288 (1992)
20. Rajgopal, J., Bricker, D.L.: Posynomial geometric programming as a special case of semi-infinite linear programming. J. Optim. Theory Appl. **66**, 455–475 (1990)
21. Rajgopal, J., Bricker, D.L.: Solving posynomial geometric programming problems via generalized linear programming. Comput. Optim. Appl. **21**, 95–109 (2002)
22. Yang, H.H., Bricker, D.L.: Investigation of path-following algorithms for signomial geometric programming problems. Eur. J. Oper. Res. **103**, 230–241 (1997)
23. Zhu, J., Kortanek, K.O., Huang, S.: Controlled dual perturbations for central path trajectories in geometric programming. Eur. J. Oper. Res. **73**, 524–531 (1992)
24. Avriel, M., Wilde, D.J.: Engineering design under uncertainty. I&EC Process Des. Dev. **8**(1), 127–131 (1969)
25. Dupačová, J.: Stochastic geometric programming with an application. Kybernetika **46**(3), 374–386 (2010)
26. Liu, S.T.: A geometric programming approach to profit maximization. Appl. Math. Comput. **182**(2), 1093–1097 (2006)
27. Liu, S.T.: Posynomial geometric programming with parametric uncertainty. Eur. J. Oper. Res. **168**, 345–353 (2006b)
28. Liu, S.T.: Geometric programming with fuzzy parameters in engineering optimization. Int. J. Approx. Reason. **46**(3), 484–498 (2007)
29. Liu, S.T.: Posynomial geometric programming with interval exponents and coefficients. Eur. J. Oper. Res. **186**(1), 17–27 (2008)
30. Liu, S.T.: Fuzzy measures for profit maximization with fuzzy parameters. J. Comput. Appl. Math. **236**(6), 1333–1342 (2011)
31. Tsai, J.F., Li, H.L., Hu, N.Z.: Global optimization for signomial discrete programming problems in engineering design. Eng. Optim. **34**, 613–622 (2002)
32. Li, H.L., Tsai, J.F.: Treating free variables in generalized geometric global optimization programs. J. Glob. Optim. **33**, 1–13 (2005)
33. Tsai, J.F., Lin, M.H., Hu, Y.C.: On generalized geometric programming problems with non-positive variables. Eur. J. Oper. Res. **178**(1), 10–19 (2007)
34. Tsai, J.F.: Treating free variables in generalized geometric programming problems. Comput. Chem. Eng. **33**, 239–243 (2009)
35. Tsai, J.F., Lin, M.H.: An optimization approach for solving signomial discrete programming problems with free variables. Comput. Chem. Eng. **30**, 1256–1263 (2006)
36. Lin, M.H., Tsai, J.F.: Range reduction techniques for improving computational efficiency in global optimization of signomial geometric programming problems. Eur. J. Oper. Res. **216**, 17–25 (2012)
37. Liu, B.: Uncertainty Theory, 4th edn. Springer, Berlin (2015)
38. Liu, B.: Some research problems in uncertainty theory. J. Uncertain Syst. **3**(1), 3–10 (2009)
39. Peng, J., Yao, K.: A new option pricing model for stocks in uncertainty markets. Int. J. Oper. Res. **8**(2), 18–26 (2011)
40. Liu, B.: Uncertain risk analysis and uncertain reliability analysis. J. Uncertain Syst. **4**(3), 163–170 (2010)

41. Wang, X.S., Gao, Z.C., Guo, H.Y.: Delphi method for estimating uncertainty distributions. Information **15**(2), 449–460 (2012)
42. Wang, X.S., Gao, Z.C., Guo, H.Y.: Uncertain hypothesis testing for expert's empirical data. Math. Comput. Model. **55**(3–4), 1478–1482 (2012)
43. Liu, B.: Uncertain set theory and uncertain inference rule with application to uncertain control. J. Uncertain Syst. **4**(2), 83–98 (2010)
44. Zhu, Y.: Uncertain optimal control with application to a portfolio selection model. Cybern. Syst. **41**(7), 535–547 (2010)
45. Li, S., Peng, J., Zhang, B.: The uncertain premium principle based on the distortion function. Insur. Math. Econ. **53**, 317–324 (2013)
46. Han, S., Peng, Z., Wang, S.: The maximum flow problem of uncertain network. Inf. Sci. **265**, 167–175 (2014)
47. Ding, S.: The $\alpha$-maximum flow model with uncertain capacities. Appl. Math. Model. **39**(7), 2056–2063 (2015)
48. Peterson, E.L.: The fundamental relations between geometric programming duality, parametric programming duality, and ordinary Lagrangian duality. Ann. Oper. Res. **105**, 109–153 (2001)
49. Liu, B.: Extreme value theorems of uncertain process with application to insurance risk model. Soft Comput. **17**(4), 549–556 (2013)
50. Liu, S.T.: Profit maximization with quantity discount: an application of geometric program. Appl. Math. Comput. **190**(2), 1723–1729 (2007a)
51. Hamidi, Sadjadi S.J., Hesarsorkh, A., Mohammadi, M., Bonyadi Naeini, A.: Joint pricing and production management: a geometric programming approach with consideration of cubic production cost function. J. Ind. Eng. Int. **11**(2), 209–223 (2015)
52. Samadi, F., Mirzazadeh, A., Pedram, M.: Fuzzy pricing, marketing and service planning in a fuzzy inventory model: a geometric programming approach. Appl. Math. Model. **37**, 6683–6694 (2013)