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Chance-constrained DEA models with random fuzzy inputs and outputs

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ABSTRACT

Data Envelopment Analysis (DEA) is a widely used mathematical programming technique for comparing the inputs and outputs of a set of homogenous Decision Making Units (DMUs) by evaluating their relative efficiency. The conventional DEA methods assume deterministic and precise values for the input and output observations. However, the observed values of the input and output data in real-world problems can potentially be both random and fuzzy in nature. We introduce Random Fuzzy (Ra-Fu) variables in DEA where randomness and vagueness coexist in the same problem. In this paper, we propose three DEA models for measuring the radial efficiency of DMUs when the input and output data are Ra-Fu variables with Poisson, uniform and normal distributions. We then extend the formulation of the possibility–probability and the necessity–probability DEA models with Ra-Fu parameters for a production possibility set where the Ra-Fu inputs and outputs have normal distributions with fuzzy means and variances. We finally propose the general possibility–probability and necessity–probability DEA models with fuzzy thresholds. A set of numerical examples and a case study are presented to demonstrate the efficacy of the procedures and algorithms.

1. Introduction

Efficiency and productivity measurement in organizations has received a great deal of attention in recent years. Data Envelopment Analysis (DEA) by Charnes et al. [6] generalizes the Farrell [16] single-input/single-output technical efficiency measure to the multiple-input/multiple-output case in order to evaluate the relative efficiency of peer units with respect to multiple performance measures ([6,5,11,12]). The units under evaluation in DEA are called Decision Making Units (DMUs) and their performance measures are grouped into inputs and outputs. Using the optimization for each DMU, DEA yields an efficient frontier or tradeoff curve that represents the relations among the multiple performance measures. Unlike parametric methods which require an explicit production function, DEA is non-parametric and does not require an explicit functional form relating inputs and outputs. Charnes et al [6] introduced the following conventional radial input-oriented DEA model of a given $DMU_\circ$ for constant returns to scale (called CRS or CCR):

$$\begin{aligned} \max & \sum_{r=1}^{s} u_r y_{r\circ} \\ \text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{i\circ} = 1, \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leqslant 0, \quad j = 1, \ldots, n, \\ & u_r, \ v_i \geqslant 0, \quad r = 1, \ldots, s; \ i = 1, \ldots, m. \end{aligned} \tag{1}$$

where $y_{rj} (r = 1,,2,\ldots,s)$ and x_{ij} $(i = 1,,2,\ldots,m)$ are the s outputs and m inputs for the jth DMU, $j = 1,,2,\ldots,n$. In addition, the u_r and v_i are the weights associated with the rth output and the ith input, respectively. The $DMU_\circ$ is [technically] efficient if the objective function of (1) equals unity, otherwise, is [technically] inefficient.

The conventional DEA evaluation process is based on well-defined, precise and deterministic data for the production set. However, the input and output data in real-world problems are often fuzzy and random. Some researchers have proposed various methods for dealing with imprecise and ambiguous data in DEA. Cooper et al. [11,12] has studied this problem in the context of interval data. However, many real-life problems use linguistic data such as good, fair or poor that cannot be mapped to interval data. Fuzzy logic and fuzzy sets can represent ambiguous, uncertain or imprecise information by formalizing inaccuracy in the decision making process [8]. Fuzzy set algebra developed by Zadeh [54] is

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the formal body of theory that allows the treatment of imprecise estimates in uncertain environments.

Various fuzzy methods have been proposed to solve DEA problems with Random Fuzzy (Ra-Fu) variables. In a recent study, Hatami-Marbini et al. [20] classified the fuzzy DEA methods in the literature into five general groups: (1) the tolerance approach (e.g. [50]), (2) the α-level based approach (e.g. [25,21,7,45]), (3) the fuzzy ranking approach (e.g. [18,35,22,53]), (4) the possibility approach (e.g. [19,36,26]), and (5) other developments (e.g. [24,43,52,57]).

Each of the above mentioned approaches has both advantages and disadvantages in the way it treats uncertain data in DEA models. For example, the tolerance approach fuzzifies the inequality or equality signs but it does not treat fuzzy coefficients directly. However, the input and output data are often imprecise. The α-level based approach provides fuzzy efficiency metrics, but requires the ranking of the fuzzy efficiency sets. The fuzzy ranking approach produces fuzzy efficiencies for the DMUs. The possibility approach, finally, requires generally relatively complicated numerical computations compared to the other approaches.

The fundamental advantage of DEA is that it does not require prior weights or explicit specification of functional relationships among the multiple outputs and inputs. However, when evaluating the efficiencies of DMUs, the conventional DEA methods do not allow stochastic variations in the data. Addressing this limitation, stochastic programming has been developed for decision problems where the input data are assumed to be random variables with known probability distributions [27,44,51]. Zadeh [56] introduced the so called possibility theory in the context of fuzzy set theory as a mathematical framework for modeling situations involving uncertainty. He introduced the notion of a "fuzzy variable," which is associated with a possibility distribution, similar to a random variable, which is associated with a probability distribution. In a fuzzy linear programming model, each fuzzy coefficient can be characterized as a fuzzy variable and each constraint can be viewed as a fuzzy event. Lai and Hwang [33] have given a systematic classification of all possible problems and existing fuzzy mathematical programming approaches. They also made the distinction between fuzzy linear programming problems and possibilistic linear programming problems.

In a random fuzzy environment, the crisp inputs and outputs of a conventional DEA model become random fuzzy (Ra-Fu) variables. Building a DEA model directly with Ra-Fu variables is not useful because the meanings of the objective and the constraints are not clear. This problem is apparent in both a stochastic environment and a fuzzy environment, whereby the Decision Makers (DMs) are faced with uncertainty from the random data and imprecision from the fuzzy data.

The α-level approach is the most popular fuzzy DEA model [20]. In this approach the main idea is to convert the fuzzy DEA model into a pair of parametric programs in order to find the lower and upper bounds of the α-level of the membership functions of the efficiency scores. Using an α-cut method proposed by Sakawa [47], Lertworasirikul [36] proposed the possibility and necessity methods for solving a fuzzy DEA-CCR model. They introduced a possibility approach in which constraints were treated as fuzzy events and transformed fuzzy DEA models into possibility DEA models by using possibility measures of the fuzzy events (fuzzy constraints). It is known that possibility theory is based upon two dual fuzzy measures - possibility and necessity measures [15,28,56].

Credibility theory, founded by Liu [37–39], is a branch of mathematics for studying the behavior of fuzzy phenomena. Fuzzy DEA models with credibility constraints are complex and difficult to solve because the proposed models contain the credibility of fuzzy events in the constraints and the expected value of a fuzzy variable in the objective.

One way to manipulate uncertain data in DEA is via probability distributions. The ground-breaking work by Sengupta [48,49] showed how stochastic variables could be included in the non-parametric framework. The seminal work by Land et al. [34], Olesen and Petersen [42] and Cooper et al. [9] lead to breakthroughs in developing stochastic non-parametric efficiency models. In stochastic DEA, the frontier is no longer a deterministic envelope englobing the production set, but a chance-constrained envelopment based on an *a priori* probability of production space feasibility. Critics of this model argue that certain technical parameters are arbitrary and that the model lacks statistical properties, in spite of its goal to incorporate stochastic variables. Moreover, probability distributions in general require either *a priori* predictable regularity or *a posteriori* frequency determinations, which are difficult to construct [14]. Classical probability theory is a popular tool for dealing with randomness and credibility theory is an appropriate tool for treating fuzziness. However, in many complex real-world problems, the analyst may encounter a hybrid uncertain environment where impreciseness occurs jointly with stochastic influences. A random-fuzzy (Ra-Fu) variable is an effective concept for representing phenomena in which fuzziness and randomness appear simultaneously. Ra-Fu variables were first introduced by Kwakernaak [31,32], and then studied by a number of researchers in the literature [46,30,41,38,39,17]. The *mean chance* of a Ra-Fu event is an important concept in Ra-Fu optimization, just like the *probability* of a stochastic event in stochastic optimization and the *credibility* of a fuzzy event in fuzzy optimization. The main contribution of this paper is twofold:

(1) To generalize and consolidate the disparate DEA models by simultaneously addressing Ra-Fu reference sets and imprecisely defined production sets. Naturally, this generality comes at the expense of notational and computational complexity.
(2) To systematically transform the quadratic programming formulations into implementable problems for actual problem solutions. To this end, we illustrate the feasibility and richness of the obtained solutions by means of a series of numerical examples and a real-world case study.

The remainder of the paper is organized as follows. In the next section, we present some preliminaries and definitions for Ra-Fu variables. In Section 3 we present the Ra-Fu CCR model with an expected value operator. In Section 4 we propose the chance-constrained Ra-Fu CCR model and in Section 5 we present the chance-constrained Ra-Fu CCR model with a fuzzy threshold level. In Section 6 we present the results of some numerical examples to demonstrate the efficacy of the procedures and algorithms. In Section 7 we present a case study to demonstrate the applicability of the proposed models. Section 8 presents our conclusion and discussion.

2. Background

In this section we present some preliminaries and definitions for Ra-Fu variables. Several researchers have considered fuzziness and randomness together to deal with uncertainty. Hence, many concepts such as the probability of a fuzzy event [55], linguistic probabilities [14], Ra-Fu variables [31,32,38,39], probabilistic sets [23] and uncertain probabilities [2,3,1] have been introduced in the literature. A Ra-Fu variable is a mapping on the universal set of random variables used to describe a Ra-Fu phenomenon. Let us consider the following definitions.

Definition 1 [14]. A fuzzy interval of LR-type is denoted by $\tilde{A} = (\alpha, m_1, m_2, \beta)_{LR}$ where α and β are the (non-negative) left and right spreads, respectively, and m_1 and m_2 are the mean values of $\tilde{A}$. The membership function of $\tilde{A}$ can be expressed as

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\frac{m_1-x}{\alpha}\right), & x \leqslant m_1, \\ 1 & m_1 \leqslant x \leqslant m_2 \\ R\left(\frac{x-m_2}{\beta}\right), & x \geqslant m_2. \end{cases} \quad (2)$$

where L and R are the left and right functions, respectively. For example, suppose that

$$L(x) = R(x) = \begin{cases} 1-x, & 0 \leqslant x \leqslant 1 \\ 0, & \text{otherwise} \end{cases}$$

$\tilde{A} = (\alpha, m_1, m_2, \beta)_{LR} = (\alpha, m_1, m_2, \beta)$ is called a trapezoidal fuzzy number. Also, if $m_1 = m_2 = m$, $\tilde{A} = (\alpha, m, \beta)_{LR} = (\alpha, m, \beta)$ is called a triangular fuzzy number. Alternatively, the trapezoidal and triangular fuzzy numbers can be indicated as $(m_1 - \alpha, m_1, m_2, m_2 + \beta)$ and $(m - \alpha, m, m + \beta)$, respectively.

Definition 2 (*Fuzzy Arithmetic [14]*). Let $\tilde{A} = (\alpha, m_1, m_2, \beta)_{LR}$ and $\tilde{B} = (\bar{\alpha}, \bar{m}_1, \bar{m}_2, \bar{\beta})_{LR}$ be two positive fuzzy numbers. Then, the fuzzy arithmetic of $\tilde{A}$ and $\tilde{B}$ can be defined as follows:

Addition:
$(\alpha, m_1, m_2, \beta)_{LR} + (\bar{\alpha}, \bar{m}_1, \bar{m}_2, \bar{\beta})_{LR} = (\alpha + \bar{\alpha}, m_1 + \bar{m}_1, m_2 + \bar{m}_2, \beta + \bar{\beta})_{LR}$.

Subtraction: $(\alpha, m_1, m_2, \beta)_{LR} - (\bar{\alpha}, \bar{m}_1, \bar{m}_2, \bar{\beta})_{LR} = (\alpha + \bar{\beta}, m_1 - \bar{m}_2, m_2 - \bar{m}_1, \beta + \bar{\alpha})_{LR}$.

Multiplication (approximation):

$\tilde{A} \otimes \tilde{B} = (\alpha, m_1, m_2, \beta)_{LR} \otimes (\bar{\alpha}, \bar{m}_1, \bar{m}_2, \quad \bar{\beta})_{LR} \simeq (m_1\bar{\alpha} + \bar{m}_1\alpha - \alpha\bar{\alpha}, m_1\bar{m}_1, m_2\bar{m}_2, m_2\bar{\beta} + \bar{m}_2\beta + \beta\bar{\beta})_{LR}$ and if k is the non-zero real number, then

$$k(\alpha, m_1, m_2, \beta)_{LR} = \begin{cases} (k\alpha, km_1, km_2, k\beta)_{LR}, & \text{if } k > 0 \\ (-k\beta, km_2, km_1, -k\alpha)_{LR}, & \text{if } k < 0 \end{cases}$$

Remark 1. Let $\tilde{\lambda} = (\alpha, \lambda_2, \lambda_3, \beta)_{LR} = (\lambda_2 - \alpha, \lambda_2, \lambda_3, \lambda_3 + \beta)_{LR}(\lambda_2 - \alpha, \lambda_2, \lambda_3 > 0)$ be a fuzzy variable, then we approximately have:

a. $(\tilde{\lambda})^k \simeq ((\lambda_2 - \alpha)^k, \lambda_2^k, \lambda_3^k, (\lambda_3 + \beta)^k)$
b. $e^{\tilde{\lambda}} \simeq (e^{(\lambda_2 - \alpha)}, e^{\lambda_2}, e^{\lambda_3}, e^{(\lambda_3 + \beta)})$

where k is any positive integer number (see Appendix A for proof).

Definition 3 ([29]) The α-cut, $\alpha \in [0,1]$, of a LR type fuzzy number $\tilde{A} = (\bar{\alpha}, a_2, a_3, \bar{\beta})_{LR}$ is a closed interval as follows:

$$A_\alpha = \{x | \mu_A(x) \geqslant \alpha\} = \left[A_\alpha^L, A_\alpha^R\right] = [a_2 - \bar{\alpha}L^{-1}(\alpha), a_3 + \bar{\beta}R^{-1}(\alpha)],$$

where A_α^L and A_α^R are the left and right extreme points, respectively.

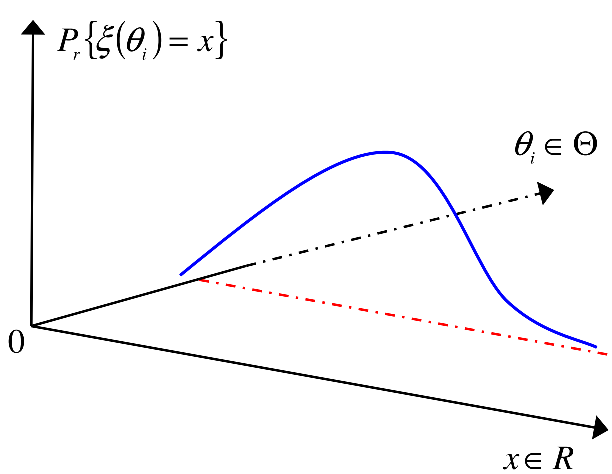

Fig. 1. Random fuzzy numbers.

Definition 4 (*[56,58]*). Let $(\Theta, P(\Theta), Pos)$ be a possibility space where Θ is a non-empty set involving all possible events, and $P(\Theta)$ is the power set of Θ. For each $A \subseteq P(\Theta)$, there is a non-negative number $Pos(A)$, the so-called *possibility measure*, with the following properties

(i) $Pos\{\emptyset\} = 1$, $Pos\{\Theta\} = 1$;
(ii) $A \subseteq B$ implies $Pos(A) \leqslant Pos(B)$ for any $A, B \in P(\Theta)$; and
(iii) $Pos\{\bigcup_k A_k\} = Sup_k Pos\{A_k\}$.

Definition 5 ([58,13,15]) The *necessity measure* of A, denoted by $Nec(A)$, is defined on $(\Theta, P(\Theta), Pos)$ as $Nec\{A\} = 1 - Pos\{A^c\}$ where A^c is the complement set of A. For any sets A and B, the properties of the necessity measure are presented as follows:

a. $Nec\{\emptyset\} = 0$, $Nec\{\Theta\} = 1$;
b. $Pos(A) \geqslant Nec(A)$[1];
c. $A \subseteq B$ implies $Nec(A) \leqslant Nec(B)$;
d. $Pos(A) < 1 \Rightarrow Nec(A) = 0$; and
e. $Nec(A) > 0 \Rightarrow Pos(A) = 1$.

Definition 6 ([40]) Let $(\Theta, P(\Theta), Pos)$ be a possibility space. The *credibility measure* of a fuzzy event A, $Cr(A)$, is defined as $Cr(A) = 0.5(Pos\{A\} + Nec\{A\})$ with the following properties:

a. $Cr\{\emptyset\} = 0$, $Cr\{\Theta\} = 1$;
b. Monotonicity: $A \subset B$ implies $Cr\{A\} \leqslant Cr\{B\}$ for any $A, B \in P(\Theta)$;
c. Self-duality: $Cr\{A\} + Cr\{A^c\} = 1$, for any $A \in P(\Theta)$;
d. $Cr\{\cup_i A_i\} = Sup_i Cr\{A_i\}$ for any subset $\{A_i\}$ in $P(\Theta)$ with $Sup_i Cr\{A_i\} < 0.5$;
e. Subadditivity: $Cr\{A \cup B\} \leqslant Cr\{A\} + Cr\{B\}$ for any A, $B \in P(\Theta)$; and
f. $Pos\{A\} \geqslant Cr\{A\} \geqslant Nec\{A\}$.

Definition 7 ([40]) Let ξ be a fuzzy variable on the a possibility space $(\Theta, P(\Theta), Pos)$. The possibility, necessity and credibility of a fuzzy event $\{\xi \geqslant r\}$ are represented by:

$$Pos\{\xi \geqslant r\} = \underset{t \geqslant r}{Sup}\, \mu_\xi(t),$$
$$Nes\{\xi \geqslant r\} = 1 - \underset{t < r}{Sup}\, \mu_\xi(t),$$
$$Cr(\xi \geqslant r) = 0.5[Pos\{\xi \geqslant r\} + Nec\{\xi \geqslant r\}].$$

where $\mu_\xi : \Re \to [0,1]$ is the membership function of ξ and r is a real number. Note here that $Cr(\xi \geqslant r) = 1 - Cr(\xi < r)$.

Definition 8 ([41]) A Ra-Fu variable is a function from a possibility space $(\Theta, P(\Theta), Pos)$ to a collection of random variables.

Definition 9 ([38,39]) An n-dimensional Ra-Fu vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ is a function from the possibility space to the set of n-dimensional random vectors $\xi_1, \xi_2, \ldots, \xi_n$.

Definition 10 ([37]) Let $f : \Re^n \to \Re$ be a measurable function, and ξ_i Ra-Fu variables on the possibility spaces $(\Theta_i, P(\Theta_i), Pos_i)$,

[1] Suppose that $B = A^c$. Then $Pos(A \cup A^c) + Pos(A \cap A^c) \leqslant Pos(A^c) + -(A \cup A^c) + Pos(A \cap A^c) \leqslant Pos(A^c) + Pos(A) \Rightarrow 1 \leqslant Pos(A^c) + Pos(A)$. In addition, $Nec\{A\} + Pos\{A^c\} = 1$. These imply that $Nec\{A\} \leqslant Pos(A)$.

$i = 1,\ldots,n$. Then $\xi = f(\xi_1,\xi_2,\ldots,\xi_n)$ is a Ra-Fu variable on the product possibility space $(\Theta,P(\Theta),Pos)$, defined as $\xi(\theta_1,\theta_2,\ldots,\theta_n) = f(\xi_1(\theta_1), \xi_2(\theta_2),\ldots,\xi_n(\theta_n))$ for all $(\theta_1,\theta_2,\ldots,\theta_n) \in \Theta$.

Theorem 1 ([38]) *Let ξ be an n-dimensional Ra-Fu vector, and $f : \Re^n \to \Re$ a measurable function. Then $f(\xi)$ is a Ra-Fu variable.*

Theorem 2 ([38]) *Assume that ξ is a Ra-Fu vector, and g_j are real-valued continuous functions for $j = 1,2,\ldots,n$. We have*

a. *The possibility $Pr\{g_j(\xi(\omega)) \leqslant 0, j = 1,\ldots,p\}$ is a fuzzy variable; and*
b. *The necessity $Nec\{g_j(\xi(\omega)) \leqslant 0, j = 1,\ldots,p\}$ is a fuzzy variable.*

Definition 11 ([1]) A Ra-Fu variable can be a random variable with a fuzzy parameter. Additional information on this definition can be found in Fig. 1. This figure can be considered as an example where $\xi \sim N(\tilde{\mu},\sigma^2)$ is a normally-distributed random variable with fuzzy mean $\tilde{\mu}$.

Definition 12 (*Discrete Ra-Fu variable [2]*). A discrete Ra-Fu variable ξ is defined on the possibility space $(\Theta,P(\Theta),Pos)$ when $\xi(\theta)$ is a discrete random variable for all $\theta \in \Theta$. For instance, let ξ be a discrete Ra-Fu variable with a Poisson distribution in which its following probability density function (PDF) involves the fuzzy parameter $\tilde{\lambda}$:

$$P\{\xi = k\} = \frac{\tilde{\lambda}^k e^{-\tilde{\lambda}}}{k!}, \quad k = 1,2,\ldots$$

Definition 13 (*Continuous Ra-Fu variable [3]*). A continuous Ra-Fu variable ξ is defined on the possibility space $(\Theta,P(\Theta),Pos)$ when $\xi(\theta)$ is a continuous random variable for each $\theta \in \Theta$. For instance, let ξ be a continuous Ra-Fu variable with a uniform distribution in which its following PDF involves the two fuzzy parameters $\tilde{a}$ and $\tilde{b}$:

$$P(x) = \begin{cases} \frac{1}{\tilde{b}-\tilde{a}}, & \tilde{a} \leqslant x \leqslant \tilde{b}, \\ 0, & \text{o.w.} \end{cases} \tag{3}$$

Also assume that ξ is a continuous Ra-Fu variable with normal distribution with either fuzzy mean or fuzzy variance, or both.

Definition 14 ([41]) If ξ is a Ra-Fu variable on the possibility space $(\Theta,P(\Theta),Pos)$, then the expected value of ξ is expressed as:

$$\mathrm{E}(\xi) = \int_0^{+\infty} Cr\{\theta \in \Theta|\ E[\xi(\theta)] \geqslant r\}dr - \int_{-\infty}^0 Cr\{\theta \in \Theta|E[\xi(\theta)] \leqslant r\}dr$$

Notice that at least one of the above two integrals is finite. If ξ is a discrete Ra-Fu variable on the possibility space $(\Theta,P(\Theta),Pos)$, then the expected value of ξ is:

$$E[\xi] = \sum_{i=1}^{\infty} p_i E[\tilde{u}_i(\theta)]$$

where $\tilde{u}_i(\theta)$ are fuzzy variables on $(\Theta_i,P(\Theta_i),Pos_i)$ and p_i is the probability of $\tilde{u}_i(\theta)$.

If ξ is a continuous Ra-Fu variable on the possibility space $(\Theta,P(\Theta),Pos)$, then the expected value of ξ is:

$$\mathrm{E}(\xi) = \int_0^{+\infty} Cr\left\{\int_{x\in\Theta} xf(x)dx \geqslant r\right\}dr - \int_{-\infty}^0 Cr\left\{\int_{x\in\Theta} xf(x)dx \leqslant r\right\}dr$$

where $f(x)$ is a density function with fuzzy parameters.

Definition 15 ([1]) Let $f(x,\tilde{\theta})$ be the PDF of X where $\tilde{\theta}$ is a fuzzy parameter. The probability of the event "$a \leqslant X \leqslant b$" is characterized by the fuzzy number whose α-cut is:

$$[\tilde{P}(a \leqslant X \leqslant b)]^{\alpha-Cut} = \left\{\int_a^b f(x,\theta)dx|\theta \in \tilde{\theta}^\alpha; \int_{-\infty}^{+\infty} f(x,\theta)dx = 1\right\}.$$

Accordingly, the α-cut of the fuzzy mean, denoted by $\tilde{\mu}_X^{\alpha-Cut}(\theta)$, and the fuzzy variance, denoted by $\tilde{\sigma}_X^{2\alpha-Cut}(\theta)$, are defined as:

$$\tilde{\mu}_X^{\alpha-Cut}(\theta) = \left\{\int_{-\infty}^{+\infty} xf(x,\theta)dx|\theta \in \tilde{\theta}^\alpha; \mu_X(\theta) \in \tilde{\mu}_X^\alpha(\theta); \int_{-\infty}^{+\infty} f(x,\theta)dx = 1\right\}$$

$$\tilde{\sigma}_X^{2\alpha-Cut}(\theta) = \left\{\int_{-\infty}^{+\infty} (x-\mu_X(\theta))^2 f(x,\theta)dx|\theta \in \tilde{\theta}^\alpha; \mu_X(\theta) \in \tilde{\mu}_X^\alpha(\theta); \sigma_X^2(\theta) \in \tilde{\sigma}_X^{2\alpha}(\theta); \int_{-\infty}^{+\infty} f(x,\theta)dx = 1\right\}$$

Theorem 3 ([47]) *Let $\bar{\lambda}_1$ and $\bar{\lambda}_2$ be two independent fuzzy numbers with continuous membership functions. For a given confidence level $\alpha \in [0,1]$,*

$$Pos\{\bar{\lambda}_1 \geqslant \bar{\lambda}_2\} \geqslant \alpha \text{ if and only if } \lambda_{1,\alpha}^R \geqslant \lambda_{2,\alpha}^L,$$
$$Nec\{\bar{\lambda}_1 \geqslant \bar{\lambda}_2\} \geqslant \alpha \text{ if and only if } \lambda_{1,1-\alpha}^L \geqslant \lambda_{2,\alpha}^R.$$

where $\lambda_{1,\alpha}^L$, $\lambda_{1,\alpha}^R$ and $\lambda_{2,\alpha}^L$, $\lambda_{2,\alpha}^R$ are the left and the right side extreme points of the α-level sets $\left[\lambda_{1,\alpha}^L,\lambda_{1,\alpha}^R\right]$ and $\left[\lambda_{2,\alpha}^L,\lambda_{2,\alpha}^R\right]$ of $\bar{\lambda}_1$ and $\bar{\lambda}_2$, respectively, and $Pos\{\bar{\lambda}_1 \geqslant \bar{\lambda}_2\}$ and $Nec\{\bar{\lambda}_1 \geqslant \bar{\lambda}_2\}$ mean the degree of possibility and necessity that $\bar{\lambda}_1$ is greater than or equal to $\bar{\lambda}_2$.

3. Ra-Fu CCR model with expected value operator

The observed values of data in real-world situations are often inexact, incomplete, vague, ambiguous, or imprecise. In order to deal with these real-world situations, we first introduce the discrete and continuous Ra-Fu expected value approach based on the definitions presented in the previous section. We then present a general Ra-Fu expected value model followed by a deterministic Ra-Fu CCR model with an expected value operator. We should note that although the proposed method is based on the CCR model, the same method could simply be used for the BCC version of the CCR model.

3.1. Discrete and continuous Ra-Fu expected value approach

In this section, we introduce an expected value operator for Ra-Fu variables and derive its equivalent crisp model. DMs often encounter Ra-Fu variables (discrete or continuous) in decision problems where the chance constraints cannot be converted into deterministic ones such as normal distributions. In these cases, the expected value operator is used to convert the chance constraints into deterministic constraints. Initially, we study models with Ra-Fu input and output variables with Poisson, uniform, and normal probability distributions. The expected value operator is used to convert these models into deterministic models.

Let us consider the following definitions for the Ra-Fu variables. The Ra-Fu variables can be classified into two distinct classes: *discrete* and *continuous* variables. There is a special class of Ra-Fu variables that are functions from the possibility space $(\Theta,P(\Theta),Pos)$ to the collection of discrete or continuous random variables.

Theorem 4. *Let ξ be a Ra-Fu variable with $E[\xi(\theta)] = \tilde{\lambda}$ and $\tilde{\lambda} = (\alpha,\lambda_2,\lambda_3,\beta)$ be a LR fuzzy variable where λ_2 and λ_3 are the mean values and α and β are the (non-negative) left and right spreads. Then the expected value of Ra-Fu variable is as follows:*

$$E[\xi] = \frac{1}{2}[(\lambda_3 + \lambda_2) + \alpha(T(0) - T(1)) + \beta(\mathrm{P}(1) - \mathrm{P}(0))]$$

where T(x) and P(x) are continuous functions on [0,1], and $\frac{\partial \mathrm{T}(x)}{\partial x} = L(x)$, *and* $\frac{\partial \mathrm{P}(x)}{\partial x} = R(x)$.

Proof. In terms of the definition of the credibility measure we have the following relationship:

$$Cr(\tilde{\lambda} \geqslant r) = \begin{cases} 1, & r \leqslant \lambda_2 - \alpha \\ 1 - \frac{1}{2}L\left(\frac{\lambda_2 - r}{\alpha}\right), & \lambda_2 - \alpha \leqslant r \leqslant \lambda_2 \\ \frac{1}{2}, & \lambda_2 \leqslant r \leqslant \lambda_3 \\ \frac{1}{2}R\left(\frac{r-\lambda_3}{\beta}\right), & \lambda_3 \leqslant r \leqslant \lambda_3 + \beta \\ 0, & r > \lambda_3 + \beta \end{cases} \quad (4)$$

$$Cr(\tilde{\lambda} \leqslant r) = \begin{cases} 0, & r \leqslant \lambda_2 - \alpha \\ \frac{1}{2}L\left(\frac{\lambda_2 - r}{\alpha}\right), & \lambda_2 - \alpha \leqslant r \leqslant \lambda_2 \\ \frac{1}{2}, & \lambda_2 \leqslant r \leqslant \lambda_3 \\ 1 - \frac{1}{2}R\left(\frac{r-\lambda_3}{\beta}\right), & \lambda_3 \leqslant r \leqslant \lambda_3 + \beta \\ 1, & r > \lambda_3 + \beta \end{cases} \quad (5)$$

Obviously, it follows from (5) that $\int_{-\infty}^{0} Cr\{\tilde{\lambda} \leqslant r\}dr = 0$ and accordingly

$$E(\tilde{\lambda}) = \int_0^{+\infty} Cr\{\tilde{\lambda} \geqslant r\}dr - \int_{-\infty}^{0} Cr\{\tilde{\lambda} \leqslant r\}dr = \int_0^{\lambda_2-\alpha} dr + \int_{\lambda_2-\alpha}^{\lambda_2}\left(1 - \frac{1}{2}L\left(\frac{\lambda_2 - r}{\alpha}\right)\right)dr + \int_{\lambda_2}^{\lambda_3}\frac{dr}{2} + \int_{\lambda_3}^{\lambda_3+\beta}\frac{1}{2}R\left(\frac{r-\lambda_3}{\beta}\right)dr = \frac{\lambda_3+\lambda_2}{2} + \frac{\alpha}{2}(T(0) - T(1)) + \frac{\beta}{2}(P(1) - P(0))$$

In particular, let $\tilde{A} = (\alpha, \lambda_2, \lambda_3, \beta)$ and $\tilde{B} = (\alpha, \lambda, \beta)$ be trapezoidal and triangular fuzzy variables, respectively, with linear membership functions. It follows that:

$E[\tilde{\lambda}] = \frac{1}{4}(a + 2b + c)$. If $\tilde{\lambda} = (a, b, c, d)$ is a trapezoidal fuzzy variable, then $E[\xi] = \frac{1}{4}(a + b + c + d)$. □

Lemma 1. *Let ξ(λ) be a discrete Ra-Fu variable with a Poisson distribution (i.e., its PDF is* $\xi(\lambda) = \frac{(\tilde{\lambda})^k}{k!}e^{-\tilde{\lambda}}$ $(k = 0, 1, 2, \ldots, \infty)$) *in which* $\tilde{\lambda} = (\alpha, \lambda_2, \lambda_3, \beta)$ *is a LR trapezoidal fuzzy variable. Then the expected value of ξ(λ) is expressed as follows:*

$$E[\xi] = \frac{1}{2}[(\lambda_3 + \lambda_2) + \alpha(T(0) - T(1)) + \beta(P(1) - p(0))]$$

where T(x) and P(x) are continuous functions on [0,1], and $\frac{\partial \mathrm{T}(x)}{\partial x} = L(x)$, *and* $\frac{\partial \mathrm{P}(x)}{\partial x} = R(x)$.

Proof. Based on Definition (14) and Remark 1, we have:

$$E[\xi] = \sum_{k=0}^{+\infty} kE[p_k] = E\left[\sum_{k=0}^{+\infty} k\frac{\tilde{\lambda}^k}{k!}e^{-\tilde{\lambda}}\right] = E\left[\tilde{\lambda}e^{-\tilde{\lambda}}\sum_{k=1}^{+\infty}\frac{\tilde{\lambda}^{k-1}}{(k-1)!}\right]$$
$$= E\left[(\lambda_2 - \alpha, \lambda_2, \lambda_3, \lambda_3 + \beta) \cdot e^{-\tilde{\lambda}} \cdot \sum_{k=1}^{+\infty}\frac{(\lambda_2 - \alpha, \lambda_2, \lambda_3, \lambda_3 + \beta)^{k-1}}{(k-1)!}\right]$$
$$= E\left[(\lambda_2 - \alpha, \lambda_2, \lambda_3, \lambda_3 + \beta) \cdot e^{-\tilde{\lambda}} \cdot \left(\sum_{k=1}^{+\infty}\frac{(\lambda_2 - \alpha)^{k-1}}{(k-1)!}, \sum_{k=0}^{+\infty}\frac{(\lambda_2)^{k-1}}{(k-1)!}, \sum_{k=0}^{+\infty}\frac{(\lambda_3)^{k-1}}{(k-1)!}, \sum_{k=0}^{+\infty}\frac{(\lambda_2 + \beta)^{k-1}}{(k-1)!}\right)\right]$$
$$= E[(\lambda_2 - \alpha, \lambda_2, \lambda_3, \lambda_3 + \beta) \cdot e^{-\tilde{\lambda}} \cdot (e^{(\lambda_2 - \alpha)}, e^{\lambda_2}, e^{\lambda_3}, e^{(\lambda_3 + \beta)})]$$
$$= E[(\lambda_2 - \alpha, \lambda_2, \lambda_3, \lambda_3 + \beta) \cdot e^{-\tilde{\lambda}} \cdot e^{\tilde{\lambda}}]$$
$$= E[(\lambda_2 - \alpha, \lambda_2, \lambda_3, \lambda_3 + \beta)] = E[\tilde{\lambda}]$$

According to Theorem 4, the expected value of $\tilde{\lambda}$ is:

$$E[\xi] = E[\tilde{\lambda}] = \frac{1}{2}[(\lambda_3 + \lambda_2) + \alpha(T(0) - T(1)) + \beta(P(1) - p(0))] \quad \square$$

Lemma 2. *Let ξ be a continuous Ra-Fu variable with uniform distribution, denoted by* $\xi \sim U(\tilde{a}, \tilde{b})$, *where* $\tilde{a}$ *and* $\tilde{b}$ *are both LR trapezoidal fuzzy numbers on the possibility space (Θ,P(Θ),Pos). Then the expected value of the Ra-Fa variable ξ is as follows:*

$$E[\xi] = \frac{1}{4}[(a_2 + a_3 + b_2 + b_3) + (\alpha_1 + \alpha_2)(T(0) - T(1)) + (\beta_1 + \beta_2) \times (P(1) - P(0))]$$

Proof. It is obvious that:

$$E[\xi] = E[E[\theta]] = E\left[\frac{\tilde{a} + \tilde{b}}{2}\right]$$

On the other hand, $\frac{\tilde{a}+\tilde{b}}{2}$ is a fuzzy variable and using fuzzy arithmetic it can be represented as follows:

$$\frac{1}{2}(\tilde{a} + \tilde{b}) = \left(\frac{\alpha_1 + \alpha_2}{2}, \frac{a_2 + b_2}{2}, \frac{a_3 + b_3}{2}, \frac{\beta_1 + \beta_2}{2}\right)$$

Then, similar to Theorem 4, we have:

$$E[\xi] = \frac{1}{4}[(a_2 + a_3 + b_2 + b_3) + (\alpha_1 + \alpha_2)(T(0) - T(1)) + (\beta_1 + \beta_2) \times (P(1) - P(0))] \quad \square$$

Lemma 3. *Assume that ξ is the Ra-Fu variable with a normal distribution, denoted by* $\xi \sim N(\tilde{\mu}, \sigma^2)$ *where* $\tilde{\mu} = (\alpha, \lambda_2, \lambda_3, \beta)$ *is a LR fuzzy variable. Then the expected value of ξ is obtained as follows:*

$$E[\xi] = \frac{1}{2}[(\lambda_3 + \lambda_2) + \alpha(T(0) - T(1)) + \beta(P(1) - P(0))].$$

Proof. Follows directly from the application of Theorem 4 to a normal distribution. □

3.2. General Ra-Fu expected value model

Uncertainty, vagueness, or imprecision arises when the outcome of an experiment cannot be determined properly. The Ra-Fu variable developed by Kwakernaak [31,32] is one way to handle the uncertainties associated with a random variable whose value is a fuzzy number. The uncertain mathematical model consisting of the Ra-Fu variables can be converted into a deterministic model using the expected value operator of Ra-Fu variables. The general Ra-Fu mathematical model can be formulated as follows:

$$\max \ f(x, \xi) \quad \text{s.t.} \begin{cases} g_j(x, \xi) \leqslant 0, & j = 1, \ldots, n \\ x \in X \end{cases} \quad (6)$$

where $f(x, \xi)$ and $g_j(x, \xi)$, $j = 1, \ldots, n$ present continuous functions in X and $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ is a Ra-Fu vector on the possibility space (Θ,P(Θ),Pos). In order to obtain the crisp form of (6), the corresponding Ra-Fu expected value can be calculated from:

$$\max \ E[f(x, \xi)] \quad \text{s.t.} \begin{cases} E[g_j(x, \xi)] \leqslant 0, & j = 1, \ldots, n \\ x \in X \end{cases} \quad (7)$$

Using the expected value operator, Model (6) has been converted into a certain programming model in order to obtain a numerical solution.

3.3. Deterministic Ra-Fu CCR model with expected value operator

In this sub-section, we develop a DEA-based method for evaluating the efficiencies of DMUs where the inputs and outputs are characterized by the Ra-Fu variables with three different distributions: Poisson, normal and uniform.

Assume that there are n DMUs ($j = 1,\ldots,n$) to be assessed, each using amounts $\tilde{\bar{x}}_{ij}$ of m Ra-Fu inputs ($i = 1,\ldots,m$) to produce amounts $\tilde{\bar{y}}_{rj}$ of s Ra-Fu outputs ($r = 1,\ldots,s$). The following Ra-Fu CCR model results from consideration of the Ra-Fu inputs and outputs for $DMU_\circ$:

$$\begin{aligned} \max \quad & \sum_{r=1}^{s} u_r \tilde{\bar{y}}_{r\circ} \\ \text{s.t.} \quad & \sum_{i=1}^{m} v_i \tilde{\bar{x}}_{i\circ} = 1, \\ & \sum_{r=1}^{s} u_r \tilde{\bar{y}}_{rj} - \sum_{i=1}^{m} v_i \tilde{\bar{x}}_{ij} \leqslant 0, \quad j = 1,\ldots,n, \\ & u_r,\ v_i \geqslant 0, \quad r = 1,\ldots,s;\ i = 1,\ldots,m. \end{aligned} \tag{8}$$

The Ra-Fu expected value form of the above model is expressed as:

$$\begin{aligned} \max \quad & E\left[\sum_{r=1}^{s} u_r \tilde{\bar{y}}_{r\circ}\right] \\ \text{s.t.} \quad & E\left[\sum_{i=1}^{m} v_i \tilde{\bar{x}}_{i\circ}\right] = 1, \\ & E\left[\sum_{r=1}^{s} u_r \tilde{\bar{y}}_{rj} - \sum_{i=1}^{m} v_i \tilde{\bar{x}}_{ij}\right] \leqslant 0, \quad j = 1,\ldots,n, \\ & u_r,\ v_i \geqslant 0, \quad r = 1,\ldots,s;\ i = 1,\ldots,m. \end{aligned} \tag{9}$$

Let us assume that $\tilde{\bar{x}}_{ij} \sim P(\bar{k}_{ij})$ and $\tilde{\bar{y}}_{rj} \sim P(\bar{\lambda}_{rj})$ in Model (8) are Poisson-distributed Ra-Fu variables on the possibility space (Θ, $P(\Theta)$,Pos) where $\bar{k}_{ij} = \left(x_{ij}^{\alpha}, x_{ij}^{m_1}, x_{ij}^{m_2}, x_{ij}^{\beta}\right)_{LR}$ and $\bar{\lambda}_{rj} = \left(y_{rj}^{\alpha}, y_{rj}^{m_1}, y_{rj}^{m_2}, y_{rj}^{\beta}\right)_{LR}$. We substitute the following expected value of Poisson-distributed Ra-Fu variables (see Lemma 1 into Model (9) to obtain a deterministic CCR model for assessing:

$$E\left[\sum_{r=1}^{s} u_r \tilde{\bar{y}}_{rj}\right] = \frac{1}{2}\sum_{r=1}^{s} u_r \left[\left(y_{rj}^{m_2} + y_{rj}^{m_1}\right) + y_{rj}^{\alpha}(T(0) - T(1)) + y_{rj}^{\beta}(P(1) - P(0))\right], \quad \text{and}$$

$$E\left[\sum_{i=1}^{m} v_i \tilde{\bar{x}}_{ij}\right] = \frac{1}{2}\sum_{i=1}^{m} v_i \left[\left(x_{ij}^{m_2} + x_{ij}^{m_1}\right) + x_{ij}^{\alpha}(T(0) - T(1)) + x_{ij}^{\beta}(P(1) - P(0))\right].$$

Let us further assume that the uncertain input and output in Model (8) are the uniformly-distributed Ra-Fu variables, denoted by $\tilde{\bar{x}}_{ij} \sim U(\bar{x}_{ij}, \hat{x}_{ij})$ and $\tilde{\bar{y}}_{rj} \sim U(\bar{y}_{rj}, \hat{y}_{rj})$, where $\bar{x}_{ij} = \left(x_{ij}^{\alpha}, x_{ij}^{m_1}, x_{ij}^{m_2}, x_{ij}^{\beta}\right)_{LR}$, $\hat{x}_{ij} = \left(x_{ij}^{\bar{\alpha}}, x_{ij}^{\bar{m}_1}, x_{ij}^{\bar{m}_2}, x_{ij}^{\bar{\beta}}\right)_{LR}$, $\bar{y}_{rj} = \left(y_{rj}^{\alpha}, y_{rj}^{m_1}, y_{rj}^{m_2}, y_{rj}^{\beta}\right)_{LR}$, and $\hat{y}_{rj} = \left(y_{rj}^{\bar{\alpha}}, y_{rj}^{\bar{m}_1}, y_{rj}^{\bar{m}_2}, y_{rj}^{\bar{\beta}}\right)_{LR}$ are the LR trapezoidal fuzzy numbers.

By applying the following expected value of the uniformly-distributed Ra-Fu variables (see Lemma 2 to model (9), the definitive CCR model can be obtained in which the inputs and outputs have uniform distributions on the possibility space (Θ,$P(\Theta)$,Pos):

$$\begin{aligned} E\left[\sum_{r=1}^{s} u_r \tilde{\bar{y}}_{rj}\right] = \frac{1}{4}\sum_{r=1}^{s} u_r \Big[&\left(y_{rj}^{m_1} + y_{rj}^{m_2} + y_{rj}^{\bar{m}_1} + y_{rj}^{\bar{m}_2}\right) + \left(y_{rj}^{\alpha} + y_{rj}^{\bar{\alpha}}\right)(T(0) \\ &- T(1)) + \left(y_{rj}^{\beta} + y_{rj}^{\bar{\beta}}\right)(P(1) - P(0))\Big] \end{aligned}$$

$$\begin{aligned} E\left[\sum_{i=1}^{m} v_i \tilde{\bar{x}}_{ij}\right] = \frac{1}{4}\sum_{i=1}^{m} v_i \Big[&\left(x_{ij}^{m_1} + x_{ij}^{m_2} + x_{ij}^{\bar{m}_1} + x_{ij}^{\bar{m}_2}\right) + \left(x_{ij}^{\alpha} + x_{ij}^{\bar{\alpha}}\right)(T(0) \\ &- T(1)) + \left(x_{ij}^{\beta} + x_{ij}^{\bar{\beta}}\right)(P(1) - P(0))\Big] \end{aligned}$$

We ultimately assume that the input and output parameters in Model (8) are normally-distributed Ra-Fu variables, denoted by $\tilde{\bar{x}}_{ij} \sim N\left(\bar{x}_{ij}, \sigma_{ij}^2\right)$ and $\tilde{\bar{y}}_{rj} \sim N\left(\bar{y}_{rj}, \sigma_{rj}^2\right)$, where the mean parameter of the input and output are the LR fuzzy numbers $\bar{x}_{ij} = \left(x_{ij}^{\alpha}, x_{ij}^{m_1}, x_{ij}^{m_2}, x_{ij}^{\beta}\right)_{LR}$ and $\bar{y}_{rj} = \left(y_{rj}^{\alpha}, y_{rj}^{m_1}, y_{rj}^{m_2}, y_{rj}^{\beta}\right)_{LR}$.

By applying the following expected value of the normally-distributed Ra-Fu variables (see Lemma 3), we can construct the deterministic linear programming model to evaluate the efficiency of each DMU in the uncertain environment:

$$E\left[\sum_{r=1}^{s} u_r \tilde{\bar{y}}_{rj}\right] = \frac{1}{2}\sum_{r=1}^{s} u_r \left(y_{rj}^{m_2} + y_{rj}^{m_1} + y_{rj}^{\alpha}(T(0) - T(1)) + y_{rj}^{\beta}(P(1) - P(0))\right)$$

$$E\left[\sum_{i=1}^{m} v_i \tilde{\bar{x}}_{ij}\right] = \frac{1}{2}\sum_{i=1}^{m} v_i \left[\left(x_{ij}^{m_2} + x_{ij}^{m_1}\right) + x_{ij}^{\alpha}(T(0) - T(1)) + x_{ij}^{\beta}(P(1) - P(0))\right]$$

We should note that in all three models, the under assessment is said to be efficient if the corresponding optimal solution is equal to unity; otherwise, $DMU_\circ$ is inefficient.

4. Chance-constrained Ra-Fu CCR model

In this section we develop an imprecise DEA-based formulation for dealing with the randomness of fuzzy parameters on a possibility space (Θ,$P(\Theta)$,Pos) through efficiency measurement. Let us consider n DMUs, indexed by $j = 1,\ldots,n$, where each of them consumes m different Ra-Fu inputs, indexed by $\tilde{\bar{x}}_{ij} (i = 1,\ldots,m)$, to secure s different Ra-Fu outputs indexed by $\tilde{\bar{y}}_{rj}$ ($r = 1,\ldots,s$). We construct the following generic DEA model, called the Possibility–Probability Constrained Programming (PPCP) model as follows:

$$\begin{aligned} & \max \varphi \\ \text{s.t.} \quad & Pos\left[P\left(\varphi \leqslant \sum_{r=1}^{s} u_r \tilde{\bar{y}}_{r\circ}\right) \geqslant \delta\right] \geqslant \gamma, \\ & Pos\left[P\left(\sum_{i=1}^{m} v_i \tilde{\bar{x}}_{i\circ} \geqslant 1\right) \geqslant \delta\right] \geqslant \gamma, \\ & Pos\left[P\left(\sum_{i=1}^{m} v_i \tilde{\bar{x}}_{i\circ} \leqslant 1\right) \geqslant \delta\right] \geqslant \gamma, \\ & Pos\left[P\left(\sum_{r=1}^{s} u_r \tilde{\bar{y}}_{rj} - \sum_{i=1}^{m} v_i \tilde{\bar{x}}_{ij} \leqslant 0\right) \geqslant \delta\right] \geqslant \gamma, \quad j = 1,\ldots,n, \\ & u_r,\ v_i \geqslant 0, \quad r = 1,\ldots,s;\ i = 1,\ldots,m. \end{aligned} \tag{10}$$

where δ and γ are a pre-specified minimum probability level and a pre-specified acceptable level of possibility, respectively, and vary between $[0,1]$. These parameters, assumed known *a priori*, are also called the threshold (or aspiration) levels. Let us assume that the inputs and outputs have normal distributions $\tilde{\bar{x}}_{ij} \sim N\left(\bar{x}_{ij}, \bar{\sigma}_{ij}^2\right)$ and $\tilde{\bar{y}}_{rj} \sim N\left(\bar{y}_{rj}, \bar{\sigma}_{rj}^2\right)$ where $\tilde{\bar{y}}_{rj} \sim N\left(\bar{y}_{rj}, \bar{\sigma}_{rj}^2\right)$, $\bar{y}_{rj} = \left(y_{rj}^{\alpha}, y_{rj}^{m_1}, y_{rj}^{m_2}, y_{rj}^{\beta}\right)_{LR}$, $\bar{x}_{ij} = \left(x_{ij}^{\alpha}, x_{ij}^{m_1}, x_{ij}^{m_2}, x_{ij}^{\beta}\right)_{LR}$, $\bar{\sigma}_{ij}^2 = \left(\hat{x}_{ij}^{\alpha}, \hat{x}_{ij}^{m_1}, \hat{x}_{ij}^{m_2}, \hat{x}_{ij}^{\beta}\right)_{LR}$, and $\bar{\sigma}_{rj}^2 = \left(\hat{y}_{rj}^{\alpha}, \hat{y}_{rj}^{m_1}, \hat{y}_{rj}^{m_2}, \hat{y}_{rj}^{\beta}\right)_{LR}$ are characterized by the LR trapezoidal fuzzy numbers. Notice that in Model (10) the *Principle Extension* method (see Definition 11) enables us to generalize the membership functions of $\sum_{r=1}^{s} u_r \bar{y}_{rj}$ and $\sum_{r=1}^{s} v_i \bar{x}_{ij}$ as follows:

$$\mu_{\sum_{i=1}^{m} v_i \bar{x}_{ij}}(t) = \begin{cases} L\left(\dfrac{\sum_{i=1}^{m} v_i x_{ij}^{m_1} - t}{\sum_{i=1}^{m} v_i x_{ij}^{\alpha}}\right) & t \leqslant \sum_{i=1}^{m} v_i x_{ij}^{m_1} \\ R\left(\dfrac{t - \sum_{i=1}^{m} v_i x_{ij}^{m_2}}{\sum_{i=1}^{m} v_i x_{ij}^{\beta}}\right) & t \geqslant \sum_{i=1}^{m} v_i x_{ij}^{m_2} \end{cases} \tag{11}$$

and

$$\mu_{\sum_{r=1}^{s} u_r \tilde{y}_{rj}}(t) = \begin{cases} L\left(\frac{\sum_{r=1}^{s} u_r y_{rj}^{m_1} - t}{\sum_{r=1}^{s} u_r y_{rj}^{\alpha}}\right) & t \leqslant \sum_{r=1}^{s} u_r y_{rj}^{m_1} \\ R\left(\frac{t - \sum_{r=1}^{s} u_r y_{rj}^{m_2}}{\sum_{r=1}^{s} u_r y_{rj}^{\beta}}\right) & t \geqslant \sum_{r=1}^{s} u_r y_{rj}^{m_2} \end{cases} \quad (12)$$

We use Theorem 5 to solve PPCP Model (10).

Theorem 5. *Suppose that $\tilde{\bar{a}}_j$ and $\tilde{\bar{b}}$ are the Ra-Fu generalized left right fuzzy numbers with normal distributions and the mean and variance of $\tilde{\bar{a}}_j$ and $\tilde{\bar{b}}$ are, respectively, $\left(\bar{\mu}_j, \bar{\sigma}_j^2\right)$ and $(\hat{\mu}, \hat{\sigma}^2)$ where $\bar{\mu}_j = \left(a_j^{\alpha}, a_j^{m_1}, a_j^{m_2}, a_j^{\beta}\right)$, $\hat{\mu} = (b^{\alpha}, b^{m_1}, b^{m_2}, b^{\beta})$, $\bar{\sigma}_j^2 = \left(\bar{a}_j^{\alpha}, \bar{a}_j^{m_1}, \bar{a}_j^{m_2}, \bar{a}_j^{\beta}\right)$ and $\hat{\sigma}^2 = (\hat{a}^{\alpha}, \hat{a}^{m_1}, \hat{a}^{m_2}, \hat{a}^{\beta})$. Then we have:*

(a) $Pos\left[P\left(\sum_{j=1}^{n} \tilde{\bar{a}}_j x_j \leqslant \tilde{\bar{b}}\right) \geqslant \delta\right] \geqslant \gamma$ if and only if

$$\begin{cases} \sum_{j=1}^{n}\left(\bar{\mu}_j^{m_1} - L^{-1}(\gamma)\bar{\mu}_j^{\alpha}\right)x_j + \sqrt{\sum_{j=1}^{n} x_j^2\left(a_j^{m_1} - L^{-1}(\alpha)\bar{a}_j^{\alpha}\right) + \left(\hat{a}_j^{m_1} - L^{-1}(\alpha)\hat{a}_j^{\alpha}\right)} \\ \Phi^{-1}(\delta) \leqslant \\ \quad b^{m_2} + R^{-1}(\gamma)b^{\beta}, \quad \delta > 0.5, \\ \sum_{j=1}^{n}\left(\bar{\mu}_j^{m_1} - L^{-1}(\gamma)\bar{\mu}_j^{\alpha}\right)x_j + \sqrt{\sum_{j=1}^{n} x_j^2\left(\bar{a}_j^{m_2} + R^{-1}(\alpha)\bar{a}_j^{\beta}\right) + \left(\hat{a}_j^{m_2} + R^{-1}(\alpha)\hat{a}_j^{\beta}\right)} \\ \Phi^{-1}(\delta) \leqslant \\ \quad b^{m_2} + R^{-1}(\gamma)b^{\beta}, \quad \delta \leqslant 0.5, \end{cases}$$

(b) $Nec\left[P\left(\sum_{j=1}^{n} \tilde{\bar{a}}_j x_j \leqslant \tilde{\bar{b}}\right) \geqslant \delta\right] \geqslant \gamma$ *if and only if*

$$\begin{cases} \sum_{j=1}^{n}\left(\bar{\mu}_j^{m_2} + R^{-1}(\gamma)\bar{\mu}_j^{\beta}\right)x_j + \sqrt{\sum_{j=1}^{n} x_j^2\left(\bar{a}_j^{m_2} + R^{-1}(\alpha)\bar{a}_j^{\beta}\right) + (\hat{a}_j^{m_2} + R^{-1}(\alpha)\hat{a}_j^{\beta})}\,\Phi^{-1}(\delta) \leqslant \\ b^{m_1} - L^{-1}(1-\gamma)b^{\alpha}, \quad \delta > 0.5, \\ \sum_{j=1}^{n}\left(\bar{\mu}_j^{m_2} + R^{-1}(\gamma)\bar{\mu}_j^{\beta}\right)x_j + \sqrt{\sum_{j=1}^{n} x_j^2\left(a_j^{m_1} - L^{-1}(\alpha)\bar{a}_j^{\alpha}\right) + \left(\hat{a}_j^{m_1} - L^{-1}(\alpha)\hat{a}_j^{\alpha}\right)}\,\Phi^{-1}(\delta) \leqslant \\ b^{m_1} - L^{-1}(1-\gamma)b^{\alpha}, \quad \delta \leqslant 0.5, \end{cases}$$

where "P" presents probability, δ is a pre-specified minimum probability, "Pos" and "Nec" stands for possibility and necessity, and γ is a pre-specified acceptable level of possibility or necessity.

Proof. See Appendix A. □

We apply the first part of Theorem 5 to the Ra-Fu CCR Model (10) to obtain the following two deterministic equivalent models in the presence of Ra-Fu inputs and outputs:

$$\begin{aligned} \max_{\delta > 0.5} \quad & \varphi \\ \text{s.t.} \quad & \varphi + \left(\sigma_{\circ}^{O}(u,\varphi)\right)_{\delta}^{L}\Phi^{-1}(\delta) \leqslant \sum_{r=1}^{s} u_r\left(y_{r\circ}^{m_2} + R^{-1}(\gamma)y_{r\circ}^{\beta}\right), \\ & \sum_{i=1}^{m} v_i\left(x_{i\circ}^{m_2} + R^{-1}(\gamma)x_{i\circ}^{\beta}\right) - \left(\sigma_{\circ}^{I}(v)\right)_{\delta}^{L}\Phi^{-1}(\delta) \geqslant 1, \\ & \sum_{j=1}^{n} v_i\left(x_{i\circ}^{m_1} - L^{-1}(\gamma)x_{i\circ}^{\alpha}\right) + \left(\sigma_{\circ}^{I}(v)\right)_{\delta}^{L}\Phi^{-1}(\delta) \leqslant 1, \\ & \sum_{r=1}^{s} u_r\left(y_{rj}^{m_1} - L^{-1}(\gamma)y_{rj}^{\alpha}\right) - \sum_{i=1}^{m} v_i\left(x_{ij}^{m_2} + R^{-1}(\gamma)x_{ij}^{\beta}\right) \\ & \quad + \Phi^{-1}(\delta)(\sigma_j(u,v))_{\delta}^{L} \leqslant 0, \quad j = 1,\ldots,n, \\ & u_r,\ v_i \geqslant 0, \quad r = 1,\ldots,s;\ i = 1,\ldots,m. \end{aligned} \quad (13)$$

Table 12
Normalized necessity–probability score of the DMUs.

δ	0.5				0.1	0.3	0.5	0.9
γ	0.1	0.3	0.5	0.9	0.5			
DMU 1	0.7198	0.7211	0.7224	0.7251	0.7250	0.7940	0.7224	0.7203
DMU 2	0.6820	0.6767	0.6710	0.6629	0.6880	0.7452	0.671	0.6650
DMU 3	0.6323	0.6276	0.6226	0.6124	0.6232	0.6838	0.6226	0.6221
DMU 4	0.8709	0.866	0.8609	0.8501	0.8646	0.9460	0.8609	0.8592
DMU 5	0.5535	0.5495	0.5454	0.5364	0.5410	0.5969	0.5454	0.5497
DMU 6	0.4918	0.4842	0.4796	0.4708	0.4920	0.5308	0.4796	0.4746
DMU 7	0.7284	0.7313	0.7354	0.7453	0.7591	0.8192	0.7354	0.7080
DMU 8	0.8189	0.8158	0.8124	0.8054	0.8014	0.8870	0.8124	0.8249
DMU 9	0.8548	0.8540	0.8543	0.8547	0.8771	0.9477	0.8543	0.8534
DMU 10	0.7499	0.7508	0.7518	0.7541	0.7442	0.8221	0.7518	0.8981
DMU 11	0.6080	0.6115	0.6149	0.6223	0.6175	0.6765	0.6149	0.6115
DMU 12	0.5721	0.5715	0.571	0.5697	0.5751	0.6289	0.571	0.5658
DMU 13	0.4931	0.497	0.5011	0.5097	0.5045	0.5521	0.5011	0.4964
DMU 14	0.6345	0.6421	0.6498	0.6657	0.6686	0.7229	0.6498	0.6257
DMU 15	0.7853	0.7771	0.77	0.7540	0.7886	0.8545	0.77	0.7462
DMU 16	0.6042	0.5999	0.5954	0.5859	0.5963	0.6522	0.5954	0.5986
DMU 17	0.6342	0.6296	0.6247	0.6146	0.6175	0.6828	0.6247	0.6323
DMU 18	0.6786	0.6644	0.6506	0.6216	0.6651	0.7217	0.6506	0.6305
DMU 19	0.7660	0.7487	0.7309	0.6978	0.7474	0.8109	0.7309	0.7006
DMU 20	0.8389	0.8278	0.8199	0.8109	0.8416	0.9093	0.8199	0.8189
DMU 21	0.9013	0.8990	0.8964	0.8911	0.9264	0.9988	0.8964	0.8601
DMU 22	0.6662	0.6674	0.6676	0.6682	0.6901	0.7441	0.6676	0.6404
DMU 23	0.8524	0.8477	0.8417	0.8293	0.8663	0.9362	0.8417	0.8111
DMU 24	1.0000	1.0000	1.0000	1.0000	1.0000	0.9902	1.0000	1.7633
DMU 25	0.8330	0.8273	0.8219	0.8113	0.8489	0.9152	0.8219	0.7924
DMU 26	0.6244	0.6338	0.6438	0.6657	0.6528	0.7050	0.6438	0.7810
DMU 27	0.6288	0.6344	0.6392	0.6494	0.6629	0.6460	0.6392	0.6110
DMU 28	0.6819	0.6806	0.6778	0.6715	0.6626	0.7559	0.6778	0.6493
DMU 29	0.8751	0.8669	0.8613	0.8554	0.8900	0.9590	0.8613	1.0429
DMU 30	0.9019	0.9001	0.8983	0.8947	0.9259	1.0000	0.8983	0.8637

$$\max_{\delta\leqslant 0.5} \ \varphi$$
$$\text{s.t.} \ \ \varphi + (\sigma^O_\circ(u,\varphi))^R_\delta \Phi^{-1}(\delta) \leqslant \sum_{r=1}^{s} u_r\left(y^{m_2}_{r\circ} + R^{-1}(\gamma) y^{\beta}_{r\circ}\right),$$
$$\sum_{i=1}^{m} v_i\left(x^{m_2}_{i\circ} + R^{-1}(\gamma)x^{\beta}_{i\circ}\right) - (\sigma^I_\circ(v))^R_\delta \Phi^{-1}(\delta) \geqslant 1,$$
$$\sum_{j=1}^{n} v_i\left(x^{m_1}_{i\circ} - L^{-1}(\gamma)x^{\alpha}_{i\circ}\right) + (\sigma^I_\circ(v))^R_\delta \Phi^{-1}(\delta) \leqslant 1, \qquad (14)$$
$$\sum_{r=1}^{s} u_r\left(y^{m_1}_{rj} - L^{-1}(\gamma) y^{\alpha}_{rj}\right) - \sum_{i=1}^{m} v_i\left(x^{m_2}_{ij} + R^{-1}(\gamma)x^{\beta}_{ij}\right) + \Phi^{-1}(\delta)(\sigma_j(u,v))^R_\delta \leqslant 0, \quad j=1,\ldots,n,$$
$$u_r,\ v_i \geqslant 0, \quad r=1,\ldots,s;\ i=1,\ldots,m.$$

where in Model (13), $(\sigma^O_\circ(u,\varphi))^L_\delta$, $(\sigma^I_\circ(v))^L_\delta$ and $(\sigma_j(u,v))^L_\delta$ are the lower bounds and in Model (14), $(\sigma^O_\circ(u,\varphi))^R_\delta$, $(\sigma^I_\circ(v))^R_\delta$ and $(\sigma_j(u,v))^R_\delta$ are the upper bounds of the δ-cut of the following intervals:

$$\tilde{\sigma}^O_\circ(u,\varphi) = \left[(\sigma^O_\circ(u,\varphi))^L_\delta, (\sigma^O_\circ(u,\varphi))^R_\delta\right],$$
$$\tilde{\sigma}^I_P(v) = \left[(\sigma^I_P(v))^L_\delta, (\sigma^I_P(v))^R_\delta\right],$$
$$\tilde{\sigma}_j(u,v) = \left[(\sigma_j(u,v))^L_\delta, (\sigma_j(u,v))^R_\delta\right]$$

Obviously, Models (13) and (14) are non-linear programming models because of the lower and upper bounds of the above intervals. These non-linear programming models can be transformed into quadratic programming models by using the following substitutions:

(a) In Model (13): $(\theta^O_\circ)^2 = \left((\sigma^O_\circ(u,\varphi))^L_\delta\right)^2$, $(\theta^I_\circ)^2 = \left((\sigma^I_\circ(v))^L_\delta\right)^2$ and $(\lambda_j)^2 = \left((\sigma_j(u,v))^L_\delta\right)^2$

(b) In Model (14): $(\bar{\theta}^O_\circ)^2 = \left(\sigma^O_\circ(u,\varphi)^R_\delta\right)^2$, $(\bar{\theta}^I_\circ)^2 = \left((\sigma^I_\circ(v))^R_\delta\right)^2$ and $(\bar{\lambda}_j)^2 = \left((\sigma_j(u,v))^R_\delta\right)^2$

By applying the above substitutions to Models (13) and (14), respectively, the following quadratic programming problems can be formulated:

$$\max_{\delta>0.5} \ \varphi$$
$$\text{s.t.} \ \ \varphi + \theta^O_\circ \Phi^{-1}(\delta) \leqslant \sum_{r=1}^{s} u_r\left(y^{m_2}_{r\circ} + R^{-1}(\gamma) y^{\beta}_{r\circ}\right),$$
$$\sum_{i=1}^{m} v_i\left(x^{m_2}_{i\circ} + R^{-1}(\gamma)x^{\beta}_{i\circ}\right) - \theta^I_\circ \Phi^{-1}(\delta) \geqslant 1,$$
$$\sum_{j=1}^{n} v_i\left(x^{m_1}_{i\circ} - L^{-1}(\gamma)x^{\alpha}_{i\circ}\right) + \theta^I_\circ \Phi^{-1}(\delta) \leqslant 1,$$
$$\sum_{r=1}^{s} u_r\left(y^{m_1}_{rj} - L^{-1}(\gamma) y^{\alpha}_{rj}\right) - \sum_{i=1}^{m} v_i\left(x^{m_2}_{ij} + R^{-1}(\gamma)x^{\beta}_{ij}\right) + \Phi^{-1}(\delta)\lambda_j \leqslant 0, \ \ j=1,\ldots,n,$$
$$(\theta^O_\circ)^2 = \sum_{i=1}^{m} u_r^2\left(\hat{y}^{m_1}_{r\circ} - L^{-1}(\gamma)\hat{y}^{\alpha}_{r\circ}\right),$$
$$(\theta^I_\circ)^2 = \sum_{i=1}^{m} v_i^2\left(\hat{x}^{m_1}_{i\circ} - L^{-1}(\gamma)\hat{x}^{\alpha}_{i\circ}\right),$$
$$(\lambda_j)^2 = \sum_{r=1}^{s} u_r^2 \hat{y}^{m_2}_{rj} + \sum_{i=1}^{m} v_i^2 \hat{x}^{m_2}_{ij} - L^{-1}(\gamma)\left(\sum_{r=1}^{s} u_r^2\hat{y}^{\beta}_{rj} + \sum_{i=1}^{m} v_i^2\hat{x}^{\beta}_{ij}\right),$$
$$j=1,\ldots,n,$$
$$u_r, v_i, \theta^O_p, \theta_p, \bar{\theta}^I_p, \lambda_j \geqslant 0, \ \ r=1,\ldots,s;\ i=1,\ldots,m;\ j=1,\ldots,n. \qquad (15)$$

$$\max_{\delta\leqslant 0.5} \ \varphi$$
$$\text{s.t.} \ \ \varphi - \sum_{r=1}^{s} u_r\left(y^{m_2}_{r\circ} + R^{-1}(\gamma)\tilde{\bar{y}}^{\beta}_{r\circ}\right) + \bar{\theta}^O_\circ \Phi^{-1}(\delta) \leqslant 0,$$
$$\sum_{i=1}^{m} v_i\left(x^{m_2}_{i\circ} + R^{-1}(\gamma)x^{\beta}_{i\circ}\right) - \bar{\theta}^I_\circ \Phi^{-1}(\delta) \geqslant 1,$$
$$\sum_{j=1}^{n} v_i\left(x^{m_1}_{i\circ} - L^{-1}(\gamma)x^{\alpha}_{i\circ}\right) + \bar{\theta}^I_\circ \Phi^{-1}(\delta) \leqslant 1,$$
$$\sum_{r=1}^{s} u_r\left(y^{m_1}_{rj} - L^{-1}(\gamma)\sum_{r=1}^{s} u_r y^{\alpha}_{rj}\right) - \sum_{i=1}^{m} v_i\left(\tilde{\bar{x}}^{m_2}_{ip} + R^{-1}(\gamma)\sum_{i=1}^{m} v_i \tilde{\bar{x}}^{\beta}_{ip}\right) + \bar{\lambda}_j \Phi^{-1}(\delta) \leqslant 0, \ \ j=1,\ldots,n,$$
$$(\bar{\theta}^O_\circ)^2 = \sum_{i=1}^{m} u_r^2\left(\hat{y}^{m_2}_{r\circ} + R^{-1}(\gamma)\hat{y}^{\beta}_{r\circ}\right),$$
$$(\bar{\theta}^I_\circ)^2 = \sum_{i=1}^{m} v_i^2\left(\hat{x}^{m_2}_{i\circ} + R^{-1}(\gamma)\hat{x}^{\beta}_{i\circ}\right),$$
$$(\bar{\lambda}_j)^2 = \sum_{r=1}^{s} u_r^2 \hat{y}^{m_2}_{rj} + \sum_{i=1}^{m} v_i^2 \hat{x}^{m_2}_{ij} + R^{-1}(\gamma)\left(\sum_{r=1}^{s} u_r^2\hat{y}^{\beta}_{rj} + \sum_{i=1}^{m} v_i^2\hat{x}^{\beta}_{ij}\right),$$
$$j=1,\ldots,n,$$
$$u_r, v_i, \bar{\theta}^I_\circ, \bar{\theta}^O_\circ, \bar{\lambda}_j \geqslant 0, \ \ r=1,\ldots,s;\ i=1,\ldots,m;\ j=1,\ldots,n. \qquad (16)$$

We should note that the $DMU_\circ$ is called possibilistic–probabilistic γ–δ efficient if the objective function of Model (15) or Model (16), φ, is greater than or equal to one at the possibility level γ and probability level δ; otherwise, it is called possibilistic–probabilistic γ–δ inefficient.

In addition to the PPCP Model (10) with possibility–probability constraints, we present a Necessity–Probability Constrained Programming (NPCP) model under fuzzy probability necessity constraints as follows:

$$\max \ \ \varphi$$
$$\text{s.t.} \ \ Nec\left[P\left(\varphi \leqslant \sum_{r=1}^{s} u_r \tilde{\bar{y}}_{r\circ}\right) \geqslant \delta\right] \geqslant \gamma,$$
$$Nec\left[P\left(\sum_{i=1}^{m} v_i \tilde{\bar{x}}_{i\circ} \leqslant 1\right) \geqslant \delta\right] \geqslant \gamma,$$
$$Nec\left[P\left(\sum_{i=1}^{m} v_i \tilde{\bar{x}}_{i\circ} \geqslant 1\right) \geqslant \delta\right] \geqslant \gamma,$$
$$Nec\left[P\left(\sum_{r=1}^{s} u_r \tilde{\bar{y}}_{rj} - \sum_{i=1}^{m} v_i \tilde{\bar{x}}_{ij} \leqslant 0\right) \geqslant \delta\right] \geqslant \gamma, \quad j=1,\ldots,n,$$
$$u_r, v_i \geqslant 0, \quad r=1,\ldots,s;\ i=1,\ldots,m. \qquad (17)$$

Similar to the solution method of PPCP in Model (10), we use the second part of Theorem 5 for the constraints of NPCP in Model (17) to obtain the following quadratic deterministic models:

$$\max_{\delta>0.5} \ \bar{\varphi}$$
$$\text{s.t.} \ \ \bar{\varphi} - \sum_{r=1}^{s} u_r y^{m_1}_{r\circ} + L^{-1}(1-\gamma)\sum_{r=1}^{s} u_r y^{\alpha}_{r\circ} + \tilde{\theta}^O_\circ \Phi^{-1}(\delta) \leqslant 0,$$
$$\sum_{i=1}^{m} v_i\left(x^{m_1}_{i\circ} - L^{-1}(1-\gamma)x^{\alpha}_{i\circ}\right) - \Phi^{-1}(\delta)\tilde{\theta}^I_\circ \geqslant 1,$$
$$\sum_{i=1}^{m} v_i x^{m_2}_{i\circ} + R^{-1}(\gamma)\sum_{i=1}^{m} v_i x^{\beta}_{i\circ} + \Phi^{-1}(\delta)\tilde{\theta}^I_\circ \leqslant 1,$$
$$\sum_{r=1}^{s} u_r\left(y^{m_2}_{rj} + R^{-1}(\gamma)y^{\beta}_{rj}\right) - \sum_{i=1}^{m} v_i\left(x^{m_1}_{ij} - L^{-1}(1-\gamma)x^{\alpha}_{ij}\right) + \Phi^{-1}(\delta)\tilde{\lambda}_j \leqslant 0, \ \ j=1,\ldots,n,$$
$$\left(\tilde{\theta}^O_\circ\right)^2 = \sum_{i=1}^{m} u_r^2\left(\hat{y}^{m_1}_{r\circ} + R^{-1}(\gamma)\hat{y}^{\alpha}_{r\circ}\right),$$
$$\left(\tilde{\theta}^I_\circ\right)^2 = \sum_{i=1}^{m} v_i^2\left(\hat{x}^{m_2}_{i\circ} + R^{-1}(\gamma)\hat{x}^{\beta}_{i\circ}\right),$$
$$(\tilde{\lambda}_j)^2 = \sum_{r=1}^{s} u_r^2 \hat{y}^{m_1}_{rj} + \sum_{i=1}^{m} v_i^2 \hat{x}^{m_1}_{ij} + R^{-1}(\gamma)\left(\sum_{r=1}^{s} u_r^2\hat{y}^{\alpha}_{rj} + \sum_{i=1}^{m} v_i^2\hat{x}^{\alpha}_{ij}\right),$$
$$j=1,\ldots,n,$$
$$u_r, v_i, \tilde{\theta}^O_\circ, \tilde{\theta}^I_\circ, \tilde{\lambda}_j \geqslant 0, \ \ r=1,\ldots,s;\ \ i=1,\ldots,m;\ j=1,\ldots,n. \qquad (18)$$

$$\begin{aligned}
\max_{\delta\leqslant 0.5}\ & \bar{\varphi} \\
\text{s.t.}\ & \bar{\varphi}-\sum_{r=1}^{s}u_r y_{r\circ}^{m_1}+L^{-1}(1-\gamma)\sum_{r=1}^{s}u_r y_{r\circ}^{\alpha}+\hat{\theta}_{\circ}^{O}\Phi^{-1}(\delta)\leqslant 0,\\
& \sum_{i=1}^{m}v_i x_{i\circ}^{m_1}-L^{-1}(1-\gamma)\sum_{i=1}^{m}v_i x_{i\circ}^{\alpha}-\Phi^{-1}(\delta)\hat{\theta}_{\circ}^{I}\geqslant 1,\\
& \sum_{i=1}^{m}v_i x_{i\circ}^{m_2}+R^{-1}(\gamma)\sum_{i=1}^{m}v_i x_{i\circ}^{\beta}+\Phi^{-1}(\delta)\hat{\theta}_{\circ}^{I}\leqslant 1,\\
& \sum_{r=1}^{s}u_r\left(y_{rj}^{m_2}+R^{-1}(\gamma)y_{rj}^{\beta}\right)-\sum_{i=1}^{m}v_i\left(x_{ij}^{m_1}-L^{-1}(1-\gamma)x_{ij}^{\alpha}\right)\\
& +\Phi^{-1}(\delta)\hat{\lambda}_j\leqslant 0,\ \ j=1,\ldots,n,\\
& \left(\hat{\theta}_{\circ}^{O}\right)^2=\sum_{i=1}^{m}u_r^2\left(\hat{y}_{r\circ}^{m_1}-L^{-1}(1-\gamma)\hat{y}_{r\circ}^{\alpha}\right),\\
& \left(\hat{\theta}_{\circ}^{I}\right)^2=\sum_{i=1}^{m}v_i^2\left(\hat{x}_{i\circ}^{m_1}-L^{-1}(1-\gamma)\hat{x}_{i\circ}^{\beta}\right),\\
& \left(\hat{\lambda}_j\right)^2=\sum_{r=1}^{s}u_r^2\hat{y}_{rj}^{m_1}+\sum_{i=1}^{m}v_i^2\hat{x}_{ij}^{m_1}-L^{-1}(1-\gamma)\\
& \left(\sum_{i=1}^{m}v_i^2\hat{x}_{ij}^{\alpha}+\sum_{r=1}^{s}u_r^2\hat{y}_{rj}^{\alpha}\right),\ \ j=1,\ldots,n,\\
& u_r,v_i,\hat{\theta}_{\circ}^{O},\hat{\theta}_{\circ}^{I},\hat{\lambda}_j\geqslant 0,\ \ r=1,\ldots,s;\ i=1,\ldots,m;\ j=1,\ldots,n.
\end{aligned}\tag{19}$$

In order to define the necessity-probabilistic γ–δ efficient $DMU_\circ$, we can normalize the efficiency as $\varphi_\circ=\bar{\varphi}_\circ/\max_j(\bar{\varphi}_j)$. As a result, $DMU_\circ$ is necessity-probabilistic γ–δ efficient if $\varphi_\circ=1$; otherwise, it is called necessity-probabilistic γ–δ inefficient.

5. Chance-constrained Ra-Fu CCR model with fuzzy threshold level

In the previous section, we discussed the CCR model in which the inputs and outputs are Ra-Fu variables on a possibility space $(\Theta, P(\Theta), Pos)$ with a normal distribution. In a general setting, the probability levels may be unknown or imprecise, in particular for fuzzy and stochastic production sets. Hence, we develop the proposed models with the fuzzy threshold level to take into account the generalized DEA model. Let us keep our earlier assumptions as well as the fuzzy δ, denoted by $\tilde{\delta}=(\delta^{\alpha},\delta^{m_1},\delta^{m_2},\delta^{\beta})$, to derive the following PPCP model:

$$\begin{aligned}
\max\ & \varphi \\
\text{s.t.}\ & Pos\left[P\left(\varphi\leqslant\sum_{r=1}^{s}u_r\tilde{\bar{y}}_{r\circ}\right)\geqslant\tilde{\delta}\right]\geqslant\gamma,\\
& Pos\left[P\left(\sum_{i=1}^{m}v_i\tilde{\bar{x}}_{i\circ}\leqslant 1\right)\geqslant\tilde{\delta}\right]\geqslant\gamma,\\
& Pos\left[P\left(\sum_{i=1}^{m}v_i\tilde{\bar{x}}_{i\circ}\geqslant 1\right)\geqslant\tilde{\delta}\right]\geqslant\gamma,\\
& Pos\left[P\left(\sum_{r=1}^{s}u_r\tilde{\bar{y}}_{rj}-\sum_{i=1}^{m}v_i\tilde{\bar{x}}_{ij}\leqslant 0\right)\geqslant\tilde{\delta}\right]\geqslant\gamma,\quad j=1,\ldots,n,\\
& u_r,v_i\geqslant 0,\quad r=1,\ldots,s;\ i=1,\ldots,m.
\end{aligned}\tag{20}$$

We propose Theorem 6 below in order to achieve the deterministic form of Model (20) involving the fuzzy threshold level.

Theorem 6. *Let $\bar{\lambda}_1$ and $\bar{\lambda}_2$ be two independently Ra-Fu generalized left right fuzzy numbers with normal distributions, denoted by $\tilde{\bar{\lambda}}_1\sim N(\bar{\mu}_1,\bar{\sigma}_1^2)$ and $\tilde{\bar{\lambda}}_2\sim N(\bar{\mu}_2,\bar{\sigma}_2^2)$ where $\bar{\mu}_1=(\lambda_1^{\alpha},\lambda_1^{m_1},\lambda_1^{m_2},\lambda_1^{\beta})$, $\bar{\mu}_2=(\lambda_2^{\alpha},\lambda_2^{m_1},\lambda_2^{m_2},\lambda_2^{\beta})$, $\bar{\sigma}_1^2=\left(\hat{\lambda}_1^{\alpha},\hat{\lambda}_1^{m_1},\hat{\lambda}_1^{m_2},\hat{\lambda}_1^{\beta}\right)$, $\bar{\sigma}_2^2=\left(\hat{\lambda}_2^{\alpha},\hat{\lambda}_2^{m_1},\hat{\lambda}_2^{m_2},\hat{\lambda}_2^{\beta}\right)$ and $\tilde{\delta}=(\delta^{\alpha},\delta^{m_1},\delta^{m_2},\delta^{\beta})$. Then, for a given $\alpha\in[0,1]$, we have:*

(a) *$Pos[P(\tilde{\bar{\lambda}}_1\leqslant\tilde{\bar{\lambda}}_2)\geqslant\tilde{\delta}]\geqslant\alpha$ if and only if*

$$\begin{aligned}
&\left(\lambda_1^{m_1}-\lambda_2^{m_2}-L^{-1}(\alpha)\left(\lambda_1^{\alpha}+\lambda_2^{\beta}\right)\right)\\
&+\left(\sqrt{\left(\hat{\lambda}_1^{m_1}+\hat{\lambda}_2^{m_1}\right)-L^{-1}(\alpha)\left(\hat{\lambda}_1^{\alpha}+\hat{\lambda}_2^{\alpha}\right)}\right)\Phi^{-1}(\delta^{m_1}-L^{-1}(\alpha)\delta^{\alpha})\leqslant 0
\end{aligned}$$

(b) *$Nec[P(\tilde{\bar{\lambda}}_1\leqslant\tilde{\bar{\lambda}}_2)\geqslant\tilde{\delta}]\geqslant\beta$ if and only if*

$$\begin{aligned}
&\lambda_1^{m_2}-\lambda_2^{m_1}+R^{-1}(1-\alpha)\left(\lambda_1^{\beta}+\lambda_2^{\alpha}\right)\\
&+\left(\sqrt{\hat{\lambda}_1^{m_2}+\hat{\lambda}_2^{m_2}+R^{-1}(1-\alpha)\left(\hat{\lambda}_1^{\beta}+\hat{\lambda}_2^{\beta}\right)}\right)\Phi^{-1}(\delta^{m_2}+R^{-1}(\alpha)\delta^{\beta})\leqslant 0
\end{aligned}$$

Proof. See Appendix B. □

Assume that there are n DMUs ($j=1,\ldots,n$) with m Ra-Fu inputs ($i=1,\ldots,m$) and s Ra-Fu outputs ($r=1,\ldots,s$). The Ra-Fu inputs and Ra-Fu outputs are normally distributed with fuzzy mean and fuzzy variance as follows:

$$\tilde{\bar{x}}_{ij}\sim N\left(\bar{x}_{ij},\bar{\sigma}_{ij}^2\right),\quad \text{where } \bar{x}_{ij}=\left(x_{ij}^{\alpha},x_{ij}^{m_1},x_{ij}^{m_2},x_{ij}^{\beta}\right)_{LR},$$
$$\bar{\sigma}_{ij}^2=\left(\hat{x}_{ij}^{\alpha},\hat{x}_{ij}^{m_1},\hat{x}_{ij}^{m_2},\hat{x}_{ij}^{\beta}\right)_{LR}$$

$$\tilde{\bar{y}}_{rj}\sim N\left(\bar{y}_{rj},\bar{\sigma}_{rj}^2\right),\quad \text{where } \bar{y}_{rj}=\left(y_{rj}^{\alpha},y_{rj}^{m_1},y_{rj}^{m_2},y_{rj}^{\beta}\right)_{LR},$$
$$\bar{\sigma}_{rj}^2=\left(\hat{y}_{rj}^{\alpha},\hat{y}_{rj}^{m_1},\hat{y}_{rj}^{m_2},\hat{y}_{rj}^{\beta}\right)_{LR}$$

After incorporating Theorem 6 into the constraints in Model (20), the following deterministic model is derived for $DMU_\circ$:

$$\begin{aligned}
\max\ & \varphi \\
\text{s.t.}\ & \varphi-\sum_{r=1}^{s}u_r\left(y_{r\circ}^{m_2}-L^{-1}(\gamma)\tilde{\bar{y}}_{r\circ}^{\beta}\right)+(\sigma_\circ(u))_\gamma^L\Phi^{-1}(\delta)_\gamma^L\leqslant 0,\\
& \sum_{i=1}^{m}v_i\left(x_{i\circ}^{m_1}-L^{-1}(\gamma)x_{i\circ}^{\alpha}\right)+(\sigma_\circ(v))_\gamma^L\Phi^{-1}(\delta)_\gamma^L\leqslant 1,\\
& \sum_{i=1}^{m}v_i\left(x_{i\circ}^{m_1}+L^{-1}(\gamma)x_{i\circ}^{\alpha}\right)-(\sigma_\circ(v))_\gamma^L\Phi^{-1}(\delta)_\gamma^L\geqslant 1,\\
& \sum_{r=1}^{s}u_r y_{rj}^{m_1}-\sum_{i=1}^{m}v_i x_{ij}^{m_2}-L^{-1}(\gamma)\left(\sum_{r=1}^{s}u_r\tilde{\bar{y}}_{rj}^{\alpha}+\sum_{i=1}^{m}v_i\tilde{\bar{x}}_{ij}^{\beta}\right)\\
& +(\sigma_j(u,v))_\gamma^L\Phi^{-1}(\delta)_\gamma^L\leqslant 0,\quad j=1,\ldots,n,\\
& u_r,v_i,(\sigma_\circ(u))^{\gamma L},(\sigma_\circ(v))^{\gamma L},(\sigma_j(u,v))^{\gamma L}\geqslant 0,\\
& r=1,\ldots,s;\ i=1,\ldots,m;\ j=1,\ldots,n.
\end{aligned}\tag{21}$$

where $(\sigma_\circ(u))_\gamma^L$, $(\sigma_\circ(v))_\gamma^L$, $(\sigma_j(u,v))_\gamma^L$ and $(\delta)_\gamma^L$ are the lower bounds of the γ-cut (i.e., Definition 3) of the following intervals:

$$\begin{aligned}
\tilde{\sigma}_\circ(u)&=\left[(\sigma_\circ(u))_\gamma^L,(\sigma_\circ(u))_\gamma^U\right]\\
&=\left[\sqrt{\sum\nolimits_{i=1}^{m}u_r^2\left(\hat{y}_{r\circ}^{m_1}-L^{-1}(\gamma)\hat{y}_{r\circ}^{\alpha}\right)},\sqrt{\sum\nolimits_{i=1}^{m}u_r^2\left(\hat{y}_{r\circ}^{m_2}+R^{-1}(\gamma)\hat{y}_{r\circ}^{\beta}\right)}\right],
\end{aligned}$$

$$\begin{aligned}
\tilde{\sigma}_\circ(v)&=\left[(\sigma_\circ(v))_\gamma^L,(\sigma_\circ(v))_\gamma^U\right]\\
&=\left[\sqrt{\sum\nolimits_{i=1}^{m}v_i^2(\hat{x}_{i\circ}^{m_1}-L^{-1}(\gamma)\hat{x}_{i\circ}^{\alpha})},\sqrt{\sum\nolimits_{i=1}^{m}v_i^2\left(\hat{x}_{i\circ}^{m_2}+R^{-1}(\gamma)\hat{x}_{i\circ}^{\beta}\right)}\right],
\end{aligned}$$

$$\begin{aligned}
\tilde{\sigma}_j(u,v)&=\left[(\sigma_j(u,v))_\gamma^L,(\sigma_j(u,v))_\gamma^U\right]\\
&=\left[\sqrt{\sum\nolimits_{r=1}^{s}u_r^2\hat{y}_{rj}^{m_1}+\sum\nolimits_{i=1}^{m}v_i^2\hat{x}_{ij}^{m_1}-L^{-1}(\gamma)\left(\sum\nolimits_{r=1}^{s}u_r^2\hat{y}_{rj}^{\alpha}+\sum\nolimits_{i=1}^{m}v_i^2\hat{x}_{ij}^{\alpha}\right)},\right.\\
&\left.\sqrt{\sum\nolimits_{r=1}^{s}u_r^2\hat{y}_{rj}^{m_2}+\sum\nolimits_{i=1}^{m}v_i^2\hat{x}_{ij}^{m_2}+R^{-1}(\gamma)\left(\sum\nolimits_{r=1}^{s}u_r^2\hat{y}_{rj}^{\beta}+\sum\nolimits_{i=1}^{m}v_i^2\hat{x}_{ij}^{\beta}\right)}\right],\\
&j=1,\ldots,n,
\end{aligned}$$

$\tilde{\delta} = \left[(\delta)^L_\gamma, (\delta)^U_\gamma\right] = \left[\delta^{m_1} - L^{-1}(\gamma)\delta^\alpha, \delta^{m_2} + R^{-1}(\gamma)\delta^\beta\right].$

Because of $(\sigma_\circ(u))^L_\gamma$, $(\sigma_\circ(v))^L_\gamma$ and $(\sigma_j(u,v))^L_\gamma$, Model (21) is a non-linear programming model. We derive the quadratic programming Model (22) by using the substitutions $\left(\theta^O_p\right)^2 = \left((\sigma_\circ(u))^L_\gamma\right)^2$, $\left(\theta^I_\circ\right)^2 = \left((\sigma_\circ(v))^L_\gamma\right)^2$ and $(\lambda_j)^2 = \left((\sigma_j(u,v))^L_\gamma\right)^2$ as follows:

$$\begin{aligned}
\max\quad & \varphi\\
& \varphi - \sum_{r=1}^{s} u_r\left(y^{m_2}_{r\circ} - L^{-1}(\gamma)\tilde{\bar{y}}^\beta_{r\circ}\right) + \theta^O_\circ \Phi^{-1}(\delta^{m_1} - L^{-1}(\gamma)\delta^\alpha) \leqslant 0,\\
& \sum_{i=1}^{m} v_i\left(x^{m_1}_{i\circ} - L^{-1}(\gamma)x^\alpha_{i\circ}\right) + \theta^I_\circ \Phi^{-1}(\delta^{m_1} - L^{-1}(\gamma)\delta^\alpha) \leqslant 1,\\
& \sum_{i=1}^{m} v_i\left(x^{m_1}_{i\circ} + L^{-1}(\gamma)x^\alpha_{i\circ}\right) - \theta^I_\circ \Phi^{-1}(\delta^{m_1} - L^{-1}(\gamma)\delta^\alpha) \geqslant 1,\\
& \sum_{r=1}^{s} u_r\tilde{\bar{y}}^{m_1}_{rj} - \sum_{i=1}^{m} v_i\tilde{\bar{x}}^{m_2}_{ij} - L^{-1}(\gamma)\left(\sum_{r=1}^{s} u_r\tilde{\bar{y}}^\alpha_{rj} + \sum_{i=1}^{m} v_i\tilde{\bar{x}}^\beta_{ij}\right)\\
& \quad + \lambda_j\Phi^{-1}(\delta^{m_1} - L^{-1}(\gamma)\delta^\alpha) \leqslant 0, \quad j = 1,\ldots,n,\\
& \left(\theta^O_\circ\right)^2 = \sum_{r=1}^{s} u_r^2\left(\hat{y}^{m_1}_{r\circ} - L^{-1}(\gamma)\hat{y}^\alpha_{r\circ}\right),\\
& \left(\theta^I_\circ\right)^2 = \sum_{i=1}^{m} v_i^2\left(\hat{x}^{m_1}_{i\circ} - L^{-1}(\gamma)\hat{x}^\alpha_{i\circ}\right),\\
& (\lambda_j)^2 = \sum_{r=1}^{s} u_r^2\hat{y}^{m_1}_{rj} + \sum_{i=1}^{m} v_i^2\hat{x}^{m_1}_{ij} - L^{-1}(\gamma)\left(\sum_{r=1}^{s} u_r^2\hat{y}^\alpha_{rj} + \sum_{i=1}^{m} v_i^2\hat{x}^\alpha_{ij}\right),\\
& j = 1,\ldots,n,\\
& u_r, v_i, \theta^I_\circ, \theta^O_\circ, \lambda_j \geqslant 0, \quad r = 1,\ldots,s;\ i = 1,\ldots,m;\ j = 1,\ldots,n.
\end{aligned} \tag{22}$$

The DMU under consideration is called possibilistic–probabilistic $\gamma - \tilde{\delta}$ efficient if the objective function of Model (22), φ, is greater than or equal to one at the possibility level γ and fuzzy probability level $\tilde{\delta}$; otherwise, it is called possibilistic–probabilistic $\gamma - \tilde{\delta}$ inefficient.

Here we extend the NPCP Model (17) with the fuzzy threshold for performance evaluation of a group of DMUs. The corresponding generic NPCP model with fuzzy δ, denoted by $\tilde{\delta} = (\delta^\alpha, \delta^{m_1}, \delta^{m_2}, \delta^\beta)$, is:

$$\begin{aligned}
\max\quad & \varphi\\
\text{s.t.}\quad & Nec\left[P(\varphi \leqslant \sum_{r=1}^{s} u_r\tilde{\bar{y}}_{rp}) \geqslant \tilde{\delta}\right] \geqslant \gamma,\\
& Nec\left[P\left(\sum_{i=1}^{m} v_i\tilde{\bar{x}}_{ip} \leqslant 1\right) \geqslant \tilde{\delta}\right] \geqslant \gamma,\\
& Nec\left[P\left(\sum_{i=1}^{m} v_i\tilde{\bar{x}}_{ip} \geqslant 1\right) \geqslant \tilde{\delta}\right] \geqslant \gamma,\\
& Nec\left[P\left(\sum_{r=1}^{s} u_r\tilde{\bar{y}}_{rj} - \sum_{i=1}^{m} v_i\tilde{\bar{x}}_{ij} \leqslant 0\right) \geqslant \tilde{\delta}\right] \geqslant \gamma, \quad j = 1,\ldots,n,\\
& u_r, v_i \geqslant 0, \quad r = 1,\ldots,s;\ i = 1,\ldots,m.
\end{aligned} \tag{23}$$

Let us also assume that there are n DMUs ($j = 1,\ldots,n$) to be evaluated, each consuming m Ra-Fu inputs $\tilde{\bar{x}}_{ij}$ ($i = 1,\ldots,m$) to produce s Ra-Fu outputs $\tilde{\bar{y}}_{rj}$ ($r = 1,\ldots,s$) and the inputs and outputs have normal distributions with fuzzy mean and fuzzy variance as denoted by:

$$\tilde{\bar{x}}_{ij} \sim N\left(\bar{x}_{ij}, \bar{\sigma}^2_{ij}\right), \quad \text{where } \bar{x}_{ij} = \left(x^\alpha_{ij}, x^{m_1}_{ij}, x^{m_2}_{ij}, x^\beta_{ij}\right)_{LR},$$
$$\bar{\sigma}^2_{ij} = \left(\hat{x}^\alpha_{ij}, \hat{x}^{m_1}_{ij}, \hat{x}^{m_2}_{ij}, \hat{x}^\beta_{ij}\right)_{LR}$$

$$\tilde{\bar{y}}_{rj} \sim N\left(\bar{y}_{rj}, \bar{\sigma}^2_{rj}\right), \quad \text{where } \bar{y}_{rj} = \left(y^\alpha_{rj}, y^{m_1}_{rj}, y^{m_2}_{rj}, y^\beta_{rj}\right)_{LR},$$
$$\bar{\sigma}^2_{rj} = \left(\hat{y}^\alpha_{rj}, \hat{y}^{m_1}_{rj}, \hat{y}^{m_2}_{rj}, \hat{y}^\beta_{rj}\right)_{LR}$$

In order to solve the proposed NPCP Model (23), we apply Part (b) of Theorem 6 to determine the deterministic model expressed as follows:

$$\sum_{i=1}^{m} v_i\left(x^{m_2}_{ip} + x^\beta_{ip}R^{-1}(\gamma)\right) - 1 + \sum_{i=1}^{m} v_i^2\left(\hat{x}^{m_2}_{ip} + R^{-1}(\gamma)\hat{x}^\beta_{ip}\right)\Phi^{-1}(\tilde{\delta}^{\gamma R}) \leqslant 0$$

$$\begin{aligned}
\max\quad & \bar{\varphi}\\
\text{s.t.}\quad & \bar{\varphi} - \sum_{r=1}^{s} u_r y^{m_2}_{r\circ} + R^{-1}(1-\gamma)\sum_{r=1}^{s} u_r y^\beta_{r\circ} + \rho^O_\circ \Phi^{-1}(\delta^{m_2} + R^{-1}(\gamma)\delta^\beta) \leqslant 0,\\
& \sum_{i=1}^{m} v_i x^{m_2}_{i\circ} + R^{-1}(1-\gamma)\sum_{i=1}^{m} v_i x^\beta_{i\circ} + \rho^I_\circ \Phi^{-1}(\delta^{m_2} + R^{-1}(\gamma)\delta^\beta) \leqslant 1,\\
& \sum_{i=1}^{m} v_i x^{m_1}_{i\circ} - R^{-1}(1-\gamma)\sum_{i=1}^{m} v_i x^\alpha_{i\circ} - \rho^I_\circ \Phi^{-1}(\delta^{m_2} + R^{-1}(\gamma)\delta^\beta) \geqslant 1,\\
& \sum_{r=1}^{s} u_r y^{m_2}_{rj} - \sum_{i=1}^{m} v_i x^{m_1}_{ij} + R^{-1}(1-\gamma)\left(\sum_{r=1}^{s} u_r y^\beta_{rj} + \sum_{i=1}^{m} v_i x^\alpha_{ij}\right)\\
& \quad + h_j\Phi^{-1}(\delta^{m_2} + R^{-1}(\gamma)\delta^\beta) \leqslant 0, \quad j = 1,\ldots,n,\\
& \left(\rho^O_\circ\right)^2 = \sum_{r=1}^{s} u_r^2\left(\hat{y}^{m_2}_{r\circ} + R^{-1}(1-\gamma)\hat{y}^\beta_{r\circ}\right),\\
& \left(\rho^I_\circ\right)^2 = \sum_{i=1}^{m} v_i^2\left(\hat{x}^{m_2}_{i\circ} + R^{-1}(1-\gamma)\hat{x}^\beta_{i\circ}\right),\\
& (h_j)^2 = \sum_{r=1}^{s} u_r^2\left(\hat{y}^{m_2}_{rj} + R^{-1}(1-\gamma)\hat{y}^\beta_{rj}\right) + \sum_{i=1}^{m} v_i^2\left(\hat{x}^{m_2}_{ij} + R^{-1}(1-\gamma)\hat{x}^\beta_{ij}\right),\\
& j = 1,\ldots,n,\\
& u_r, v_i, \rho^O_\circ, \rho^I_\circ, h_j \geqslant 0, \quad r = 1,\ldots,s;\ i = 1,\ldots,m;\ j = 1,\ldots,n.
\end{aligned} \tag{24}$$

The $DMU_\circ$ is called necessity-probabilistic $\gamma - \tilde{\delta}$ efficient if $\varphi_\circ = 1$ where $\varphi_\circ = \bar{\varphi}_\circ / \max_j(\bar{\varphi}_j)$.

We should note that although we use the value based DEA model in this paper, we could alternatively use the envelopment model presented in Appendix C.

6. Numerical illustrations

In this paper, we initially developed three DEA-based models to measure the efficiency when the inputs and outputs are the Ra-Fu variables with Poisson, uniform and normal distributions. We then developed the possibility–probability and the necessity–probability DEA models with Ra-Fu parameters on a possibility space where the Ra-Fu inputs and Ra-Fu outputs are normal distributed with fuzzy means and fuzzy variances. We finally developed the general possibility–probability and necessity–probability DEA models with fuzzy thresholds.

In this section, we provide three numerical examples to illustrate the applications of the proposed random fuzzy expected value operator and possibility–probability models in performance assessment. We use the hypothetical example 1 to exemplify model (9) for three cases where the inputs and outputs are Ra-Fu variables with Poisson, uniform and normal distributions presented in Section 3. To illustrate Models (15) and (16) we deploy the numeric example 2 where inputs and outputs have a normal distribution presented in Section 4. We present the numerical example 3 to demonstrate Models (21) and (24) presented in Section 5.

Example 1. In this example we examine three cases in which five DMUs are evaluated with two Ra-Fu inputs and the two Ra-Fu outputs. In Case I, we assume that the inputs and outputs are Ra-Fu variables with Poisson distributions, denoted respectively by $\tilde{\tilde{x}}_{ij} \sim P(\bar{k}_{ij})$ and $\tilde{\tilde{y}}_{rj} \sim P(\bar{\lambda}_{rj})$, where $\bar{k}_{ij} = \left(x^{\alpha}_{ij}, x^{m_1}_{ij}, x^{m_2}_{ij}, x^{\beta}_{ij}\right)_{LR}$ and $\bar{\lambda}_{rj} = \left(y^{\alpha}_{rj}, y^{m_1}_{rj}, y^{m_2}_{rj}, y^{\beta}_{rj}\right)_{LR}$ are presented in Table 1. In Case II, we assume the uniformly-distributed Ra-Fu inputs and outputs that are denoted by $\tilde{\tilde{x}}_{ij} \sim U(\bar{x}_{ij}, \hat{x}_{ij})$ and $\tilde{\tilde{y}}_{rj} \sim U(\bar{y}_{rj}, \hat{y}_{rj})$, where $\tilde{\bar{x}}_{ij} = \left(x^{\alpha}_{ij}, x^{m_1}_{ij}, x^{m_2}_{ij}, x^{\beta}_{ij}\right)_{LR}$, $\hat{x}_{ij} = \left(x^{\bar{\alpha}}_{ij}, x^{\bar{m}_1}_{ij}, x^{\bar{m}_2}_{ij}, x^{\bar{\beta}}_{ij}\right)_{LR}$, $\bar{y}_{rj} = \left(y^{\alpha}_{rj}, y^{m_1}_{rj}, y^{m_2}_{rj}, y^{\beta}_{rj}\right)_{LR}$, $\hat{y}_{rj} = \left(y^{\bar{\alpha}}_{rj}, y^{\bar{m}_1}_{rj}, y^{\bar{m}_2}_{rj}, y^{\bar{\beta}}_{rj}\right)_{LR}$ are reported in Table 1. Finally, in Case III, the Ra-Fu inputs and Ra-Fu outputs are normally-distributed as denoted by $\tilde{\tilde{x}}_{ij} \sim N\left(\bar{x}_{ij}, \sigma^2_{ij}\right)$ and $\tilde{\tilde{y}}_{rj} \sim N\left(\bar{y}_{rj}, \sigma^2_{rj}\right)$ where their means are characterized by the fuzzy numbers $\bar{x}_{ij} = \left(x^{\alpha}_{ij}, x^{m_1}_{ij}, x^{m_2}_{ij}, x^{\beta}_{ij}\right)_{LR}$ and $\bar{y}_{rj} = \left(y^{\alpha}_{rj}, y^{m_1}_{rj}, y^{m_2}_{rj}, y^{\beta}_{rj}\right)_{LR}$ as shown in Table 1. We apply the Ra-Fu expected value Model (9) to calculate the efficiency measure of the DMUs as reported in the last column of Table 1. From Table 1, DMUs 3 and 5 in Case I, DMUs 1 and 5 in Case II and DMUs 2, 4 and 5 in Case III are identified as the efficient units with a unity score. We do not compare these results because different Ra-Fu variables in these three cases lead to incomparability with each other.

The proposed Ra-Fu expected value model is used for measuring the efficiency of a group of DMUs as well as providing adequate discriminatory power in the presence of the Ra-Fu inputs and outputs. In addition to the three distributions considered in this study, the proposed Ra-Fu expected value model can be easily extended to any probability distribution for performance assessment in a random fuzzy environment.

Example 2. In this example we examine ten DMUs with two Ra-Fu inputs and one Ra-Fu output on a possibility space. The Ra-Fu inputs, $\tilde{\tilde{x}}_{ij}$, and Ra-Fu outputs, $\tilde{\tilde{y}}_{rj}$, are normally-distributed with triangular fuzzy means and triangular fuzzy variances as $\tilde{\tilde{x}}_{ij} \sim N\left(\bar{x}_{ij}, \bar{\sigma}^2_{ij}\right)$ where $\bar{x}_{ij} = \left(x^{\alpha}_{ij}, x^{m}_{ij}, x^{\beta}_{ij}\right)_{LR}$ and $\bar{\sigma}^2_{ij} = \left(\hat{x}^{\alpha}_{ij}, \hat{x}^{m}_{ij}, \hat{x}^{\beta}_{ij}\right)_{LR}$; and $\tilde{\tilde{y}}_{rj} \sim N\left(\bar{y}_{rj}, \bar{\sigma}^2_{rj}\right)$ where $\bar{y}_{rj} = \left(y^{\alpha}_{rj}, y^{m}_{rj}, y^{\beta}_{rj}\right)_{LR}$ and $\bar{\sigma}^2_{rj} = \left(\hat{y}^{\alpha}_{rj}, \hat{y}^{m}_{rj}, \hat{y}^{\beta}_{rj}\right)_{LR}$. The data are shown in Table 2.

Using "what if" analysis in performance evaluation for the possibility–probability developed models, we first assume δ =0.5 when γ assumes the four different measures of 0.1, 0.3, 0.5 and 0.9 (presented as Case I). We then assume γ = 0.5 when δ assumes the four different measures as 0.1, 0.3, 0.5 and 0.9 (presented as Case II). Table 3 reports the results of Models (15) and (16) for the above-mentioned levels.

For all pre-defined levels of Cases I and II, DMU 1, DMU 2 and DMU 10 are possibilistic–probabilistic γ–δ-efficient while DMU 9 is always γ–δ-inefficient. In addition, the ranking order of units is also reported in Table 3 (see in the parenthesis of each cell). It is interesting that in Cases I and II, DMU 1 and DMU 9 take the superior and inferior positions, respectively, for all levels. Note that (γ = 0.5, δ = 0.5) is reported twice in Table 3 for each case in order to provide a comprehensive comparison. Let us consider (γ = 0.5, δ = 0.5) as an index for behavioral analysis of γ and δ in the change of efficiency scores. We hence provide the efficiency scores with a δ-increase (γ = 0.5, δ = 0.9) and with a γ-increase (γ = 0.9, δ = 0.5). As is shown in Table 3, the scores with (γ = 0.9, δ = 0.5) are generally smaller than the scores with (γ = 0.5, δ =0.9). In short, when δ is fixed at 0.5 and γ increases slightly, the corresponding score for each unit is mostly decreasing. The same behavior can be observed when γ is fixed and δ is decreased (see Table 3).

In addition to the possibility–probability models, we can take into account the necessity–probability models in order to provide

Table 1
Fuzzy parameters and efficiencies for two Ra-Fu inputs and two Ra-Fu outputs.

Distribution		Input 1		Input 2		Output 1		Output 2		Efficiency
Poisson (Case I)	DMU	$\bar{k}$		$\bar{k}$		$\bar{\lambda}$		$\bar{\lambda}$		
	1	(5.5,17,18,10)		(7,8,9,10)		(15,26,30,15)		(12,15,25,35)		0.91
	2	(6,7.5,9.5,10)		(1.5,3.7,5,6)		(9,12,13,10)		(10,16,17,19)		0.97
	3	(10,13,15,12)		(3.5,5.5,7,8)		(9,18,20,24)		(23,26,28,25)		1.00
	4	(17,24,30,17)		(6,7,9,6)		(15,17,28,15)		(13.5,15.5,26,15.5)		0.56
	5	(21,22,30,26)		(2.3,4,6,5)		(23,30,39,40)		(29,35,44,46)		1.00
Uniform (Case II)	DMU	$\bar{x}$	$\hat{x}$	$\bar{x}$	$\hat{x}$	$\bar{y}$	$\hat{y}$	$\bar{y}$	$\hat{y}$	
	1	(4,5,6,7)	(1.2,2,4,5.5)	(2,5,6,9)	(2.2,4.6,7,8.5)	(25,30,35,25)	(20,25,30,35)	(50,56,60,65)	(30,43,64,66)	1.00
	2	(3.1,4.1,5.1,7.1)	(1.5,2.4,3.4,4.5)	(11,14,15,11)	(9,6,7,9)	(19,25,30,40)	(10,16,17,19)	(29,35,39,49)	(17,26,37,40)	0.90
	3	(6,8,9,6)	(4.5,6.5,7.5,5)	(13,18,19,13)	(4.5,16.5,18,5)	(40,48,60,40)	(23,26,28,25)	(40,48,60,40)	(33,36,38,35)	0.84
	4	(3.5,4,10,12)	(2.5,3.5,6,2.5)	(5,7,10,8)	(3,4.5,6,5.5)	(15,27,30,24)	(13,14.5,16,15.5)	(15,77,80,24)	(33,54,56,40)	0.91
	5	(2.6,4.6,5.6)	(3.3,4.3,4,5)	(5,6.6,9.6,5)	(7,8,9,7)	(10,14,19.6,15)	(9,19,19,12)	(10,94,99.6,15)	(15,89,90,30)	1.00
Normal (Case III)	DMU	$\bar{x}$		$\bar{x}$		$\bar{y}$		$\bar{y}$		
	1	((4,14,7),2)		((3,8,6),1.5)		((12,14,12),2)		((6,12,6),0.01)		0.71
	2	((5,6,7),3)		((1.5.4,9,8),2)		((17,19,17),1)		((7,11,7),1)		1.00
	3	((10,12,13),1)		((4,6,10),1.5)		((7,16,9),1.2)		((7.5,16,9.5),2)		0.70
	4	((7,19,9),2)		((5,8,9),0.1)		((10,15,16),1.5)		((10,25,16),3)		1.00
	5	((8,12,8),1.1)		((5,6,7),0.1)		((12,18,14),1)		((12,18,14),1)		1.00

Table 2
Fuzzy mean and fuzzy variance of the DMUs with normal distribution.

DMU	Ra-Fu input 1	Ra-Fu input 2	Ra-Fu output
1	N ((12,16,20),(0.9,1,1.1))	N ((0.3,7,13.7),(1,2,3))	N ((36.3,45.3,54.3),(5,10,15))
2	N ((10,15,20),(0.3,1,1.7))	N ((19.7,25,30.3),(1.5,2,2.5))	N ((32.1,40.1,54.3),(3,5,7))
3	N ((12,16,20),(0.4,2.1,3.8))	N ((11,17,23),(2.5,3,3.5))	N ((32.6,39.6,46.6),(9,15,21))
4	N ((15,17,19),(0.4,1.5,2.6))	N ((11.5,15,18.57),(1.5,3,4.5))	N ((31.5,39.,46.5),(.4,10,13.6))
5	N ((14,18,22),(0.2,0.9,1.6))	N ((10,13,16),(3,4,5))	N ((30.7,34.2,37.7),(5,8,11))
6	N ((7,11,15),(0.2,1,1.8))	N ((6,11,16),(0.3,2,3.7))	N ((14.1,20.1,26.1),(3.5,5,6.5))
7	N ((22,25,28),(1,2,3))	N ((5,9,13),(1,2,3))	N ((21,26.5,32),(5,7,9))
8	N ((11,15,19),(2.4,3,3.6))	N ((10.7,13.8,16.9),(5.3,6.5,7.7))	N ((29.9,35.9,41.9),(10,12,14))
9	N ((11,16,21),(3.5,5,6.5))	N ((13.9,18.1,22.3),(2.25,3,3.75))	N ((10.9,17.4,23.9),(2.25,3,3.75))
10	N ((8,12,16),(1.5,2,2.5))	N ((10.8,16.5,22.2),(1.9,2.5,3.1))	N ((27.7,34.3,40.9),(2.5,4,5.5))

Table 3
Possibility–probability scores of the DMUs and their rankings.

δ	0.5				0.1	0.3	0.5	0.9
γ	0.1	0.3	0.5	0.9	0.5			
DMU 1	13.000 (1)	6.6956 (1)	3.4611 (1)	1.2608 (1)	8.5054 (1)	5.0654 (1)	3.4611 (1)	1.8738 (1)
DMU 2	1.6534 (3)	1.9156 (3)	1.5565 (3)	1.0397 (3)	1.7363 (3)	1.7771 (3)	1.5565 (3)	1.1965 (2)
DMU 3	1.3586 (5)	1.6251 (4)	1.3582 (4)	0.9529 (4)	1.6730 (5)	1.6105 (4)	1.3582 (4)	0.9863 (4)
DMU 4	0.9671 (8)	1.2098 (8)	1.0547 (8)	0.8007 (6)	1.2247 (8)	1.2214 (8)	1.0547 (8)	0.7909 (7)
DMU 5	0.9520 (9)	1.1623 (9)	0.9913 (9)	0.7247 (7)	1.1141 (9)	1.1347 (9)	0.9913 (9)	0.7617 (8)
DMU 6	1.2648 (7)	1.4285 (7)	1.1324 (6)	0.7205 (8)	1.4223 (7)	1.3529 (7)	1.1324 (6)	0.8067 (6)
DMU 7	1.4211 (4)	1.4673 (6)	1.0600 (7)	0.5434 (9)	1.6818 (4)	1.375 (6)	1.0600 (7)	0.6437 (9)
DMU 8	1.3297 (6)	1.5844 (5)	1.3201 (5)	0.9225 (5)	1.6585 (6)	1.5737 (5)	1.3201 (5)	0.9044 (5)
DMU 9	0.7421 (10)	0.8465 (10)	0.6748 (10)	0.4295 (10)	0.8972 (10)	0.8215 (10)	0.6748 (10)	0.4525 (10)
DMU 10	1.7583 (2)	2.0393 (2)	1.6588 (2)	1.1109 (2)	2.0337 (2)	1.9573 (2)	1.6588 (2)	1.1832 (3)

practical and insightful information to the DM. We thus used the proposed necessity–probability Models (18) and (19) and applied values of δ and γ similar to the ones used in the above cases and obtained the results presented in Table 4.

According to Definition 5b, the efficiency of the necessity approach must be less than or equal to the efficiency of the possibility approach at a given threshold level as is shown in Tables (3) and (4). Interestingly, in Case I (or Case II), the efficiency measure strictly decreases when γ (or δ) is higher and δ (or γ) is fixed (see Table 4). Seemingly smaller γ and δ give more opportunities for DMUs in Cases I and II, respectively, to achieve the higher scores. The efficiency scores are adjusted using the normalization formulation as shown in Table 5.

In Case I, similar to the possibilistic–probabilistic, DMUs 1 and 9 have the highest and the lowest scores, respectively, at the different levels. We should note that the ranking order of the DMUs in Case I is indifferent to various levels. In other words, each DMU is placed in the same position without consideration of the pre-defined thresholds. Analogously, in Case II the same ranking order for DMUs 5, 6 and 9 is derived at the different levels (see Table 5).

Example 3. In this example we use the same data provided in Example 2. However, the threshold level δ is characterized with the triangular fuzzy number as $\tilde{\delta} = (\delta^{\alpha}, \delta^{m}, \delta^{\beta})$. In this example, we also take into account two Cases I and II and apply "what if" analysis to the DM. Case I involves the fixed $\tilde{\delta} = (0.10, 0.5, 0.10)$ and four values for $\gamma \in \{0.1, 0.3, 0.5, 0.9\}$ while Case II consists of the fixed $\gamma = 0.5$ and four fuzzy numbers for $\tilde{\delta} \in \{(0.1, 0.1, 0.1), (0.1, 0.3, 0.1), (0.1, 0.5, 0.1), (0.1, 0.7, 0.1)\}$. In order to compare the four $\tilde{\delta}$s, we select $\tilde{\delta}$ such that $(0.1,0.1,0.1) \leqslant (0.1,0.3,0.1) \leqslant (0.1,0.5,0.1) \leqslant (0.1,0.7,0.1)$ as is shown in Fig. 2. The corresponding results derived from Model (21) are reported in Table 6.

Table 4
Necessity–probability scores of the DMUs.

δ	0.5				0.1	0.3	0.5	0.9
γ	0.1	0.3	0.5	0.9	0.5			
DMU 1	0.6594	0.6440	0.6270	0.5878	0.7885	0.6535	0.6270	0.4488
DMU 2	0.5810	0.5655	0.5486	0.5100	0.7350	0.6210	0.5486	0.376
DMU 3	0.5739	0.5578	0.5401	0.4994	0.7671	0.6239	0.5401	0.3453
DMU 4	0.5482	0.5296	0.5095	0.4639	0.6988	0.5809	0.5095	0.3417
DMU 5	0.4558	0.4514	0.4454	0.4276	0.5887	0.4997	0.4454	0.3127
DMU 6	0.3867	0.3717	0.3555	0.3189	0.5156	0.3937	0.3555	0.2211
DMU 7	0.2735	0.2622	0.2500	0.2226	0.3391	0.2833	0.2500	0.1716
DMU 8	0.5507	0.5396	0.5266	0.4936	0.7824	0.6184	0.5266	0.3396
DMU 9	0.2365	0.2219	0.2062	0.1714	0.3207	0.2466	0.2062	0.1251
DMU 10	0.6216	0.6060	0.5889	0.5495	0.8476	0.6783	0.5889	0.3934

Table 5
Normalized necessity–probability score of the DMUs with their rankings.

δ	0.5				0.1	0.3	0.5	0.9
γ	0.1	0.3	0.5	0.9	0.5			
DMU 1	1.0000 (1)	1.0000 (1)	1.0000 (1)	1.0000 (1)	0.9303 (2)	0.9634 (2)	1.0000 (1)	1.0000 (1)
DMU 2	0.8811 (3)	0.8781 (3)	0.8750 (3)	0.8676 (3)	0.8672 (5)	0.9155 (4)	0.8750 (3)	0.8378 (3)
DMU 3	0.8703 (4)	0.8661 (4)	0.8614 (4)	0.8496 (4)	0.9050 (4)	0.9198 (3)	0.8614 (4)	0.7694 (4)
DMU 4	0.8314 (6)	0.8224 (6)	0.8126 (6)	0.7892 (6)	0.8244 (6)	0.8564 (6)	0.8126 (6)	0.7614 (5)
DMU 5	0.6912 (7)	0.7009 (7)	0.7104 (7)	0.7275 (7)	0.6945 (7)	0.7367 (7)	0.7104 (7)	0.6967 (7)
DMU 6	0.5864 (8)	0.5772 (8)	0.5670 (8)	0.5425 (8)	0.6083 (8)	0.5804 (8)	0.5670 (8)	0.4926 (8)
DMU 7	0.4148 (9)	0.4071 (9)	0.3987 (9)	0.3787 (9)	0.4001 (9)	0.4177 (9)	0.3987 (9)	0.3824 (9)
DMU 8	0.8352 (5)	0.8379 (5)	0.8399 (5)	0.8397 (5)	0.9231 (3)	0.9117 (5)	0.8399 (5)	0.7567 (6)
DMU 9	0.3587 (10)	0.3446 (10)	0.3289 (10)	0.2916 (10)	0.3784 (10)	0.3636 (10)	0.3289 (10)	0.2787 (10)
DMU 10	0.9427 (2)	0.9410 (2)	0.9392 (2)	0.9348 (2)	1.0000 (1)	1.0000 (1)	0.9392 (2)	0.8766 (2)

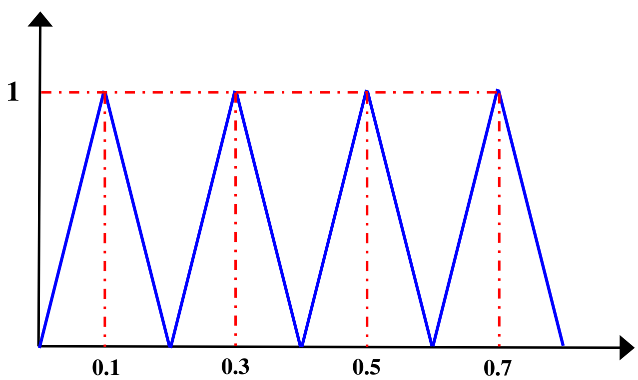

Fig. 2. Triangular fuzzy numbers for δ in Example 3.

In both cases, DMUs 1, 2 and 10 are possibilistic–probabilistic γ–$\bar{\delta}$-efficient at all the given levels whereas DMU 9 is generally γ–$\bar{\delta}$-inefficient. When $\bar{\delta}$ (or γ) is fixed and γ (or $\bar{\delta}$) takes the higher value, the efficiency score strictly decreases (see Table 6). To provide more insight, we applied the necessity-probabilistic Model (24) for this example as well and the obtained results reported in Table 7.

The results of the necessity-probabilistic method reported in Table 7 are significantly lower compared to the results of the possibilistic–probabilistic method presented in Table 6. Note here that we expected to achieve this result with respect to Definition 5b. In addition, when γ (or γ) is increased and $\bar{\delta}$ (or γ) is fixed, the efficiency always decreases (see Table 7). To specify the necessity-probabilistic efficient units for the given threshold, the normalization scores are presented in Table 8.

In Case I, DMUs 1 and 9 had the highest and lowest scores, respectively, at different levels. In addition, as was shown in Table 8, the first three γs in Case I had the same ranking order but there was a slight change in the ranking of DMUs when $\gamma = 0.9$. For Case II, the DMU 9 again had the worst ranking for various lev-

Table 6
Possibility–probability scores of the DMUs with their fuzzy thresholds.

δ	(0.1,0.5,0.1)				(0.1,0.1,0.1)	(0.1,0.3,0.1)	(0.1,0.5,0.1)	(0.1,0.7,0.1)
γ	0.1	0.3	0.5	0.9	0.5			
DMU 1	15.000	5.3000	3.0389	1.3669	12.000	4.1791	3.0389	2.3198
DMU 2	2.1040	2.0679	1.3079	1.0563	1.9225	1.4638	1.3079	1.1771
DMU 3	2.0701	1.967	1.1761	0.9777	1.9053	1.3558	1.1761	1.0283
DMU 4	1.5540	1.4957	0.9584	0.8319	1.4656	1.0868	0.9584	0.8509
DMU 5	1.9414	1.7129	0.9191	0.7516	1.5610	1.0303	0.9191	0.8247
DMU 6	1.6435	1.5441	0.8700	0.7242	1.4823	1.019	0.8700	0.7485
DMU 7	2.8398	2.3421	0.9073	0.5836	2.0269	1.1395	0.9073	0.7343
DMU 8	2.4371	1.8419	1.1619	0.9548	2.1135	1.3802	1.1619	0.9901
DMU 9	1.0110	0.9712	0.4837	0.4343	0.9608	0.582	0.4837	0.4039
DMU 10	2.9379	2.6381	1.3510	1.1467	2.4457	1.6525	1.3510	1.2308

Table 7
Necessity–probability scores of the DMUs with their fuzzy thresholds.

δ	(0.1,0.5,0.1)				(0.1,0.1,0.1)	(0.1,0.3,0.1)	(0.1,0.5,0.1)	(0.1,0.7,0.1)
γ	0.1	0.3	0.5	0.9	0.5			
DMU 1	0.5733	0.9567	0.7787	0.6643	0.7559	0.7828	0.7787	0.7990
DMU 2	0.4934	0.8013	0.7079	0.5806	0.7389	0.7296	0.7079	0.7140
DMU 3	0.4688	0.7746	0.6951	0.6004	0.7783	0.7034	0.6951	0.7343
DMU 4	0.4179	0.6957	0.6473	0.5975	0.7046	0.7113	0.6473	0.6741
DMU 5	0.3799	0.6071	0.5579	0.5034	0.5906	0.5653	0.5579	0.5732
DMU 6	0.3389	0.5343	0.4569	0.3588	0.4806	0.5084	0.4569	0.4874
DMU 7	0.2179	0.4939	0.3657	0.2763	0.3354	0.4481	0.3657	0.3914
DMU 8	0.4577	0.7493	0.6711	0.5810	0.7561	0.7444	0.6711	0.7131
DMU 9	0.1978	0.3017	0.2667	0.2022	0.3126	0.2961	0.2667	0.2892
DMU 10	0.5246	0.8659	0.7641	0.6316	0.8330	0.7952	0.7641	0.7902

Table 8
Normalized necessity–probability score of the DMUs and their rankings.

δ	(0.1,0.5,0.1)				(0.1,0.1,0.1)	(0.1,0.3,0.1)	(0.1,0.5,0.1)	(0.1,0.7,0.1)
γ	0.1	0.3	0.5	0.9	0.5			
DMU 1	1.0000 (1)	1.0000 (1)	1.0000 (1)	1.0000 (1)	0.9074 (4)	0.9844 (2)	1.0000 (1)	1.0000 (1)
DMU 2	0.8606 (3)	0.8376 (3)	0.9091 (3)	0.8740 (6)	0.8870 (5)	0.9175 (4)	0.9091 (3)	0.8936 (4)
DMU 3	0.8177 (4)	0.8097 (4)	0.8926 (4)	0.9038 (3)	0.9343 (2)	0.8846 (6)	0.8926 (4)	0.9190 (3)
DMU 4	0.7289 (6)	0.7272 (6)	0.8313 (6)	0.8994 (4)	0.8459 (6)	0.8945 (5)	0.8313 (6)	0.8437 (6)
DMU 5	0.6627 (7)	0.6346 (7)	0.7165 (7)	0.7578 (7)	0.7090 (7)	0.7109 (7)	0.7165 (7)	0.7174 (7)
DMU 6	0.5911 (8)	0.5585 (8)	0.5867 (8)	0.5401 (8)	0.5770 (8)	0.6393 (8)	0.5867 (8)	0.6100 (8)
DMU 7	0.3801 (9)	0.5163 (9)	0.4696 (9)	0.4159 (9)	0.4026 (9)	0.5635 (9)	0.4696 (9)	0.4899 (9)
DMU 8	0.7984 (5)	0.7832 (5)	0.8618 (5)	0.8746 (5)	0.9077 (3)	0.9361 (3)	0.8618 (5)	0.8925 (5)
DMU 9	0.3450 (10)	0.3154 (10)	0.3425 (10)	0.3044 (10)	0.3753 (10)	0.3724 (10)	0.3425 (10)	0.3620 (10)
DMU 10	0.9151 (2)	0.9051 (2)	0.9813 (2)	0.9508 (2)	1.0000 (1)	1.0000 (1)	0.9813 (2)	0.9890 (2)

Table 9
Fuzzy mean and fuzzy variance of the DMUs with normal distribution.

DMU	Ra-Fu input 1	Ra-Fu input 2	Ra-Fu output 1	Ra-Fu output 2
DMU 1	N ((9,48,9),1)	N ((0.62,12.46,0.62),1)	N ((13,110,13),1)	N ((14,144,14),1)
DMU 2	N (4,40,4),1)	N ((0.64,12.81,0.64),1)	N ((8,88,8),1)	N ((10,142,10),1)
DMU 3	N ((3,47,3),1)	N ((0.64,12.72,0.64),1)	N ((9,90,9),1)	N ((13,134,13),1)
DMU 4	N ((7,42,7),1)	N ((0.61,12.1,0.61),1)	N ((17,120,17),1)	N ((10,194,10),1)
DMU 5	N ((11,55,11),1)	N ((0.5,10.07,0.5),1)	N ((15,88,15),1)	N ((17,173,17),1)
DMU 6	N ((6,54,6),1)	N ((0.83,16.53,0.83),1)	N ((11,86,11),1)	N ((20,168,20),1)
DMU 7	N ((7,46,7),1)	N ((0.79,15.7,0.79),1)	N ((7,112,7),1)	N ((22,183,22),1)
DMU 8	N ((3,42,3),1)	N ((0.43,8.68,0.43),1)	N ((8,95,8),1)	N ((16,173,16),1)
DMU 9	N ((6,38,6),1)	N ((0.61,12.14,0.61),1)	N ((12,110,12),1)	N ((11,163,11),1)
DMU 10	N ((8,58,8),1)	N ((0.4,7.96,0.4),1)	N ((11,112,11),1)	N ((13,187,13),1)
DMU 11	N ((5,54,5),1)	N ((0.68,13.68,0.68),1)	N ((5,98,5),1)	N ((13,162,13),1)
DMU 12	N ((3,56,3),1)	N ((0.76,15.1,0.76),1)	N ((6,96,6),1)	N ((18,150,18),1)
DMU 13	N ((6,58,6),1)	N ((0.79,15.73,0.79),1)	N ((4,88,4),1)	N ((13,159,13),1)
DMU 14	N ((7,44,7),1)	N ((0.79,15.82,0.79),1)	N ((4,94,4),1)	N ((13,153,13),1)
DMU 15	N ((2,38,2),1)	N ((0.72,14.39,0.72),1)	N ((5,92,5),1)	N ((16,177,16),1)
DMU 16	N ((3,42,3),1)	N ((0.58,11.69,0.58),1)	N ((8,78,8),1)	N ((22,180,22),1)
DMU 17	N ((4,46,4),1)	N ((0.48,9.69,0.48),1)	N ((9,82,9),1)	N ((22,184,22),1)
DMU 18	N ((2,40,2),1)	N ((0.7,14.02,0.7),1)	N ((9,84,9),1)	N ((14,180,14),1)
DMU 19	N ((3,38,3),1)	N ((0.78,15.64,0.78),1)	N ((12,92,12),1)	N ((21,179,21),1)
DMU 20	N ((3,40,3),1)	N ((0.61,12.28,0.61),1)	N ((10,106,10),1)	N ((17,144,17),1)
DMU 21	N ((6,42,6),1)	N ((0.73,14.55,0.73),1)	N ((10,110,10),1)	N ((19,161,19),1)
DMU 22	N ((8,51,8),1)	N ((0.66,13.17,0.66),1)	N ((10,95,10),1)	N ((12,151,12),1)
DMU 23	N ((4,43,4),1)	N ((0.56,11.15,0.56),1)	N ((9,98,9),1)	N ((12,154,12),1)
DMU 24	N ((3,38,3),1)	N ((0.2,3.92,0.2),1)	N ((6,86,6),1)	N ((11,177,11),1)
DMU 25	N ((3,42,3),1)	N ((0.6,11.92,0.6),1)	N ((6,92,6),1)	N ((18,151,18),1)
DMU 26	N ((11,58,11),1)	N ((0.4,8,0.4),1)	N ((6,96,6),1)	N ((7,149,7),1)
DMU 27	N ((9,60,9),1)	N ((0.67,13.86,0.67),1)	N ((7,104,7),1)	N ((13,157,13),1)
DMU 28	N ((11,54,11),1)	N ((0.62,12.38,0.62),1)	N ((18,106,18),1)	N ((15,166,15),1)
DMU 29	N ((5,42,5),1)	N ((0.29,5.858,0.29),1)	N ((15,92,15),1)	N ((12,177,12),1)
DMU 30	N ((6,38,6),1)	N ((0.51,10.11,0.51),1)	N ((13,80,13),1)	N ((13,180,13),1)

els while DMUs 10 and 1 were placed first in the ranking for the first two and the second two levels, respectively.

7. Case study

In this section we present a case study to demonstrate the applicability of the proposed models in a supply chain for Technopower[2] Company with 30 suppliers, two Ra-Fu inputs and two Ra-Fu outputs on a possibility space. Technopower is a manufacturer of emergency power generator sets, based in the Middle East, with more than 15 years in the business. The company sells their emergency power generators to retail and commercial customers. In this supply chain evaluation problem the shipping costs and the demand imposed by the retail and commercial customers are random variables taking fuzzy parameters. The two inputs in this problem are the shipping costs for the retail and commercial customers. The two outputs in this problem are the retail and commercial demands for the emergency power generators. The shipping cost is a normally distributed variable with an unknown expected value μ because of the changing gasoline prices while the shipping time is a fuzzy variable because of the uncertainties associated with the traffic patterns. Furthermore, the amount of demand imposed by the customer is a normally distributed variable with an unknown expected value μ because of the seasonal variations while the price of the product is a fuzzy variable driven by product supply and demand in the market.

In this case study, we used Ra-Fu variables to deal with these uncertain parameters by combining randomness and fuzziness. The Ra-Fu inputs, $\tilde{\bar{x}}_{ij}$, and Ra-Fu outputs, $\tilde{\bar{y}}_{rj}$, are normally-distributed with triangular fuzzy means and crisp variances as $\tilde{\bar{x}}_{ij} \sim N\left(\bar{x}_{ij}, \sigma^2_{ij}\right)$ where $\bar{x}_{ij} = \left(x^{\alpha}_{ij}, x^{m}_{ij}, x^{\beta}_{ij}\right)_{LR}$; and $\tilde{\bar{y}}_{rj} \sim N\left(\bar{y}_{rj}, \sigma^2_{rj}\right)$ where $\bar{y}_{rj} = \left(y^{\alpha}_{rj}, y^{m}_{rj}, y^{\beta}_{rj}\right)_{LR}$. The data are shown in Table 9.

We performed a "what if" analysis on the performance evaluation of the possibility–probability models by first assuming δ = 0.5 and γ assuming the four different measures of 0.1, 0.3, 0.5 and 0.9 (presented as *Case I*). We then assumed γ = 0.5 with δ assuming the

[2] The name is changed to protect the anonymity of the company.

Table 10
Possibility–probability scores of the DMUs.

δ	0.5				0.1	0.3	0.5	0.9
γ	0.1	0.3	0.5	0.9	0.5			
DMU 1	1.3487	1.2012	1.0780	0.8731	0.9985	1.0424	1.0780	1.1732
DMU 2	1.1553	1.0540	0.9610	0.7975	0.8871	0.9294	0.9610	1.0420
DMU 3	0.9850	0.9081	0.8410	0.7131	0.7771	0.8122	0.8410	0.9165
DMU 4	1.6691	1.4848	1.3211	1.0603	1.2266	1.2808	1.3211	1.4308
DMU 5	1.0421	0.9318	0.8353	0.6705	0.7665	0.8058	0.8353	0.9128
DMU 6	0.8804	0.7962	0.7195	0.5882	0.6702	0.6984	0.7195	0.7729
DMU 7	1.3266	1.2007	1.0870	0.8907	1.0101	1.0542	1.0870	1.1706
DMU 8	1.2061	1.1239	1.0479	0.9046	0.9508	1.0086	1.0479	1.1522
DMU 9	1.6347	1.4654	1.3139	1.0562	1.2120	1.2707	1.3139	1.4249
DMU 10	1.1659	1.0752	0.9928	0.8466	0.9119	0.9582	0.9928	1.1711
DMU 11	0.9242	0.8591	0.7996	0.6869	0.7382	0.7734	0.7996	0.8681
DMU 12	0.8507	0.7903	0.7371	0.6363	0.6858	0.7141	0.7371	0.7984
DMU 13	0.7737	0.7128	0.6590	0.5632	0.6130	0.6387	0.6590	0.7141
DMU 14	1.1445	1.0377	0.9414	0.7756	0.8707	0.9115	0.9414	1.0179
DMU 15	1.1845	1.0994	1.0196	0.8746	0.8707	0.9859	1.0196	1.1058
DMU 16	0.9751	0.8958	0.8225	0.6932	0.7608	0.7964	0.8225	0.8893
DMU 17	0.9803	0.9062	0.8383	0.7122	0.7662	0.8074	0.8383	0.9200
DMU 18	1.0699	0.9845	0.9049	0.7619	0.7662	0.8754	0.9049	0.9806
DMU 19	1.2988	1.1830	1.0763	0.8873	0.9928	1.0409	1.0763	1.1671
DMU 20	1.3676	1.2542	1.1492	0.9658	1.0652	1.1132	1.1492	1.2423
DMU 21	1.5387	1.3997	1.2726	1.0499	1.1848	1.2355	1.2726	1.3670
DMU 22	1.1314	1.0267	0.9343	0.7783	0.8731	0.9082	0.9343	1.0008
DMU 23	1.2838	1.1879	1.0998	0.9439	1.0229	1.0673	1.0998	1.1827
DMU 24	1.3044	1.2290	1.1583	1.0297	1.0714	1.1202	1.1583	2.2708
DMU 25	1.1941	1.1155	1.0417	0.9069	0.9683	1.0107	1.0417	1.1205
DMU 26	1.0247	0.9359	0.8569	0.7256	0.7989	0.8322	0.8569	0.9826
DMU 27	1.0165	0.9328	0.8574	0.7272	0.8031	0.8345	0.8574	0.9153
DMU 28	1.3159	1.1709	1.0445	0.8365	0.9765	1.0158	1.0445	1.1174
DMU 29	1.3874	1.2723	1.1675	0.9843	1.0799	1.1303	1.1675	1.4550
DMU 30	1.4323	1.3251	1.2273	1.0486	1.1368	1.1890	1.2273	1.3253

four different measures of 0.1, 0.3, 0.5 and 0.9 (presented as *Case II*). Table 10 presents the results of Models (15) and (16) for the above-mentioned levels.

Here again we considered the necessity–probability models in order to provide practical and insightful information to the DMs. We proposed the necessity–probability Models (18) and (19) and applied values of δ and γ similar to the ones used in Cases I and II and obtained the results presented in Table 11.

As discussed earlier, according to Definition 5b, the efficiency of the necessity approach must be less than or equal to the efficiency of the possibility approach at a given threshold level. Again, in Case I or II, the efficiency measure strictly decreases when γ (or δ) is increased and δ (or γ) is fixed. Seemingly, smaller γ and δ give more opportunities to the DMUs in Cases I and II, respectively, to achieve the higher scores. The efficiency scores are adjusted using the normalization formulation as shown in Table 12.

We then calculated the efficiency scores of the cases where the threshold level δ is characterized with the triangular fuzzy number $\tilde{\delta} = (\delta^{\alpha}, \delta^{m}, \delta^{\beta})$. In this case study, we also took into account Cases I and II to provide "what if" analysis to the DM. Case I involves the fixed $\tilde{\delta} = (0.10, 0.5, 0.10)$ and four values for $\gamma \in \{0.1, 0.3, 0.5, 0.9\}$ while Case II consists of the fixed $\gamma = 0.5$ and four fuzzy numbers for $\tilde{\delta} \in \{(0.1,0.1,0.1),(0.1,0.3,0.1),(0.1,0.5,0.1),(0.1,0.7,0.1)\}$. In order to compare the four $\tilde{\delta}$s, we select $\tilde{\delta}$ such that $(0.1,0.1,0.1) \leqslant (0.1,0.3,0.1) \leqslant (0.1,0.5,0.1) \leqslant (0.1,0.7,0.1)$ as is shown in Fig. 2. The corresponding results derived from Model (21) are reported in Table 13.

The results from the necessity-probabilistic method reported in Table 7 are significantly lower than the results from the possibilistic–probabilistic method presented in Table 6. Note here that we expected to achieve this result with respect to Definition 5b. In addition, when γ (or $\bar{\delta}$) is increased and $\bar{\delta}$ (or γ) is fixed, the efficiency always decreases (see Table 7). To specify the necessity-probabilistic efficient units for the given threshold, the normalization scores are presented in Table 14.

As shown in Table 14, for $\tilde{\delta} = (0.1, 0.5, 0.1)$ and γ = 0.1, 0.3, 0.5, and 0.9, DMUs 24 and 6 have the highest and the lowest scores, respectively. Also, for $\tilde{\delta} = (0.1,0.1,0.1)$, $\tilde{\delta} = (0.1,0.3,0.1)$, $\tilde{\delta} = (0.1,0.5,0.1)$, $\tilde{\delta} = (0.1,0.7,0.1)$ and γ = 0.5, DMUs 24 and 6 have the highest and the lowest scores, respectively. The results of the necessity-probabilistic method with the fuzzy threshold reported in Table 14 are significantly lower compared to the results of the possibilistic–probabilistic method with the fuzzy threshold reported in Table 13. For example, the efficiency results of the possibilistic–probabilistic method with the fuzzy threshold $\tilde{\delta} = (0.1,0.5,0.1)$ and γ = 0.1, 0.3, 0.5 and 0.9 for DMU 3 are 0.8329, 0.8002, 0.7691, and 0.7020, but efficiency results of the necessity-probabilistic method with the fuzzy threshold for DMU 3 are 0.5238, 0.5356, 0.5474, and 0.5626. This shows that the efficiency of the necessity–probability method is less than the efficiency of the possibilistic-probability method result. The efficiency based necessity–probability method under $\tilde{\delta} = (0.1,0.1,0.1)$ and γ = 0.5 is higher than other threshold levels. When $\tilde{\delta} = (0.1,0.5,0.1)$ is kept unchanged and γ is increased from γ = 0.1 to 0.9, most of the DMUs necessity–probability results are increased, although not for all DMUs. It is possible that the efficiency of some DMUs is reduced when γ is increased. Also, when γ = 0.5 is kept unchanged and $\tilde{\delta}$ is increased, the necessity–probability results for all DMUs is decreased.

8. Conclusion and discussion

The well-documented popularity of performance evaluation methods using non-parametric programming such as DEA is due largely to their simplicity and applicability. DEA has resulted in an entirely new field branching economics, operations

Table 11
Necessity–probability scores of the DMUs.

δ	0.5				0.1	0.3	0.5	0.9
γ	0.1	0.3	0.5	0.9	0.5			
DMU 1	0.6444	0.6447	0.6451	0.6458	0.5833	0.6177	0.6451	0.6750
DMU 2	0.6105	0.6050	0.5992	0.5904	0.5535	0.5798	0.5992	0.6650
DMU 3	0.5660	0.5611	0.5560	0.5454	0.5014	0.5320	0.5560	0.6221
DMU 4	0.7796	0.7743	0.7688	0.7571	0.6956	0.7360	0.7688	0.8243
DMU 5	0.4955	0.4913	0.4870	0.4777	0.4352	0.4644	0.4870	0.5408
DMU 6	0.4403	0.4329	0.4283	0.4193	0.3958	0.4130	0.4283	0.4551
DMU 7	0.6521	0.6539	0.6567	0.6638	0.6107	0.6373	0.6567	0.6563
DMU 8	0.7331	0.7294	0.7255	0.7173	0.6447	0.6901	0.7255	0.8249
DMU 9	0.7652	0.7636	0.7629	0.7612	0.7056	0.7373	0.7629	0.8360
DMU 10	0.6713	0.6713	0.6714	0.6716	0.5987	0.6396	0.6714	0.8981
DMU 11	0.5443	0.5467	0.5491	0.5542	0.4968	0.5263	0.5491	0.6115
DMU 12	0.5121	0.5110	0.5099	0.5074	0.4627	0.4893	0.5099	0.5658
DMU 13	0.4414	0.4444	0.4475	0.4539	0.4059	0.4295	0.4475	0.4813
DMU 14	0.5680	0.5741	0.5803	0.5929	0.5379	0.5624	0.5803	0.5660
DMU 15	0.7030	0.6948	0.6876	0.6715	0.6344	0.6648	0.6876	0.7462
DMU 16	0.5409	0.5364	0.5317	0.5218	0.4797	0.5074	0.5317	0.5986
DMU 17	0.5677	0.5629	0.5579	0.5474	0.4968	0.5312	0.5579	0.6323
DMU 18	0.6075	0.5940	0.5810	0.5536	0.5351	0.5615	0.5810	0.6118
DMU 19	0.6857	0.6694	0.6527	0.6215	0.6013	0.6309	0.6527	0.6564
DMU 20	0.7510	0.7401	0.7322	0.7222	0.6771	0.7074	0.7322	0.8189
DMU 21	0.8068	0.8038	0.8005	0.7936	0.7453	0.7771	0.8005	0.7231
DMU 22	0.5964	0.5967	0.5962	0.5951	0.5552	0.5789	0.5962	0.5640
DMU 23	0.7631	0.7579	0.7516	0.7386	0.6969	0.7284	0.7516	0.7718
DMU 24	0.8952	0.8941	0.8930	0.8906	0.8045	0.7704	0.8930	1.7633
DMU 25	0.7457	0.7397	0.7340	0.7225	0.6829	0.7120	0.7340	0.7692
DMU 26	0.5590	0.5667	0.5749	0.5929	0.5252	0.5485	0.5749	0.7810
DMU 27	0.5629	0.5672	0.5708	0.5784	0.5333	0.5026	0.5708	0.5662
DMU 28	0.6104	0.6085	0.6053	0.5980	0.5331	0.5881	0.6053	0.5847
DMU 29	0.7834	0.7751	0.7691	0.7618	0.7160	0.7461	0.7691	1.0430
DMU 30	0.8074	0.8048	0.8022	0.7968	0.7449	0.7780	0.8022	0.7408

Table 12
Normalized necessity–probability score of the DMUs.

δ	0.5				0.1	0.3	0.5	0.9
γ	0.1	0.3	0.5	0.9	0.5			
DMU 1	0.7198	0.7211	0.7224	0.7251	0.7250	0.7940	0.7224	0.7203
DMU 2	0.6820	0.6767	0.6710	0.6629	0.6880	0.7452	0.671	0.6650
DMU 3	0.6323	0.6276	0.6226	0.6124	0.6232	0.6838	0.6226	0.6221
DMU 4	0.8709	0.866	0.8609	0.8501	0.8646	0.9460	0.8609	0.8592
DMU 5	0.5535	0.5495	0.5454	0.5364	0.5410	0.5969	0.5454	0.5497
DMU 6	0.4918	0.4842	0.4796	0.4708	0.4920	0.5308	0.4796	0.4746
DMU 7	0.7284	0.7313	0.7354	0.7453	0.7591	0.8192	0.7354	0.7080
DMU 8	0.8189	0.8158	0.8124	0.8054	0.8014	0.8870	0.8124	0.8249
DMU 9	0.8548	0.8540	0.8543	0.8547	0.8771	0.9477	0.8543	0.8534
DMU 10	0.7499	0.7508	0.7518	0.7541	0.7442	0.8221	0.7518	0.8981
DMU 11	0.6080	0.6115	0.6149	0.6223	0.6175	0.6765	0.6149	0.6115
DMU 12	0.5721	0.5715	0.571	0.5697	0.5751	0.6289	0.571	0.5658
DMU 13	0.4931	0.497	0.5011	0.5097	0.5045	0.5521	0.5011	0.4964
DMU 14	0.6345	0.6421	0.6498	0.6657	0.6686	0.7229	0.6498	0.6257
DMU 15	0.7853	0.7771	0.77	0.7540	0.7886	0.8545	0.77	0.7462
DMU 16	0.6042	0.5999	0.5954	0.5859	0.5963	0.6522	0.5954	0.5986
DMU 17	0.6342	0.6296	0.6247	0.6146	0.6175	0.6828	0.6247	0.6323
DMU 18	0.6786	0.6644	0.6506	0.6216	0.6651	0.7217	0.6506	0.6305
DMU 19	0.7660	0.7487	0.7309	0.6978	0.7474	0.8109	0.7309	0.7006
DMU 20	0.8389	0.8278	0.8199	0.8109	0.8416	0.9093	0.8199	0.8189
DMU 21	0.9013	0.8990	0.8964	0.8911	0.9264	0.9988	0.8964	0.8601
DMU 22	0.6662	0.6674	0.6676	0.6682	0.6901	0.7441	0.6676	0.6404
DMU 23	0.8524	0.8477	0.8417	0.8293	0.8663	0.9362	0.8417	0.8111
DMU 24	1.0000	1.0000	1.0000	1.0000	1.0000	0.9902	1.0000	1.7633
DMU 25	0.8330	0.8273	0.8219	0.8113	0.8489	0.9152	0.8219	0.7924
DMU 26	0.6244	0.6338	0.6438	0.6657	0.6528	0.7050	0.6438	0.7810
DMU 27	0.6288	0.6344	0.6392	0.6494	0.6629	0.6460	0.6392	0.6110
DMU 28	0.6819	0.6806	0.6778	0.6715	0.6626	0.7559	0.6778	0.6493
DMU 29	0.8751	0.8669	0.8613	0.8554	0.8900	0.9590	0.8613	1.0429
DMU 30	0.9019	0.9001	0.8983	0.8947	0.9259	1.0000	0.8983	0.8637

Table 13
Possibility–probability scores of the DMUs with their fuzzy thresholds.

δ	(0.1,0.5,0.1)				(0.1,0.1,0.1)	(0.1,0.3,0.1)	(0.1,0.5,0.1)	0.1,0.7,0.1)
γ	0.1	0.3	0.5	0.9	0.5			
DMU 1	1.0951	1.0262	0.9679	0.8563	1.0938	1.0014	0.9679	0.9329
DMU 2	0.9878	0.9328	0.8809	0.7850	0.9893	0.9103	0.8809	0.8497
DMU 3	0.8329	0.8002	0.7691	0.7020	0.8709	0.7962	0.7691	0.7408
DMU 4	1.3042	1.2261	1.1530	1.0337	1.3044	1.1895	1.1530	1.1144
DMU 5	0.7783	0.7438	0.7121	0.6509	0.8107	0.7382	0.7121	0.6847
DMU 6	0.6962	0.6628	0.6309	0.5734	0.7000	0.6498	0.6309	0.6107
DMU 7	1.1837	1.0977	1.0191	0.8804	1.1329	1.0501	1.0191	0.9860
DMU 8	1.0549	1.0138	0.9747	0.8938	1.1177	1.0124	0.9747	0.9355
DMU 9	1.3543	1.2657	1.1836	1.0360	1.3297	1.2231	1.1836	1.1417
DMU 10	0.9928	0.9499	0.9098	0.8339	1.3484	0.9424	0.9098	0.8757
DMU 11	0.8563	0.8103	0.7676	0.6827	0.8634	0.7932	0.7676	0.7407
DMU 12	0.7667	0.7321	0.6994	0.6309	0.7843	0.7221	0.6994	0.6754
DMU 13	0.7162	0.6744	0.6360	0.5603	0.7134	0.6568	0.6360	0.6142
DMU 14	1.0664	0.9817	0.9048	0.7706	1.0112	0.9338	0.9048	0.8741
DMU 15	1.0757	1.0192	0.9657	0.8664	1.0845	0.9979	0.9657	0.9315
DMU 16	0.8068	0.7738	0.7426	0.6799	0.8479	0.7705	0.7426	0.7134
DMU 17	0.8185	0.7885	0.7599	0.7001	0.8695	0.7889	0.7599	0.7297
DMU 18	0.8843	0.8485	0.8137	0.7470	0.9137	0.8409	0.8137	0.7297
DMU 19	1.0245	0.9830	0.9427	0.8653	1.0601	0.9745	0.9427	0.9089
DMU 20	1.1567	1.1004	1.0466	0.9492	1.1784	1.0802	1.0466	1.0109
DMU 21	1.3153	1.2374	1.1647	1.0325	1.2941	1.1998	1.1647	1.1281
DMU 22	0.9576	0.9035	0.8524	0.7650	0.9388	0.8761	0.8524	0.8271
DMU 23	1.1086	1.0590	1.0139	0.9303	1.1249	1.0441	1.0139	0.9818
DMU 24	1.2570	1.1992	1.1443	1.0424	3.2938	1.5012	1.1443	1.0573
DMU 25	1.0753	1.0281	0.9828	0.8977	1.0897	1.0120	0.9828	0.9517
DMU 26	0.9312	0.8693	0.8134	0.7199	1.1725	0.8434	0.8134	0.7884
DMU 27	0.9107	0.8566	0.8070	0.7195	0.8857	0.8286	0.8070	0.7840
DMU 28	1.0007	0.9461	0.8956	0.8126	0.9871	0.9206	0.8956	0.8690
DMU 29	1.0959	1.0511	1.0209	0.9588	1.6648	1.0526	1.0209	0.9872
DMU 30	1.2778	1.2131	1.1528	1.0357	1.2856	1.1889	1.1528	1.1145

Table 14
Necessity–probability scores of the DMUs with their fuzzy thresholds.

δ	(0.1,0.5,0.1)				(0.1,0.1,0.1)	(0.1,0.3,0.1)	(0.1,0.5,0.1)	0.1,0.7,,0.1)
γ	0.1	0.3	0.5	0.9	0.5			
DMU 1	0.6336	0.6360	0.6367	0.6364	0.7282	0.6671	0.6367	0.5570
DMU 2	0.5615	0.5768	0.5932	0.6142	0.6723	0.6145	0.5932	0.5746
DMU 3	0.5238	0.5356	0.5474	0.5626	0.6289	0.5754	0.5474	0.5254
DMU 4	0.7562	0.7578	0.7587	0.7587	0.8686	0.7953	0.7587	0.7278
DMU 5	0.5006	0.4906	0.4791	0.4259	0.5563	0.5053	0.4791	0.4411
DMU 6	0.4130	0.4180	0.4228	0.4335	0.4793	0.4420	0.4228	0.4058
DMU 7	0.6221	0.6357	0.6508	0.6793	0.7106	0.6717	0.6508	0.6323
DMU 8	0.6802	0.6379	0.7145	0.6799	0.8354	0.7543	0.7145	0.6539
DMU 9	0.7348	0.7444	0.7538	0.7616	0.8628	0.7893	0.7538	0.6850
DMU 10	0.6660	0.6641	0.6490	0.6113	0.9366	0.6972	0.6490	0.6318
DMU 11	0.5132	0.5281	0.5421	0.5717	0.6180	0.5674	0.5421	0.5128
DMU 12	0.4730	0.4886	0.5036	0.5271	0.5716	0.5263	0.5036	0.4365
DMU 13	0.4183	0.4244	0.4420	0.4675	0.5015	0.4619	0.4420	0.4243
DMU 14	0.5437	0.5591	0.5748	0.6023	0.6301	0.5942	0.5748	0.5578
DMU 15	0.6226	0.6505	0.6806	0.7366	0.7519	0.7054	0.6806	0.6474
DMU 16	0.5019	0.5126	0.5232	0.5416	0.6056	0.5512	0.5232	0.5004
DMU 17	0.5019	0.5126	0.5342	0.5221	0.6401	0.5795	0.5342	0.5058
DMU 18	0.5370	0.5557	0.5342	0.6083	0.6354	0.5960	0.5342	0.5567
DMU 19	0.6200	0.6331	0.6460	0.6083	0.7137	0.6696	0.6460	0.6253
DMU 20	0.6884	0.6911	0.7234	0.7663	0.8279	0.7576	0.7234	0.6572
DMU 21	0.7378	0.7507	0.7934	0.7990	0.8659	0.8186	0.7934	0.7714
DMU 22	0.5899	0.5548	0.5911	0.5479	0.6447	0.6097	0.5911	0.5744
DMU 23	0.7239	0.7360	0.7445	0.7556	0.8169	0.7696	0.7445	0.7224
DMU 24	0.8410	0.8278	0.8074	0.8677	1.8903	1.0981	0.8074	0.8281
DMU 25	0.6403	0.6656	0.7144	0.7533	0.7981	0.7517	0.7144	0.7058
DMU 26	0.5698	0.5289	0.5536	0.5615	0.8146	0.5972	0.5536	0.5254
DMU 27	0.5555	0.5623	0.5253	0.5439	0.6150	0.5831	0.5253	0.5492
DMU 28	0.6228	0.6133	0.5807	0.5577	0.6536	0.6187	0.5807	0.5732
DMU 29	0.7658	0.7717	0.7622	0.7333	1.0970	0.7869	0.7622	0.6819
DMU 30	0.7411	0.7253	0.7809	0.8005	0.8697	0.8210	0.7809	0.7335

research and statistics. DEA applications during the last decade have increasingly ventured into the fields of organizational control, marketing, public service provision and even health care. As the DEA method is becoming popular for assessing the relative efficiency of organizational units in various business entities, more work is needed to address complex evaluation settings.

A DEA model basically draws three critical elements: the model specification, the reference set itself, and the definition of the production possibility set. Starting from the latter, the production possibility set can either be defined as complete and known (like in conventional DEA) or as potentially extending beyond or excluding the reference set (like in stochastic DEA). The reference set, the very observations that form the engine of the non-parametric approach, can be either precise (as in conventional DEA), outcomes of stochastic processes (as in stochastic frontier analysis), or imprecise (as in the fuzzy DEA models). We generalized and consolidated the disparate DEA models by simultaneously addressing random and fuzzy (Ra-Fu) reference sets and imprecisely defined production sets. We also systematically transform the quadratic programming formulations into implementable problems for actual problem solutions. To this end, we illustrated the feasibility and richness of the obtained solutions through a series of numerical examples and a real-world case study.

As for future research, the model specification properties present itself as an avenue for further work. The choice of variables, the classification into inputs, outputs or categorical variables, and the direction for the radial projection: all of these questions may also be subject to behavioral bias, uncertainty and impreciseness. A second direction for further work lies in the reduction of the obtained solutions to obtain the special cases with respect to the conventional models of DEA, stochastic DEA and fuzzy DEA. Work in this direction may also shed more light on the economic and statistical interpretations of the rankings obtained.

Appendix A

Proof. First we obtain the algebraic operation for the multiplication of two generalized left right fuzzy numbers fuzzy variables. Let $\tilde{A} = (\alpha, a_2, a_3, \beta)_{LR}$ and $\tilde{B} = (\bar{\alpha}, b_2, b_3, \bar{\beta})_{LR}$ be two generalized left right fuzzy numbers fuzzy variables.

$$\begin{aligned}\tilde{A}(\cdot)\tilde{B} &= (a_2-\alpha, a_2, a_3, a_3+\beta)_{LR}(\cdot)(b_2-\bar{\alpha}, b_2, b_3, b_3+\bar{\beta})_{LR}\\ &\simeq ((a_2-\alpha)(b_2-\bar{\alpha}), a_2b_2, a_3b_3, (a_3+\beta)(b_3+\bar{\beta}))_{LR}\\ &= (a_2b_2 - a_2\bar{\alpha} - b_2\alpha + \alpha\bar{\alpha}, a_2b_2, a_3b_3, a_3b_3 + a_3\bar{\beta} + b_3\beta + \beta\bar{\beta})_{LR}\\ &= (a_2\bar{\alpha} + b_2\alpha - \alpha\bar{\alpha}, a_2b_2, a_3b_3, a_3\bar{\beta} + b_3\beta + \beta\bar{\beta})_{LR}\end{aligned}$$

Then, for $(\tilde{\lambda})^2$ where $\tilde{\lambda} = (\alpha, \lambda_1, \lambda_2, \beta)$ we have the following:

$$\begin{aligned}\tilde{A}(\cdot)\tilde{A} &= (\alpha, \lambda_2, \lambda_3, \beta)_{LR}(\cdot)(\alpha, \lambda_2, \lambda_3, \beta)_{LR}\\ &\simeq \left(2\lambda_2\alpha - \alpha^2, \lambda_2^2, \lambda_3^2, 2\lambda_3\beta + \beta^2\right)_{LR}\\ &= \left(\lambda_2^2 - 2\lambda_2\alpha + \alpha^2, \lambda_2^2, \lambda_3^2, \lambda_3^2 + 2\lambda_3\beta + \beta^2\right)_{LR}\\ &\simeq \left((\lambda_2-\alpha)^2, \lambda_2^2, \lambda_3^2, (\lambda_3+\beta)^2\right)_{LR}\end{aligned}$$

We then obtain $(\tilde{\lambda})^k \simeq \left((\lambda_2-\alpha)^k, \lambda_2^k, \lambda_3^k, (\lambda_3+\beta)^k\right)$. Similarly,

$$\begin{aligned}e^{\tilde{\lambda}} &\simeq \sum_{k=0}^{+\infty}\frac{(\tilde{\lambda})^k}{k!} \simeq \sum_{k=0}^{+\infty}\left(\frac{(\lambda_2-\alpha)^k}{k!}, \frac{(\lambda_2)^k}{k!}, \frac{(\lambda_3)^k}{k!}, \frac{(\lambda_2+\beta)^k}{k!}\right)\\ &\simeq \left(\sum_{k=0}^{+\infty}\frac{(\lambda_2-\alpha)^k}{k!}, \sum_{k=0}^{+\infty}\frac{(\lambda_2)^k}{k!}, \sum_{k=0}^{+\infty}\frac{(\lambda_3)^k}{k!}, \sum_{k=0}^{+\infty}\frac{(\lambda_2+\beta)^k}{k!}\right)\\ &\simeq (e^{(\lambda_2-\alpha)}, e^{\lambda_2}, e^{\lambda_3}, e^{(\lambda_3+\beta)}) \qquad \square\end{aligned}$$

Proof (a) of Theorem 5:

Let $Pos[P(\tilde{\bar{h}} \leqslant 0) \geqslant \delta] \geqslant \gamma$ and $\tilde{\bar{h}} = \sum_{j=1}^{n}\tilde{\bar{a}}_j x_j - \tilde{\bar{b}}$ in which $\tilde{\bar{a}}_j \sim N\left(\bar{\mu}_j, \bar{\sigma}_j^2\right)$ and $\tilde{\bar{b}} \sim N(\hat{\mu}, \hat{\sigma}^2)$. Obviously, $\tilde{\bar{h}}$ has also normal distribution with $\mu_{\tilde{\bar{h}}} = \sum_{j=1}^{n}\bar{\mu}_j x_j - \hat{\mu}$ and $\sigma_{\tilde{\bar{h}}}^2 = \sum_{j=1}^{n}\bar{\sigma}_j^2 x_j + \hat{\sigma}^2$.

Accordingly, the standardized normal distribution, (see, e.g., [4,10]), can be used to transform the probability relationship to a deterministic form as follows:

$$Pos\left[P\left(z \leqslant \frac{-\mu_{\tilde{\bar{h}}}}{\sigma_{\tilde{\bar{h}}}}\right) \geqslant \delta\right] \geqslant \gamma$$

where $z = \frac{\tilde{\bar{h}} - \mu_{\tilde{\bar{h}}}}{\sigma_{\tilde{\bar{h}}}}$ has a normal standard distribution (with zero mean and unit variance). Using the characteristic of the cumulative distribution function (CDF) of z, we have:

$$Pos\left[\sum_{j=1}^{n}\bar{\mu}_j x_j + \sigma_{\tilde{\bar{h}}}\Phi^{-1}(\delta) \leqslant \hat{\mu}\right] \geqslant \gamma$$

where Φ^{-1} is the inverse of CDF. Using Theorem 3, the deterministic equivalent form can be represented from the possibilistic relationship as follows:

$$\left[\sum_{j=1}^{n}\bar{\mu}_j x_j + \sigma_{\tilde{\bar{h}}}\Phi^{-1}(\delta)\right]_{\gamma}^{L} \leqslant (\hat{\mu})_{\gamma}^{R}$$

The above relationship is divided into the following two relationships with respect to the sign of the CDF of the normal standard distribution:

If $\delta > 0.5$ (i.e., $\Phi^{-1}(\delta) > 0$), we then have:

$$\begin{aligned}&\sum_{j=1}^{n}\left(\bar{\mu}_j^{m_2} - L^{-1}(\gamma)\bar{\mu}_j^{\beta}\right)x_j + \Phi^{-1}(\delta)\\ &\times\sqrt{\sum_{j=1}^{n}x_j^2\left(a_j^{m_1} - L^{-1}(\alpha)\bar{a}_j^{\alpha}\right) + \left(\hat{a}_j^{m_1} - L^{-1}(\alpha)\hat{a}_j^{\alpha}\right)}\\ &\leqslant b^{m_2} + R^{-1}(\gamma)b^{\beta}\end{aligned}$$

If $\delta \leqslant 0.5$ (i.e., $\Phi^{-1}(\delta) \leqslant 0$), we then have:

$$\begin{aligned}&\sum_{j=1}^{n}\left(\bar{\mu}_{ij}^{m_2} - L^{-1}(\gamma)\bar{\mu}_{ij}^{\beta}\right)x_j + \Phi^{-1}(\delta)\\ &\times\sqrt{\sum_{j=1}^{n}x_j^2\left(\bar{a}_j^{m_2} + R^{-1}(\alpha)\bar{a}_j^{\beta}\right) + \left(\hat{a}_j^{m_2} + R^{-1}(\alpha)\hat{a}_j^{\beta}\right)}\\ &\leqslant b^{m_2} + R^{-1}(\gamma)b^{\beta}\end{aligned}$$

The proof (a) is complete. □

Proof (b): The proof is similar to (a). □

Appendix B

Proof (a) of Theorem 6:

Let $\bar{\lambda}_1$ and $\bar{\lambda}_2$ be the two Ra-Fu variables with normal distributions as $\tilde{\bar{\lambda}}_1 \sim N(\bar{\mu}_1, \bar{\sigma}_1^2)$ and $\tilde{\bar{\lambda}}_2 \sim N(\bar{\mu}_2, \bar{\sigma}_2^2)$, respectively. Then $\tilde{\bar{h}} = \tilde{\bar{\lambda}}_1 - \tilde{\bar{\lambda}}_2$ is also Ra-Fu variable with normal distribution involving the following mean and variance:

$$\begin{aligned}\mu_{\tilde{\bar{h}}} &= E[\tilde{\bar{\lambda}}_1 - \tilde{\bar{\lambda}}_2] = \bar{\lambda}_1 - \bar{\lambda}_2 = \left(\lambda_1^{\alpha} + \lambda_2^{\beta}, \lambda_1^{m_1} - \lambda_2^{m_2}, \lambda_1^{m_2} - \lambda_2^{m_1}, \lambda_1^{\beta} + \lambda_2^{\alpha}\right)\\ \sigma_{\tilde{\bar{h}}}^2 &= Var[\tilde{\bar{\lambda}}_1 - \tilde{\bar{\lambda}}_2] = \bar{\sigma}_1^2 + \bar{\sigma}_2^2 = \left(\hat{\lambda}_1^{\alpha} + \hat{\lambda}_2^{\alpha}, \hat{\lambda}_1^{m_1} + \hat{\lambda}_2^{m_1}, \hat{\lambda}_1^{m_2} + \hat{\lambda}_2^{m_2}, \hat{\lambda}_1^{\beta} + \hat{\lambda}_2^{\beta}\right).\end{aligned}$$

and the corresponding fuzzy PDF of $\tilde{\bar{h}}$ is $f(\tilde{\bar{h}}) = \frac{1}{\sigma_{\tilde{\bar{h}}}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\tilde{\bar{h}}-\bar{h}}{\sigma_{\tilde{\bar{h}}}}\right)^2}$. From the concept of the α-cut (i.e., Definition 3), the probability of $\tilde{\bar{\lambda}}_1 \leqslant \tilde{\bar{\lambda}}_2$ for a given $\alpha \in [0,1]$ can be obtained as follows:

$$[\tilde{P}(\tilde{\bar{\lambda}}_1 \leqslant \tilde{\bar{\lambda}}_2)]_{\alpha} = [\tilde{P}(\tilde{\bar{h}} \leqslant 0)]_{\alpha} = \left\{\int_{-\infty}^{0}\frac{1}{\sigma_{\tilde{\bar{h}}}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\tilde{\bar{h}}-\mu_{\tilde{\bar{h}}}}{\sigma_{\tilde{\bar{h}}}}\right)^2}d(\tilde{\bar{h}})\,|\,\mu_{\tilde{\bar{h}}} \in \bar{h}_{\alpha},\right.$$
$$\left.\sigma_{\tilde{\bar{h}}}^2 \in \left(\tilde{\sigma}_{\tilde{\bar{h}}}^2\right)_{\alpha}; \frac{1}{\sigma_{\tilde{\bar{h}}}\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\tilde{\bar{h}}-\mu_{\tilde{\bar{h}}}}{\sigma_{\tilde{\bar{h}}}}\right)^2}d(\tilde{\bar{h}}) = 1\right\}$$

Using the variable substitution $z = \frac{\tilde{\bar{h}} - \mu_{\tilde{\bar{h}}}}{\sigma_{\tilde{\bar{h}}}}$, we have:

$$[\tilde{P}(\tilde{\bar{h}} \leqslant 0)]_\alpha = \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{-\mu_{\tilde{\bar{h}}}}{\sigma_{\tilde{\bar{h}}}}} e^{-\frac{z^2}{2}} dz | \mu_{\tilde{\bar{h}}} \in \bar{h}_\alpha, \sigma^2_{\tilde{\bar{h}}} \in \left(\tilde{\sigma}^2_{\tilde{\bar{h}}}\right)_\alpha \right\}$$
$$= \left\{ \Phi\left(\frac{-\mu_{\tilde{\bar{h}}}}{\sigma_{\tilde{\bar{h}}}}\right) | \mu_{\tilde{\bar{h}}} \in \bar{h}_\alpha, \sigma^2_{\tilde{\bar{h}}} \in (\tilde{\sigma}^2_{\tilde{\bar{h}}})_\alpha \right\}$$

where Φ for the CDF of z[3] is an increasing and continuous function. The definition of the α-cut (see Definition 3) implies the following interval for each α-cut:

$$[\tilde{P}(\tilde{\bar{\lambda}}_1 \leqslant \tilde{\bar{\lambda}}_2)]_\alpha = \left[\Phi\left(\frac{-\mu_{\tilde{\bar{h}}}}{\sigma_{\tilde{\bar{h}}}}\right)^R_\alpha, \Phi\left(\frac{-\mu_{\tilde{\bar{h}}}}{\sigma_{\tilde{\bar{h}}}}\right)^L_\alpha \right]$$

Based on Theorem 3, $[\tilde{P}(\tilde{\bar{\lambda}}_1 \leqslant \tilde{\bar{\lambda}}_2)]^R_\alpha \geqslant (\tilde{\delta})^L_\alpha$ can be derived from $Pos[P(\tilde{\bar{\lambda}}_1 \leqslant \tilde{\bar{\lambda}}_2) \geqslant \tilde{\delta}] \geqslant \alpha$. Therefore, we have

$$\Phi\left(\frac{-\mu_{\tilde{\bar{h}}}}{\sigma_{\tilde{\bar{h}}}}\right)^L_\alpha \geqslant (\tilde{\delta})^L_\alpha \Longleftrightarrow (\mu_{\tilde{\bar{h}}})^L_\alpha + (\sigma_{\tilde{\bar{h}}})^L_\alpha \Phi^{-1}((\tilde{\delta})^L_\alpha) \leqslant 0.$$

where

$$(\mu_{\tilde{\bar{h}}})^L_\alpha = \lambda_1^{m_1} - \lambda_2^{m_2} - L^{-1}(\alpha)(\lambda_1^\alpha + \lambda_2^\beta)$$
$$(\sigma_{\tilde{\bar{h}}})^L_\alpha = \sqrt{\hat{\lambda}_1^{m_1} + \hat{\lambda}_2^{m_1} - L^{-1}(\alpha)(\hat{\lambda}_1^\alpha + \hat{\lambda}_2^\alpha)}$$
$$(\tilde{\delta})^L_\alpha = \delta^{m_1} - L^{-1}(\alpha)\delta^\alpha \qquad \square$$

Proof (b): The proof is similar to (a). □

Appendix C

The envelopment CCR model with the Possibility–Probability Constrained Programming (PPCP) model is as follows:

$$\begin{aligned} \min \quad & \theta \\ \text{s.t.} \quad & Pos\left[P\left(\sum_{j=1}^n \lambda_j \tilde{\bar{x}}_{ij} \leqslant \theta \tilde{\bar{x}}_{ip}\right) \geqslant \delta\right] \geqslant \gamma, \quad i = 1, \ldots, m \\ & Pos\left[P\left(\sum_{j=1}^n \lambda_j \tilde{\bar{y}}_{rj} \geqslant \tilde{\bar{y}}_{rp}\right) \geqslant \delta\right] \geqslant \gamma, \quad r = 1, \ldots, s \\ & \lambda_j \geqslant 0, \quad \theta \text{ is a free variable} \end{aligned}$$

By applying the first part of Theorem 5 to the Ra-Fu envelopment CCR, the following two deterministic equivalent models are obtained in the presence of the Ra-Fu inputs and outputs:

$$\begin{aligned} \min \quad & \theta_{\delta>0.5} \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j \left(x_{ij}^{m_1} - L^{-1}(\gamma) x_{ij}^\alpha\right) + \Phi^{-1}(\delta) v_i \leqslant \theta\left(x_{ip}^{m_2} + R^{-1}(\gamma) x_{ip}^\beta\right), \\ & i = 1, \ldots, m \\ & \sum_{j=1}^n \lambda_j \left(y_{rj}^{m_2} + R^{-1}(\gamma) y_{rj}^\beta\right) - \Phi^{-1}(\delta) u_r \geqslant \left(y_{rp}^{m_1} - L^{-1}(\gamma) y_{rp}^\alpha\right), \\ & r = 1, \ldots, s \\ & v_i^2 = \sum_{j=1}^n \lambda_j^2 \hat{x}_{ij}^{m_1} + \theta^2 \hat{x}_{ip}^{m_1} - 2\theta\lambda_p \hat{x}_{ip}^{m_2} - L^{-1}(\gamma) \\ & \left(\sum_{j=1}^n \lambda_j^2 \hat{x}_{ij}^\alpha + \theta^2 \hat{x}_{ip}^\alpha - 2\theta\lambda_p \hat{x}_{ip}^\beta\right) \\ & u_r^2 = \sum_{j=1}^n \lambda_j^2 \hat{y}_{rj}^{m_1} + \hat{y}_{rp}^{m_1} - 2\lambda_p \hat{y}_{rj}^{m_2} - L^{-1}(\gamma) \\ & \left(\sum_{j=1}^n \lambda_j^2 \hat{y}_{rj}^\alpha + \hat{y}_{rp}^\alpha - 2\lambda_p \hat{y}_{rj}^\beta\right) \\ & \lambda_j, v_i, u_r \geqslant 0, \theta \text{ is a free variable.} \end{aligned}$$

$$\begin{aligned} \min \quad & \theta_{\delta>0.5} \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j \left(x_{ij}^{m_1} - L^{-1}(\gamma) x_{ij}^\alpha\right) + \Phi^{-1}(\delta) v_i \leqslant \theta\left(x_{ip}^{m_2} + R^{-1}(\gamma) x_{ip}^\beta\right), \\ & i = 1, \ldots, m \\ & \sum_{j=1}^n \lambda_j \left(y_{rj}^{m_2} + R^{-1}(\gamma) y_{rj}^\beta\right) - \Phi^{-1}(\delta) u_r \geqslant \left(y_{rp}^{m_1} - L^{-1}(\gamma) y_{rp}^\alpha\right), \\ & r = 1, \ldots, s \\ & v_i^2 = \sum_{j=1}^n \lambda_j^2 \hat{x}_{ij}^{m_2} + \theta^2 \hat{x}_{ip}^{m_2} - 2\theta\lambda_p \hat{x}_{ip}^{m_2} - R^{-1}(\gamma) \\ & \left(\sum_{j=1}^n \lambda_j^2 \hat{x}_{ij}^\beta + \theta^2 \hat{x}_{ip}^\beta - 2\theta\lambda_p \hat{x}_{ip}^\alpha\right) \\ & u_r^2 = \sum_{j=1}^n \lambda_j^2 \hat{y}_{rj}^{m_2} + \hat{y}_{rp}^{m_2} - 2\lambda_p \hat{y}_{rj}^{m_1} - R^{-1}(\gamma) \\ & \left(\sum_{j=1}^n \lambda_j^2 \hat{y}_{rj}^\beta + \hat{y}_{rp}^\beta - 2\lambda_p \hat{y}_{rj}^\alpha\right), \\ & \lambda_j, v_i, u_r \geqslant 0, \theta \text{ is a free variable.} \end{aligned}$$

In addition to the envelopment CCR model with possibility–probability constraints, we present a Necessity–Probability Constrained Programming (NPCP) model with fuzzy probability necessity constraints as follows:

$$\begin{aligned} \min \quad & \theta \\ \text{s.t.} \quad & Nec\left[P\left(\sum_{j=1}^n \lambda_j \tilde{\bar{x}}_{ij} \leqslant \theta \tilde{\bar{x}}_{ip}\right) \geqslant \delta\right] \geqslant \gamma, \quad i = 1, \ldots, m \\ & Nec\left[P\left(\sum_{j=1}^n \lambda_j \tilde{\bar{y}}_{rj} \geqslant \tilde{\bar{y}}_{rp}\right) \geqslant \delta\right] \geqslant \gamma, \quad r = 1, \ldots, s \\ & \lambda_j \geqslant 0, \theta \text{ is a free variable} \end{aligned}$$

Similar to the solution method proposed for the envelopment CCR Model, we use the second part of Theorem 5 for the constraints of the NPCP to obtain the following quadratic deterministic models:

$$\begin{aligned} \min \quad & \theta_{\delta>0.5} \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j \left(x_{ij}^{m_2} + R^{-1}(\gamma) x_{ij}^\alpha\right) + \Phi^{-1}(\delta) v_i \leqslant \theta\left(x_{ip}^{m_1} - L^{-1}(1-\gamma) x_{ip}^\alpha\right), \\ & i = 1, \ldots, m \\ & \sum_{j=1}^n \lambda_j \left(y_{rj}^{m_1} - L^{-1}(1-\gamma) y_{rj}^\alpha\right) - \Phi^{-1}(\delta) u_r \geqslant \left(y_{rp}^{m_2} + R^{-1}(\gamma) y_{rp}^\beta\right), \\ & r = 1, \ldots, s \\ & v_i^2 = \sum_{j=1}^n \lambda_j^2 \hat{x}_{ij}^{m_2} + \theta^2 \hat{x}_{ip}^{m_2} - 2\theta\lambda_p \hat{x}_{ip}^{m_1} + R^{-1}(\gamma) \\ & \left(\sum_{j=1}^n \lambda_j^2 \hat{x}_{ij}^\beta + \theta^2 \hat{x}_{ip}^\beta - 2\theta\lambda_p \hat{x}_{ip}^\alpha\right) \\ & u_r^2 = \sum_{j=1}^n \lambda_j^2 \hat{y}_{rj}^{m_2} + \hat{y}_{rp}^{m_2} - 2\lambda_p \hat{y}_{rj}^{m_1} + R^{-1}(\gamma) \\ & \left(\sum_{j=1}^n \lambda_j^2 \hat{y}_{rj}^\beta + \hat{y}_{rp}^\beta - 2\lambda_p \hat{y}_{rj}^\alpha\right), \\ & \lambda_j \geqslant 0, \theta \text{ is a free variable} \end{aligned}$$

[3] z has normal standard distribution with zero mean and unit variance.

$$
\begin{aligned}
\min\quad & \theta_{\delta\leqslant 0.5}\\
\text{s.t.}\quad & \sum_{j=1}^{n}\lambda_j\left(x_{ij}^{m_2}+R^{-1}(\gamma)x_{ij}^{\alpha}\right)+\Phi^{-1}(\delta)\,v_i\leqslant\theta(x_{ip}^{m_1}-L^{-1}(1-\gamma)x_{ip}^{\alpha}),\\
& i=1,\ldots,m\\
& \sum_{j=1}^{n}\lambda_j\left(y_{rj}^{m_1}-L^{-1}(1-\gamma)y_{rj}^{\alpha}\right)-\Phi^{-1}(\delta)u_r\geqslant\left(y_{rp}^{m_2}+R^{-1}(\gamma)y_{rp}^{\beta}\right),\\
& r=1,\ldots,s\\
& v_i^2=\sum_{j=1}^{n}\lambda_j^2\hat{x}_{ij}^{m_1}+\theta^2\hat{x}_{ip}^{m_1}-2\theta\lambda_p\hat{x}_{ip}^{m_2}-L^{-1}(1-\gamma)\\
& \left(\sum_{j=1}^{n}\lambda_j^2\hat{x}_{ij}^{\alpha}+\theta^2\hat{x}_{ip}^{\alpha}-2\theta\lambda_p\hat{x}_{ip}^{\beta}\right)\\
& u_r^2=\sum_{j=1}^{n}\lambda_j^2\hat{y}_{rj}^{m_1}+\hat{y}_{rp}^{m_1}-2\lambda_p\hat{y}_{rj}^{m_2}-L^{-1}(1-\gamma)\\
& \left(\sum_{j=1}^{n}\lambda_j^2\hat{y}_{rj}^{\alpha}+\hat{y}_{rp}^{\alpha}-2\lambda_p\hat{y}_{rj}^{\beta}\right),\\
& \lambda_j\geqslant 0,\theta \text{ is a free variable}
\end{aligned}
$$

C.1. Chance-constrained Ra-Fu envelopment CCR model with a fuzzy threshold level

Similar to the value based CCR model with a fuzzy threshold level, the envelopment CCR model with a fuzzy threshold level is as follows:

$$
\begin{aligned}
\min\quad & \theta\\
\text{s.t.}\quad & Pos\left[P\left(\sum_{j=1}^{n}\lambda_j\tilde{\bar{x}}_{ij}\leqslant\theta\tilde{\bar{x}}_{ip}\right)\geqslant\tilde{\delta}\right]\geqslant\gamma,\quad i=1,\ldots,m\\
& Pos\left[P\left(\sum_{j=1}^{n}\lambda_j\tilde{\bar{y}}_{rj}\geqslant\tilde{\bar{y}}_{rp}\right)\geqslant\tilde{\delta}\right]\geqslant\gamma,\quad r=1,\ldots,s\\
& \lambda_j\geqslant 0,\theta \text{ is a free variable}
\end{aligned}
$$

By using Theorem 6, the above deterministic model for the Possibility–Probability constraint is as follows:

$$
\begin{aligned}
\min\quad & \theta_{\delta\leqslant 0.5}\\
\text{s.t.}\quad & \sum_{j=1}^{n}\lambda_j\left(x_{ij}^{m_2}+R^{-1}(\gamma)x_{ij}^{\alpha}\right)+\Phi^{-1}(\delta)\,v_i\leqslant\theta(x_{ip}^{m_1}-L^{-1}(1-\gamma)x_{ip}^{\alpha}),\\
& i=1,\ldots,m\\
& \sum_{j=1}^{n}\lambda_j\left(y_{rj}^{m_1}-L^{-1}(1-\gamma)y_{rj}^{\alpha}\right)-\Phi^{-1}(\delta)u_r\geqslant\left(y_{rp}^{m_2}+R^{-1}(\gamma)y_{rp}^{\beta}\right),\\
& r=1,\ldots,s\\
& v_i^2=\sum_{j=1}^{n}\lambda_j^2\hat{x}_{ij}^{m_1}+\theta^2\hat{x}_{ip}^{m_1}-2\theta\lambda_p\hat{x}_{ip}^{m_2}-L^{-1}(1-\gamma)\\
& \left(\sum_{j=1}^{n}\lambda_j^2\hat{x}_{ij}^{\alpha}+\theta^2\hat{x}_{ip}^{\alpha}-2\theta\lambda_p\hat{x}_{ip}^{\beta}\right)\\
& u_r^2=\sum_{j=1}^{n}\lambda_j^2\hat{y}_{rj}^{m_1}+\hat{y}_{rp}^{m_1}-2\lambda_p\hat{y}_{rj}^{m_2}-L^{-1}(1-\gamma)\\
& \left(\sum_{j=1}^{n}\lambda_j^2\hat{y}_{rj}^{\alpha}+\hat{y}_{rp}^{\alpha}-2\lambda_p\hat{y}_{rj}^{\beta}\right),\\
& \lambda_j\geqslant 0,\theta \text{ is a free variable}
\end{aligned}
$$

In addition, we extend the Necessity–Probability model with a fuzzy threshold as follows:

$$
\begin{aligned}
\min\quad & \theta\\
\text{s.t.}\quad & Nec\left[P\left(\sum_{j=1}^{n}\lambda_j\tilde{\bar{x}}_{ij}\leqslant\theta\tilde{\bar{x}}_{ip}\right)\geqslant\tilde{\delta}\right]\geqslant\gamma,\quad i=1,\ldots,m\\
& Nec\left[P\left(\sum_{j=1}^{n}\lambda_j\tilde{\bar{y}}_{rj}\geqslant\tilde{\bar{y}}_{rp}\right)\geqslant\tilde{\delta}\right]\geqslant\gamma,\quad r=1,\ldots,s\\
& \lambda_j\geqslant 0,\theta \text{ is a free variable}
\end{aligned}
$$

By using Theorem 6, the above deterministic model for the Necessity–Probability constraint is as following:

$$
\begin{aligned}
\min\quad & \theta\\
\text{s.t.}\quad & \sum_{j=1}^{n}\lambda_j\left(x_{ij}^{m_2}+R^{-1}(1-\gamma)x_{ij}^{\beta}\right)+\Phi^{-1}(\delta^{m_2}+R^{-1}(\gamma)\delta^{\beta})\,v_i\leqslant\theta\\
& \left(x_{ip}^{m_1}-R^{-1}(1-\gamma)x_{ip}^{\alpha}\right),\quad i=1,\ldots,m\\
& \sum_{j=1}^{n}\lambda_j\left(y_{rj}^{m_1}-R^{-1}(1-\gamma)y_{rj}^{\alpha}\right)-\Phi^{-1}(\delta^{m_2}+R^{-1}(\gamma)\delta^{\beta})u_r\geqslant\\
& \left(y_{rp}^{m_2}+R^{-1}(1-\gamma)y_{rp}^{\beta}\right),\quad r=1,\ldots,s\\
& v_i^2=\sum_{j=1}^{n}\lambda_j^2\hat{x}_{ij}^{m_2}+\theta\hat{x}_{ip}^{m_2}-2\theta\lambda_p\hat{x}_{ip}^{m_1}+R^{-1}(1-\gamma)\\
& (\sum_{j=1}^{n}\lambda_j^2\hat{x}_{ij}^{\beta}+\theta^2\hat{x}_{ip}^{\beta}-2\theta\lambda_p\hat{x}_{ip}^{\alpha})\\
& u_r^2=\sum_{j=1}^{n}\lambda_j^2\hat{y}_{rj}^{m_2}+\hat{y}_{rp}^{m_2}-2\lambda_p\hat{y}_{rj}^{m_1}+R^{-1}(1-\gamma)\\
& \left(\sum_{j=1}^{n}\lambda_j^2\hat{y}_{rj}^{\beta}+\hat{y}_{rp}^{\beta}-2\lambda_p\hat{y}_{rj}^{\alpha}\right)\\
& \lambda_j\geqslant 0,\theta \text{ is a free variable.}
\end{aligned}
$$

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