Soft Computing (2022) 26:327–347
https://doi.org/10.1007/s00500-021-06389-7



# Analytics under uncertainty: a novel method for solving linear programming problems with trapezoidal fuzzy variables

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Accepted: 29 September 2021 / Published online: 23 November 2021


## Abstract
Linear programming (LP) has long proved its merit as the most flexible and most widely used technique for resource allocation problems in various fields. To solve an LP problem, we have traditionally considered crisp values for the parameters, which are unrealistic in real-world decision-making under uncertainty. The fuzzy set theory has been used to model the imprecise parameter values in LP problems to overcome this shortcoming, resulting in a fuzzy LP (FLP) problem. This paper proposes a new method for solving fuzzy variable linear programming (FVLP) problems in which the decision variables and resource vectors are fuzzy numbers. We show how to use the standard simplex algorithm to solve this problem by converting the fuzzy problem into a crisp one once a linear ranking function is chosen. The novelty of the proposed model resides in that it requires less effort on fuzzy computations as opposed to the existing fuzzy methods. Furthermore, to solve the FVLP problem using the existing methods, fuzzy arithmetic operations and the solution to fuzzy systems of equations are required. By contrast, only arithmetic operations of real numbers and the solution to crisp systems of equations are required to solve the same problem with the method proposed in this study. Finally, a transportation case study in the coal industry is presented to demonstrate the applicability of the proposed algorithm.

**Keywords** Fuzzy variable linear programming · Duality results · Ranking function · Trapezoidal fuzzy number · Transportation problem

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## 1 Introduction

Analytics is the scientific process of transforming data into meaningful insights for informed decision-making. In real-life situations, and even more so in today's big data age, the data on which it relies are generally characterized by uncertainty and ambiguity. In other words, decisions must be made with limited information under uncertain and vague conditions because decision-makers (DMs) generally cannot determine a reasonable probability for alternative outcomes. Therefore, uncertainties and ambiguities must be replaced by concepts that DMs can handle. Among the ways available to model and quantify aspects of imprecision and uncertainty that characterize human-centric decision-making processes is the use of the fuzzy set theory (Zadeh, 1965). Ever since its introduction, the fuzzy set theory has advanced in a variety of ways, and it has been applied to various areas, such as artificial intelligence, computer science, medicine, control engineering, decision theory, expert systems, logic, management science, operations research, pattern recognition, and robotics (Zimmermann, 2010). The applications of this theory to real-world problems are abundant (*e.g.*, for long-term and short-term capacity planning, inventory control, product design, aggregate planning, and scheduling, among others) and the benefits that can be materialized include better performance, better efficiency, higher productivity, and lower cost.

Mathematical programming is one of the areas in which fuzzy set theory has been applied extensively. In this paper, we focus specifically on mathematical linear programming (LP), which is traditionally defined as a technique for determining the optimal allocation of scarce resources to several competing activities based on given criteria of optimality. Otherwise stated, the main aim of an LP problem is to optimize a linear objective function subject to several linear constraints. LP has long proved its merit as the most flexible and most widely used technique for numerous resource allocation problems and other applications in operations research, economics, and finance, among others.

To solve an LP problem, we have traditionally considered crisp values for the parameters, which are unrealistic in many real-world, practical applications, as we have to deal with the DMs' subjective viewpoints. The fuzzy set theory can thus be used to model the imprecise values of parameters in LP problems to overcome this shortcoming. In this situation, the resulting problem is called a fuzzy LP (FLP) problem. Therefore, FLP problems allow working with imprecise data and constraints, leading to more realistic models (Ebrahimnejad & Verdegay, 2018a). Such models have found broad applicability in real-world situations, for example, in irrigation planning (Sanchis et al., 2019), aggregate production-planning problems (Djordjevic et al., 2019), banking and finance (Hillier, 2010), environmental management (Li et al., 2010), carbon management networks (Aviso et al., 2020), supply chain management (Bilgen, 2010), transportation (Muthukumar et al., 2020), product mix (Karakas et al., 2010), product portfolio design (Dorostkar-Ahmadi et al., 2020), manufacturing (Mahdavi et al., 2011), agricultural planting structure optimization problems (Yang et al., 2020), risk management of drinking water supply (Ghandi & Roozbahani, 2020), and location selection for waste-to-energy plants (Ilbahar et al., 2021), just to name a few. For a comprehensive overview of various FLP problems based on models and solution methods, the interested reader is referred to Ghanbari et al. (2020). However, existing approaches suffer from various shortcomings, which we detail in Sect. 4.4.

In the present paper, we focus on a particular type of FLP problem, known as fuzzy variable LP (FVLP) problems, where the decision variables and the resource vectors are given as fuzzy numbers. We propose a new methodology for solving such FVLP problems. In this sense, we show that the FVLP problems under consideration can be converted into crisp ones by choosing the ranking function and can be solved using the standard LP algorithms. The novelty of the proposed method resides in the fact that it requires less effort on fuzzy computations than the existing fuzzy methods (Ebrahimnejad et al., 2010; Mahdavi-Amiri & Nasseri, 2007; Maleki, 2002; Maleki et al., 2000). Furthermore, to solve FVLP problems using the existing methods, fuzzy arithmetic operations and the solution to fuzzy systems of equations are required. By contrast, only arithmetic operations of real numbers and the solution to crisp systems of equations are required to solve the same problem with the method proposed in this study.

We further present a transportation case study in the coal industry to demonstrate the applicability of our proposed method. The coal industry plays an essential role in the economic growth of a country. One of the critical decisions about coal transportation is deciding on the optimal plan to transport the coal from the mines to different areas at the lowest cost. Mathematical LP models are generally used for this purpose. Moreover, as it is a real-world problem, some of the variables are vague and uncertain. The DM knows the exact value of the transportation cost of the coal from different mines to different cities. However, due to some uncontrollable factors, the supply and demand values may be unpredictable, which has a significant impact on the entire system's performance. Therefore, an FVLP method can be adopted to address these issues.

The remainder of this paper is organized as follows: In Sect. 2, we explore the relevant literature on the topic, and in Sect. 3, we present some basic concepts from fuzzy set theory. In Sect. 4, we first formulate the FVLP problem and then review three existing methods to solve such a problem. Subsequently, in Sect. 5, we introduce the new approach for solving the same problem. Section 6 is devoted to illustrating the proposed method using three comparative examples. Subsequently, in Sect. 7, a fuzzy coal transportation problem is solved to illustrate the real applicability of our proposed approach. Finally, Sect. 8 concludes the paper and provides future research directions.

## 2 Literature review

There are many real-world applications of deterministic LP, and one cannot avoid a probabilistic or fuzzy approach in the modeling when, unavoidably, the parameters under study may be characterized by uncertainty. One can handle such uncertainty in various ways: (a) either by modeling the application through a probabilistic orientation, more specifically with probabilistic programming or stochastic programming, or (b) by viewing the problem in terms of fuzzy logic and fuzzy theory and designing appropriate FLP models, or fuzzy mathematical programming models. Furthermore, although there are separate classifications available on probabilistic modeling and fuzzy programming, there is also the literature that combines these two approaches, comprising fuzzy stochastic programming or fuzzy probabilistic programming; an example in this sense is the study by Tanaka and Asai (1984), who were the first to formulate a probabilistic LP with crisp parameters and fuzzy decision variables.

In this section, our concentration is on FLP problems and surrounding approaches. An observation to be made at the outset is that in time, many FLP models have been proposed to incorporate fuzziness in the constraints, the resource vector, the cost coefficients, and the objective function. Researchers have focused not only on different models of the FLP problem but also on their solution techniques. The literature is vast and goes many decades back. It is beyond the purposes of the present study to cover it all here, but interested readers can refer to the works by Ebrahimnejad and Verdegay (2016a, 2018a, 2018b) for a more detailed literature review on existing techniques for solving FLP problems. For our purposes, we explore the literature over the past two decades and give a reasonably good view of what has been published on the topic and what is of recent interest to researchers.

Wang and Liang (2005) presented a new possibilistic LP approach to obtain the solution to an uncertain multi-product aggregate production-planning problem. Ganesan and Veeramani (2006) presented a fuzzy primal simplex approach to finding the solution to the FLP problem. All the parameters of the problem, except the elements of the constraints matrix, are represented by symmetric trapezoidal fuzzy numbers. Allahviranloo et al. (2008) proposed a new method for solving FLP problems with inequality constraints. In this approach, the solution to the FLP problem is provided by solving an equivalent crisp LP problem. Hosseinzadeh Lotfi et al. (2009) provided an approximate solution to the fully FLP (FFLP) problems.

Mula et al. (2010) used a fuzzy approach to model an uncertain supply chain production-planning problem. Ebrahimnejad (2011) deduced some new results about the optimal fuzzy basic solution to the FLP problem. Kumar et al. (2011) developed a novel approach to obtain the exact fuzzy solution to the FFLP problem. Kumar and Kaur (2011) presented a novel approach for solving the FLP problems formulated in Ganesan and Veeramani (2006). Hatami-Marbini and Tavana (2011) proposed finding an optimal solution under various conditions for LP problems with fuzzy parameters. Saati et al. (2012) proposed two approaches to find both crisp and fuzzy solutions to FVLP problems. Baykasoglu and Gocken (2012) presented a direct solution approach for solving the FLP problems with fuzzy decision variables. Sakawa et al. (2012) extended the concept of Stackelberg solutions and presented a computational method for bi-level LP problems in fuzzy random environments. Hatami-Marbini et al. (2013) developed a new stepwise FLP model involving fuzzy numbers in the objective function, the constraints matrix, and the resource vector. Kaur and Kumar (2013) proposed a new algorithm in order to obtain the unique solution to the FLP problems. Khan et al. (2013) used a simplex-based technique to find the solution to an FFLP problem, although Bhardwaj and Kumar (2014) then proved the existence of an error in that approach. Ezzati et al. (2013) presented an alternative approach to solving the FFLP problem, while Kumar and Kaur (2014) proposed a method based on ranking functions for solving FFLP problems. Ebrahimnejad and Tavana (2014) proposed a new approach for solving the FLP problems formulated in Ganesan and Veeramani (2006), which needs less computational effort than the existing approaches. Ebrahimnejad and Verdegay (2014a) investigated the sensitivity analysis of FLP problems. Real numbers represent the elements of the coefficient matrix of the constraints, and symmetric trapezoidal fuzzy numbers represent the remaining parameters.

Mottaghi et al. (2015) proposed simultaneously solving the primal and dual LP problems with the fuzzy objective and fuzzy resources. Ramic and Vlach (2016) proposed a level sets approach to solve LP problems with intuitionistic fuzzy parameters. Najafi et al. (2016) presented a new method for FFLP to obtain the fuzzy optimal solution with unrestricted variables and parameters. Das et al. (2017) proposed an approach based on multi-objective LP and lexicographic ordering methods to solve the FFLP problems. Considering the DM's acceptance degree that the fuzzy constraints may be violated, Dong and Wan (2018) proposed a new method for the FLP problem. All the objective function coefficients, matrix coefficients, and resources are fuzzy numbers. Zhang et al. (2018) simulated the membership functions in the FLP models enumerating the upper and lower bounds obtained from a series of fuzzy optimal objective values at given possibility levels. Kundu et al. (2019) developed a method to solve LP problems with constraints using interval type-2 fuzzy variables based on generalized credibility measures. Pérez-Cañedo and Concepción-Morales (2019) proposed a method to find the unique optimal fuzzy value of FFLP problems with inequality constraints having unrestricted L-R fuzzy parameters and decision variables. Osuna-Gómez et al. (2019) focused on solving optimization problems with fuzzy objectives and constraints and proved a necessary optimality condition for fuzzy optimization problems.

Wang and Peng (2019) developed an approach to find the fuzzy optimal solutions of fuzzy number LP problems based on the best and worst fuzzy sets concepts. Ebrahimnejad (2019) developed a new technique for reducing the computational complexity of the approach proposed in Ezzati et al. (2013) for situations where imprecise data are represented as nonnegative triangular fuzzy numbers.

Some researchers have focused on fuzzy multi-objective LP (MOLP) problems. For example, Zhang et al. (2003) converted an FLP problem with fuzzy cost coefficients into a MOLP problem. Yucel and Fuat Guneri (2011) used fuzzy MOLP approaches to handle ambiguity and fuzziness in supplier selection problems. Ezzati et al. (2014) proposed a new method for solving a fuzzy lexicographic MOLP problem. Zhang and Zuo (2014) defined new concepts for effective solutions to the models of fuzzy MOLP problems.

Yet another strand of research in the literature has been dedicated to solving FLP problems using evolutionary algorithms. For example, Lin (2008) investigated the applicability of genetic algorithms to solving FLP problems without defining membership functions for fuzzy numbers. Mansoori et al. (2017) proposed a one-layer structure recurrent neural network model for solving fuzzy non-LP problems. More recently, Abbaszadeh Sori et al. (2020) formulated a fuzzy constrained shortest path problem and proposed a computational intelligence technique called the elite artificial bee colony algorithm to find the optimal solution.

As previously indicated, in this study, we focus on a particular type of FLP problem, known as the fuzzy variable LP (FVLP) problem, where the decision variables and the resource vectors are given as fuzzy numbers. Some authors have used linear ranking functions to solve this problem. Maleki et al. (2000) and Maleki (2002) first defined a fuzzy auxiliary problem (FAP) with a fuzzy objective function. Then, they applied the crisp solution of the FAP to provide the fuzzy solution of the FVLP. Mahdavi-Amiri and Nasseri (2007) considered the relations between the FAP and the FVLP problem and deduced some duality results in a fuzzy environment. Mahdavi-Amiri and Nasseri (2007) and Ebrahimnejad et al. (2010) used these duality results to develop two new algorithms for solving this problem. These approaches do not require finding the solution to any FAP. Ebrahimnejad and Verdegay (2014b) and Ebrahimnejad (2015), respectively, extended the fuzzy primal simplex method and fuzzy dual simplex method to situations in which some or all the variables of the FVLP problem are restricted to lie within fuzzy lower and fuzzy upper bounds. Saati et al. (2015) utilized a new fuzzy ranking model and a new supplementary variable to obtain the fuzzy and crisp optimal solutions to the FVLP problem by solving one LP model. Mahapatra et al. (2016) introduced a new concept on solution technique for a fuzzy variable-based nonlinear programming problem with both decision variables and restriction being fuzzy in nature. Behera et al. (2021) proposed two methods for obtaining the solution to the FVLP problems using the concept of fuzzy center and radius. This paper shows that by choosing the ranking function, the FVLP problem under consideration is converted into a crisp one and thus can be solved by the standard LP algorithms.

The main contributions of this paper are fourfold:

(1) in contrast to the approach based on FAP, which provides the entering variable by solving fuzzy systems in all iterations, the proposed method finds the entering variable without solving any fuzzy system;
(2) in contrast to the fuzzy dual approach, which determines the leaving variable by solving fuzzy systems in all iterations, the proposed method provides the leaving variable without solving any fuzzy system;
(3) in contrast to the existing approaches that include a large number of fuzzy arithmetic operations, the proposed methodology gives the fuzzy optimal solution without any fuzzy method;
(4) in contrast to the existing complex approaches, the computational complexities are reduced significantly.

The fully fuzzy linear programming problem group is the general case of FLP problems and may not be suitable for all FLP problems with different assumptions and sources of fuzziness. The main aim of this study is to reduce the computational complexities involved in solving FVLP problems. Consequently, we have focused on the FVLP problems with fuzzy numbers representing the decision variables and the resource vectors.

## 3 Preliminaries

This section reviews some basic concepts of fuzzy numbers, which are applied throughout this paper (Ebrahimnejad et al., 2010; Kumar et al., 2011; Mahdavi-Amiri & Nasseri, 2007; Maleki, 2002; Maleki et al., 2000).

**Definition 3.1.** The characteristic function $\mu_A$ of a crisp set $A$ assigns a value of either one or zero to each individual in the universal set $X$. This function can be generalized to a function $\mu_{\tilde{A}}$ such that the values assigned to the element of the universal set $X$ fall within a specified range, *i.e.*, $\mu_{\tilde{A}} : X \to [0, 1]$. The assigned value indicates the membership grade of the element in the set $\tilde{A}$. Larger values denote the higher degrees of membership.

The function $\mu_{\tilde{A}}$ is called a membership function, and the set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$ defined by $\mu_{\tilde{A}}$ for each $x \in X$ is called a fuzzy set.

**Definition 3.2.** A fuzzy set $\tilde{A}$, defined on a universal set of real numbers $\mathbb{R}$, is said to be a fuzzy number if its membership function has the following characteristics:

i. $\tilde{A}$ is convex, *i.e.*,$\forall x, y \in \mathbb{R}, \forall \lambda \in [0, 1], \mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$,
ii. $\tilde{A}$ is normal, *i.e.*, $\exists \bar{x} \in R;\ \mu_{\tilde{A}}(\bar{x}) = 1$,
iii. $\mu_{\tilde{A}}$ is piecewise continuous.

**Definition 3.3**: A fuzzy number $\tilde{A}$ with the following membership function is called a trapezoidal fuzzy number and is denoted by $\tilde{A} = (m_1, m_2, \alpha_1, \alpha_2)$:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - (m_1 - \alpha_1)}{\alpha_1}, & m_1 - \alpha_1 \leq x \leq m_1, \\ 1, & m_1 \leq x \leq m_2, \\ \dfrac{(m_2 + \alpha_2) - x}{\alpha_2}, & m_2 \leq x \leq m_2 + \alpha_2, \\ 0, & \text{otherwise.} \end{cases}$$

We denote all trapezoidal fuzzy numbers by $F(\mathbb{R})$.

**Definition 3.4:** The arithmetic operations on $\tilde{A} = (m_1, m_2, \alpha_1, \alpha_2)$ and $\tilde{B} = (n_1, n_2, \beta_1, \beta_2)$ are defined as follows:

$$\tilde{A} \oplus \tilde{B} = (m_1 + n_1, m_2 + n_2, \alpha_1 + \beta_1, \alpha_2 + \beta_2)$$

$$k\tilde{A} = \begin{cases} (km_1, km_2, k\alpha_1, k\alpha_2), & k \geq 0 \\ (km_2, km_1, -k\alpha_2, -k\alpha_1), & k < 0 \end{cases}$$

An efficient approach to ordering fuzzy numbers is based on the concept of comparing fuzzy numbers by using ranking functions, in which a ranking function $\Re : F(\mathbb{R}) \to \mathbb{R}$ that maps each fuzzy number to the real line is defined for ranking the elements of $F(\mathbb{R})$. Thus, using the natural order of the real numbers, we can compare fuzzy numbers easily, as follows:

$$\tilde{A}_1 \succeq \tilde{A}_2 \Leftrightarrow \Re(\tilde{A}_1) \geq \Re(\tilde{A}_2)$$
$$\tilde{A}_1 \preceq \tilde{A}_2 \Leftrightarrow \Re(\tilde{A}_1) \leq \Re(\tilde{A}_2)$$
$$\tilde{A}_1 \cong \tilde{A}_2 \Leftrightarrow \Re(\tilde{A}_1) = \Re(\tilde{A}_2)$$
$$\tilde{A}_1 \cong \tilde{A}_2 \Leftrightarrow \Re(\tilde{A}_1) = \Re(\tilde{A}_2)$$

Researchers have proposed several ranking functions to suit their requirements of the problems under consideration. We restrict our attention to linear ranking functions, *i.e.*, a ranking function $\Re$ such that $\Re(k\tilde{A}_1 + \tilde{A}_2) = k\,\Re(\tilde{A}_1) + \Re(\tilde{A}_2)$ for any $\tilde{A}_1$ and $\tilde{A}_2$ belonging to $F(R)$ and any $k \in R$. For a trapezoidal fuzzy number $\tilde{A} = (m_1, m_2, \alpha_1, \alpha_2)$, a linear ranking function introduced by Yager (1981) is used to order fuzzy numbers, as follows:

$\Re(\tilde{A}) = \frac{1}{2}\left[m_1 + m_2 + \frac{\alpha_2 - \alpha_1}{2}\right]$.

**Notation 1.** The real number $A$ corresponds to the trapezoidal fuzzy number $\tilde{A} = (m_1, m_2, \alpha_1, \alpha_2)$ if $A = \Re(\tilde{A}) = \frac{1}{2}\left[m_1 + m_2 + \frac{\alpha_2 - \alpha_1}{2}\right]$.

It is worth mentioning that Notation 1 is used to choose the entering and existing variables in fuzzy primal simplex and fuzzy dual simplex algorithms, respectively, which are given in the next section.

## 4 Linear programming problems with trapezoidal fuzzy variables

In this section, we first formulate the mathematical model of the FVLP problem and then review three existing approaches to solving it. We conclude the section by highlighting the various shortcomings that these approaches suffer from.

**Definition 4.1.** An FVLP problem can be mathematically formulated as follows:

$$\begin{aligned} &\text{Min } \tilde{u} = \tilde{y}b \\ &s.t. \\ &\tilde{y}A \succeq \tilde{c} \\ &\tilde{y} \succeq \tilde{0} \end{aligned} \qquad (1)$$

where $b \in \mathbb{R}^m, c \in F(\mathbb{R})^n, A = (a_{ij})_{m\times n} \in \mathbb{R}^{m\times n}$, and $\tilde{y} \in F(\mathbb{R})^m$. The solution $\tilde{y} \in F(\mathbb{R})^m$ satisfying all the constraints of problem (1) is called a fuzzy feasible solution. The set of fuzzy feasible solutions is denoted by $\tilde{S}$, *i.e.*, $\tilde{S} = \{\tilde{y} \in F(\mathbb{R})^m : \tilde{y}A \succeq \tilde{c}, \tilde{y} \succeq \tilde{0}\}$.

In addition, $\tilde{y}^* \in \tilde{S}$ is called a fuzzy optimal solution if $\tilde{y}^* b \preceq \tilde{y}b$ for all $\tilde{y} \in \tilde{S}$.

For solving the FVLP problem (1), Maleki (2002) introduced a fuzzy auxiliary problem (FAP) and studied the relationships between these problems.

**Definition 4.2.** The FAP for the FVLP problem (1) is defined as follows:

$$\begin{aligned} &\text{Max } \tilde{z} \approx \tilde{c}x \\ &s.t. \\ &Ax \leq b \\ &x \geq 0 \end{aligned} \qquad (2)$$

where $b \in \mathbb{R}^m, c \in F(\mathbb{R})^n, A \in \mathbb{R}^{m\times n}$.

Assume matrix $A$ in the system of equality constraints (2); $A$ is a matrix of order $(m \times n)$ with $rank(A) = m$. Therefore, it can be partitioned as $[B, N]$, where $B_{m\times m}$ is an invertible matrix with $rank(B) = m$. Moreover, the following basic solution:

$$x = (x_B, x_N) = (B^{-1}b, 0) \geq (0, 0) \tag{3}$$

is called a basic feasible solution (BFS) to the system $Ax = b$. Also, the corresponding fuzzy objective function value is obtained as follows:

$$\tilde{z} \approx \tilde{c}_B x_B, \tilde{c}_B = (\tilde{c}_{B_1}, \tilde{c}_{B_2}, \ldots, \tilde{c}_{B_m}). \tag{4}$$

We denote the set of indices of the current non-basic variables by $J_N$ and define for each non-basic variable $x_j$, $j \in J_N$, the fuzzy variables $\tilde{z}_j$ and $\tilde{\bar{c}}_j$ as $\tilde{z}_j \approx \tilde{c}_B B^{-1} a_j = \tilde{c}_B y_j$ and $\tilde{\bar{c}}_j = \tilde{c}_j - \tilde{z}_j$, respectively, where $y_j$ is the solution to the system $By = a_j$.

### 4.1 The first existing approach (Maleki, 2002)

In this subsection, we review the approach proposed by Maleki (2002) for an FVLP problem. Maleki (2002) proved the following theorem to develop a fuzzy method for solving the FAP (2) with equality constraints.

**Theorem 4.1.** *Let $x = (x_B, x_N) = (B^{-1}b, 0)$ be a BFS with a basis B and fuzzy objective value $\tilde{z}$ for the auxiliary problem.*

i. Assume $\tilde{\bar{c}}_k \prec \tilde{0}$ for some non-basic variables $x_k$ and $y_k \nleq 0$. Then, the following new BFS is obtained with objective value $\tilde{z}_{new} = \tilde{z} + \tilde{\bar{c}}_k x_k = \tilde{z} + \tilde{\bar{c}}_k \frac{\bar{b}_r}{y_{rk}}$, such that $\tilde{z} \preceq \tilde{z}_{new}$.

$$\begin{aligned} x_{B_r} &= x_k = \theta = \frac{\bar{b}_r}{y_{rk}} = \min_{1\leq i\leq m}\left\{\frac{\bar{b}_i}{y_{ik}} | y_{ik} > 0\right\}, \\ x_{B_i} &= \bar{b}_i - y_{ik}\frac{\bar{b}_r}{y_{rk}}, \quad i = 1, 2, \ldots, m, \quad i \neq r, \\ x_j &= 0, \quad j \in J_N, \quad j \neq k. \end{aligned} \tag{5}$$

ii. Assume $\tilde{\bar{c}}_k \prec \tilde{0}$ for some non-basic variable $x_k$ while $y_k \nleq 0$; then, the FAP (2) is unbounded.

iii. Assume $\tilde{\bar{c}}_j = \tilde{c}_j - \tilde{z}_j \succeq \tilde{0}$, for all $j \in J_N$. Then, $x = (x_B, x_N) = (B^{-1}b, 0) = (\bar{b}, 0)$ will be an optimal solution to the FAP (2).

The fuzzy primal simplex algorithm (Maleki, 2002; Maleki et al., 2000) for solving the FAP (2) is summarized as follows.

**Algorithm 4.1. Fuzzy primal simplex algorithm (Maximization Problem):**

***Initialization step***

Choose a starting, feasible basis $B$.

***Main steps***

(1) Let $x_B = B^{-1}b = \bar{b}$ be the solution to the system $Bx_B = b$. Let $x_N = 0$ and $\tilde{z} \approx \tilde{c}_B x_B$.

(2) Let $\tilde{y} = \tilde{c}_B B^{-1}$ be the fuzzy solution to the fuzzy system $\tilde{y}B = \tilde{c}_B$.

(3) Let $\Re(\tilde{\bar{c}}_k) = \max_{j\in J_N}\{\Re(\tilde{\bar{c}}_j)\}$. If $\Re(\tilde{\bar{c}}_k) \leq 0$, then the current basic solution is optimal and stop.

(4) Let $y_k = B^{-1}a_k$ be the solution to the system $By_k = a_k$. If $y_k \leq 0$, then the FAP (2) is unbounded and stop.

(5) If $y_k \nleq 0$, then $x_k$ is an entering variable and $x_{Br}$ is a leaving variable provided that

$$\frac{\bar{b}_r}{y_{rk}} = \min_{1\leq i\leq m}\left\{\frac{\bar{b}_i}{y_{ik}} \,|\, y_{rk} > 0\right\}.$$

(6) Update the basic $B$ where $a_k$ replaces $a_{B_r}$, update the index set $J_N$, and go to (1).

In addition, Maleki et al. (2000) and Maleki (2002) proposed a method for solving the FVLP problem (1) by establishing the relation between the FVLP problem (1) and the FAP (2). This method $\tilde{y} = \tilde{c}_B B^{-1}$ provides the fuzzy optimal solution to the FVLP problem (1) by considering the basis B as the optimal basis for the FAP (2). The next section proposes an alternative approach to find the optimal solution $\tilde{y}$ by solving a crisp equivalent problem without using any fuzzy primal simplex algorithm.

### 4.2 The second existing approach (Mahdavi-Amiri and Nasseri, 2007)

In this subsection, we review the proposed method by Mahdavi-Amiri and Nasseri (2007) for solving the FVLP problem.

Mahdavi-Amiri and Nasseri (2007) concentrated on the fuzzy solutions of an FVLP problem in the following form:

$$\begin{aligned} &\text{Min } \tilde{z} \approx c\tilde{x} \\ &s.t. \\ &A\tilde{x} \succeq \tilde{b} \\ &\tilde{x} \succeq \tilde{0} \end{aligned} \tag{6}$$

where $\tilde{b} \in F(\mathbb{R})^m$, $c \in \mathbb{R}^n$, $A = (a_{ij})_{m\times n} \in \mathbb{R}^{m\times n}$, and $\tilde{x} \in F(\mathbb{R})^n$.

The fuzzy dual problem of the FVLP problem (6) is formulated as follows:

$$\begin{aligned} &\text{Max } \tilde{u} \approx y\tilde{b} \\ &s.t. \\ &yA \leq c \\ &y \geq 0 \end{aligned} \tag{7}$$

Since the coefficients of the decision variables in the objective function of this problem are fuzzy numbers, it can be considered as the FAP for the FVLP problem (6). Indeed, with regard to problems (6) and (7), we conclude that the FAP and the FVLP problems are the dual form of each other. The main relations between the primal problem (7) and dual problem (6) are summarized as follows (Mahdavi-Amiri & Nasseri, 2007; Maleki, 2002; Maleki et al., 2000):

**Property 4.1:** If $\tilde{x}_\circ$ and $y_\circ$ are feasible solutions to the FVLP problem (6) and the FAP problem (7), respectively, then $\Re(c\tilde{x}_\circ) \geq \Re(y_\circ \tilde{b})$.

**Property 4.2:** If $\tilde{x}_\circ$ and $y_\circ$ are feasible solutions to the FVLP problem (6) and FAP problem (7), respectively, and $\Re(c\tilde{x}_\circ) = \Re(y_\circ \tilde{b})$, then $\tilde{x}_\circ$ and $y_\circ$ are the optimal solutions to their respective problems.

**Property 4.3:** If the FVLP problem (6) or FAP problem (7) is unbounded, then the other problem has no feasible solution.

**Property 4.4:** If the FVLP problem (6) or FAP problem (7) has an optimal solution, then both problems have optimal solutions, and the two optimal objective fuzzy values are equal.

Mahdavi-Amiri and Nasseri (2007) deduced the above-mentioned duality results in a fuzzy environment. They introduced the following fuzzy dual simplex algorithm for solving the FVLP problem that directly works on the primary problem and does not use the solution to any FAP.

**Algorithm 4.2. Fuzzy dual simplex algorithm (Minimization Problem):**

**Initialization step**

Find a starting dual feasible basis $B$.

**Algorithm steps**

(1) Let $\tilde{x}_B = B^{-1}\tilde{b} = \tilde{\bar{b}}$ be the fuzzy solution to the fuzzy system $B\tilde{x}_B = \tilde{b}$ with $\tilde{z} \approx c_B\tilde{x}_B$.

(2) Let $\Re(\tilde{\bar{b}}_r) = \min_{1 \leq i \leq m} \left\{ \Re(\tilde{\bar{b}}_i) \right\}$. If $\Re(\tilde{\bar{b}}_r) \geq 0$, then the current basic solution is an optimal solution and stop.

(3) Let $y_j = B^{-1}a_j$ be the solution to the system $By_j = a_j$. If $y_{rj} \geq 0$, for all $j \in J_N$, then the FVLP problem is infeasible and stop.

(4) Let $y = c_B B^{-1}$ be the solution to the system $yB = c_B$. Compute $z_j = ya_j$ for all $j \in J_N$. If $y_{rj} \ngeq 0$, for all $j \in J_N$, then $x_{Br}$ is a leaving variable and $x_k$ is an entering variable provided that:

$$\frac{z_k - c_k}{y_{rk}} = \min_{j \in J_N} \left\{ \frac{z_j - c_j}{y_{rk}} \,|\, y_{rk} < 0 \right\}.$$

(5) Update the basic $B$ where $a_k$ replaces $a_{B_r}$, update the index set $J_N$, and go to (1).

## 4.3 The third existing approach (Ebrahimnejad et al., 2010)

In this subsection, we review the method proposed by Ebrahimnejad et al. (2010) for solving the FVLP problem. Ebrahimnejad et al. (2010) concentrated on the fuzzy solutions to the FVLP problem (6) with equality constraints and its corresponding dual problem. Recall that the fuzzy primal simplex method (Maleki, 2002) begins with a primal, basic solution and that the fuzzy dual simplex method (Mahdavi-Amiri & Nasseri, 2007) begins with a dual basic dual solution to the corresponding primal problems. Ebrahimnejad et al. (2010) proposed a fuzzy primal–dual algorithm that begins with dual feasibility and obtains primal feasibility while maintaining complementary slackness. An important difference between Algorithms 4.1 and 4.2 and the fuzzy primal–dual simplex method is that the fuzzy primal–dual algorithm does not require a feasible dual solution to be basic.

**Algorithm 4.3. Fuzzy primal–dual simplex algorithm (Minimization Problem):**

**Initialization step**

Choose a vector $w$ such that $wa_j - c_j = 0$, for all $j$.

**Algorithm steps**

(1) Let $\Omega = \{j : c_j - wa_j = 0\}$ and solve the following restricted FVLP problem:

$$\tilde{x}_0 = \min \sum_{j \in \Omega} 0\tilde{x}_j + 1\tilde{x}_a$$
$$s.t.$$
$$\sum_{j \in \Omega} a_j\tilde{x}_j + 1\tilde{x}_a = \tilde{b},$$
$$\tilde{x}_j \succeq \tilde{0}, \quad j \in \Omega,$$
$$\tilde{x}_a \succeq \tilde{0}.$$

If $\tilde{x}_j \cong \tilde{0}$, then stop, the current solution is optimal. Else, suppose $v^*$ is the optimal solution to the dual of the restricted FVLP problem.

(2) If $v^*a_j \leq 0$, for all $j$, then stop, the FVLP problem is infeasible. Else, let

$$\theta = \frac{-(wa_k - c_k)}{v^*a_k} = \min_j \left\{ \frac{-(wa_j - c_j)}{v^*a_j} \,|\, v^*a_j > 0 \right\},$$

and replace $w$ by $w + \theta v^*$ and go to (2).

## 4.4 The shortcomings of the existing approaches

The overall results from the previous subsection confirm that:

- To solve an FVLP problem according to the proposed methods in Maleki (2002) and Maleki et al. (2000) and to solve the restricted FVLP problem given in Algorithm 4.3, it is required to find the optimal solution to a FAP problem with a fuzzy objective function.
- According to Algorithms 4.1 and 4.3, it is necessary to solve a fuzzy system with all the iterations for solving the corresponding FAP and the dual problem of the restricted FVLP problem, respectively.
- Although Algorithm 4.2 proposed by Mahdavi-Amiri and Nasseri (2007) gives the optimal solution to the same FVLP problem without solving any FAP, it is also necessary to solve a fuzzy system for all the iterations.
- All Algorithms 4.1, 4.2, and 4.3 proposed by Maleki et al. (2000), Mahdavi-Amiri and Nasseri (2007), and Ebrahimnejad et al. (2010), respectively, require a large number of fuzzy additions, subtractions, and comparisons on the fuzzy numbers for all the iterations.
- In the primal–dual pair of Maleki (2002) and Mahdavi-Amiri and Nasseri (2007), one problem provides a crisp solution, and the other provides a fuzzy solution. However, both problems give the same fuzzy optimal objective value.

In the next section, we propose an approach for solving the FVLP problem (6) with an equivalent LP problem in a crisp environment and without solving any FAP.

## 5 Proposed approach

For simplicity, we reformulate the FVLP problem (1), with respect to the change of all the parameters and variables, as the FVLP problem (6). As mentioned earlier, Maleki et al. (2000) first introduced a FAP corresponding to the FVLP problem and then used Algorithm 4.1 to find its crisp optimal solution. The obtained crisp solution has been used to obtain the fuzzy optimal solution to the FVLP problem under consideration. It is worth noting that they generalized the primal simplex algorithm for solving the FAP. In their algorithm, a ranking function based on Notation 1 is used for choosing the entering variable.

Similarly, Mahdavi-Amiri and Nasseri (2007) extended the dual simplex method for solving the FVLP problem (6). In their algorithm, a ranking function based on Notation 1 is also used to compare the fuzzy numbers to choose the leaving variable. Based on this ranking function, a rank is defined for the trapezoidal fuzzy number; $\tilde{A} = (m_1, m_2, \alpha_1, \alpha_2)$ is defined as $A = \Re(\tilde{A}) = \frac{1}{2}\left[m_1 + m_2 + \frac{\alpha_2 - \alpha_1}{2}\right]$. According to this definition, the FVLP problem (6) is transformed into a crisp LP problem when its rank substitutes a fuzzy number of the problem. According to this substitution, an equivalent crisp LP problem is obtained. We mention that the decision variables of the FVLP problem (6) are fuzzy numbers. Thus, we require a fuzzy solution, while the equivalent problem mentioned above gives a crisp solution. However, the basis $B$ is the optimal basis for both the equivalent crisp problem and the corresponding FVLP problem. In this case, the fuzzy solution $(B^{-1}\tilde{b}, \tilde{0})$ is the optimal fuzzy solution to the FVLP problem.

It is worth mentioning that the fuzzy primal–dual simplex method proposed by Ebrahimnejad et al. (2010) needs to solve a restricted FVLP problem given in Algorithm 4.3 and its dual problem as the corresponding FAP. The restricted FVLP problem can be solved by Algorithm 4.1 or Algorithm 4.2, and its corresponding dual problem, which is in the form of FAP, can be solved by Algorithm 4.1. Therefore, the proposed approach in this paper can be used for solving the restricted FVLP problem and its corresponding dual problem with less computational effort.

In summary, by choosing a certain linear ranking function, the fuzzy problem is transformed into a crisp one, which is easily solved by the existing LP methods. Thus, the FVLP problem is solved without any auxiliary problem and any fuzzy dual approach. Such a process decreases the computational complexity significantly in our proposed approach. The main aim of this study is to reduce the computational complexity of the existing methods (Ebrahimnejad et al., 2010; Mahdavi-Amiri & Nasseri, 2007; Maleki, 2002; Maleki et al., 2000). It is shown that our proposed method needs less elementary operations such as additions, subtractions, and comparisons when compared to the existing methods mentioned above.

According to the method proposed by Maleki et al. (2000), to carry out Step 2 of Algorithm 4.1, the fuzzy system $\tilde{y}B = \tilde{c}_B$ with $m$ fuzzy equations and $m$ fuzzy variables in all iterations is solved. After that, the fuzzy value $\tilde{\bar{c}}_j$ for each non-basic variable is calculated based on $\tilde{\bar{c}}_j = \tilde{c}_j - \tilde{y}a_j$. Finally, the entering variable is determined according to the most negative rank of $\tilde{\bar{c}}_j$. As we see, Step 2 and Step 3 of this algorithm require many fuzzy additions and subtractions on trapezoidal fuzzy numbers for all the iterations. There is neither a need to solve a fuzzy system nor any fuzzy arithmetic operations in this method when choosing an entering variable.

Moreover, according to the fuzzy dual method (Mahdavi-Amiri & Nasseri, 2007), to carry out Step 1 of Algorithm 4.2, it is necessary to solve the fuzzy system $B\tilde{x}_B = \tilde{b}$ with $m$ fuzzy equations and $m$ fuzzy variables in all the iterations. After solving this fuzzy system, the leaving variable is determined according to the most negative rank of $\tilde{\bar{b}}_i$. As we see, Step 1 and Step 2 of this algorithm require many fuzzy additions and subtractions on trapezoidal fuzzy numbers for all the iterations. In this method, the leaving variable is found without solving any fuzzy system without any fuzzy arithmetic operations. In addition, in our proposed approach, the comparison of a fuzzy number is carried out just once, and all arithmetic operations use real numbers.

Finally, according to the fuzzy primal–dual simplex method (Ebrahimnejad et al., 2010), it is required to utilize both Algorithms 4.1 and 4.2 in all the iterations of Algorithm 4.3 for solving the restricted FVLP problem and the corresponding fuzzy dual problem. Thus, this method needs many fuzzy arithmetic operations and comparisons on fuzzy numbers. On the other hand, our proposed method solves the FVLP problems based on classical simplex algorithms and without solving any fuzzy systems.

These results confirm that the computational complexity of the proposed method is greatly reduced compared with the approach based on the FAP and the fuzzy dual approach.

**Theorem 5.1:** *The optimal solution to the FAP* (2) according to the existing fuzzy primal algorithm method and the proposed method is the same.

***Proof:*** According to the proposed approach, using the linear ranking function given in Notation 1, we substitute the rank of each trapezoidal fuzzy cost for the corresponding trapezoidal fuzzy cost in FAP (2). This leads to the following crisp LP problem:

$$\begin{aligned} &\text{Max} \ \ \Re(\tilde{z}) = \Re(\tilde{c})b \\ &s.t. \\ &Ax \le b \\ &x \ge 0 \end{aligned} \tag{8}$$

Let $\Re(\tilde{w})$ be the dual variable associated with constraint $Ax \le b$. In this case, the dual of problem (8) is given as follows:

$$\begin{aligned} &\text{Min} \ \ \Re(\tilde{z}) = b\Re(\tilde{w}) \\ &s.t. \\ &\Re(\tilde{w})A \ge \Re(\tilde{c}) \\ &\Re(\tilde{w}) \ge 0 \end{aligned} \tag{9}$$

Therefore, if the basis $B$ is the optimal basis for the crisp dual problem (9), we have the optimal conditions of problem (8). It should be noted that to obtain the optimal solution according to the existing fuzzy primal algorithm (Maleki, 2002; Maleki et al., 2000) and our proposed method, the FAP (2) and the crisp problem (8) have to be solved, respectively. If we show that these problems have the same optimal solution, we can conclude that the results of our proposed approach match those obtained based on the existing fuzzy primal algorithm (Maleki, 2002; Maleki et al., 2000). Note that both problems have the same feasible space. Thus, it is sufficient to show that both problems have the same optimality conditions. If the basis $B$ is the optimal basis for the FAP (2), then it will be the optimal basis for the equivalent crisp problem (8). To do this, assume $x^*$ is the optimal solution to the FAP (2) with the optimal basis $B$ derived from the existing fuzzy primal algorithm (Maleki, 2002; Maleki et al., 2000). Thus, $\tilde{\bar{c}}_j = \tilde{c}_j - \tilde{z}_j \succeq \tilde{0}$, for all $j = 1, 2, ..., n$, according to Theorem 4.1. This condition is equivalent to:

$$\begin{aligned} \Re(\tilde{\bar{c}}_j) = \Re(\tilde{c}_j) - \Re(\tilde{z}_j) \ge 0 &\Leftrightarrow \Re(\tilde{c}_j) - \Re(\tilde{c}_B B^{-1} a_j) \ge 0 \\ &\Leftrightarrow \Re(\tilde{c}_j) - \Re(\tilde{w} a_j) \ge 0 \Leftrightarrow \Re(\tilde{c}_j) - \Re(\tilde{w}) a_j \ge 0 \\ &\Leftrightarrow \Re(\tilde{w}) a_j \le \Re(\tilde{c}_j) \end{aligned}$$

Thus, we conclude that $x^*$ is the optimal solution to the crisp problem (8). This completes the proof. □

Similarly, we can prove that the optimal solution to the FVLP (6) according to the existing fuzzy dual and primal–dual algorithms method and the proposed method is the same.

**Theorem 5.2:** *The optimal solution to the FVLP* (6) according to the existing fuzzy primal and fuzzy primal–dual simplex algorithms method and the proposed method is the same.

***Proof*** It is similar to the proof of Theorem 5.1. □

***Remark 5.1*** The fuzzy primal, fuzzy dual, and fuzzy primal–dual simplex algorithms are designed to solve FVLP problems where the decision variables are nonnegative fuzzy numbers. The proposed approach applies to any fuzzy resource vector involving nonnegative or nonpositive fuzzy numbers and nonnegative fuzzy decision variables. In case there exists a nonpositive fuzzy decision variable $\tilde{y} \preceq \tilde{0}$ in the FVLP problem (1), it can be replaced by $-\tilde{y}' \succeq \tilde{0}$ where $\tilde{y}' \succeq \tilde{0}$.

## 6 Comparative examples

In this section, we solve three numerical examples with the existing methods and the proposed method.

***Example 6.1.*** Consider the following FVLP problem (Mahdavi-Amiri & Nasseri, 2007; Maleki, 2002; Maleki et al., 2000):

$$\begin{aligned} &\text{Min} \ \ \tilde{z} \approx 6\tilde{y}_1 + 10\tilde{y}_2 \\ &s.t. \\ &2\tilde{y}_1 + 5\tilde{y}_2 \succeq (5,\ 8,\ 2,\ 5) \\ &3\tilde{y}_1 + 4\tilde{y}_2 \succeq (6,\ 10,\ 2,\ 6) \\ &\tilde{y}_1, \tilde{y}_2 \succeq \tilde{0} \end{aligned} \tag{10}$$

**Solution based on the first existing approach (Maleki, 2002; Maleki et al., 2000)**

For solving the FVLP problem (10), Maleki et al. (2000) used the following FAP:

$$\begin{aligned}&\text{Max}\ \ \tilde{u} \approx (5,8,2,5)x_1 + (6,10,2,6)\,x_2\\&s.t.\\&2x_1 + 3x_2 \le 6\\&5x_1 + 4x_2 \le 10\\&x_1,\ \ x_2 \ge 0\end{aligned} \tag{11}$$

Problem (11) can be reformulated as follows:

$$\begin{aligned}&\text{Max}\ \ \tilde{u} \approx (5,8,2,5)x_1 + (6,10,2,6)x_2\\&s.t.\\&2x_1 + 3x_2 + x_3 = 6\\&5x_1 + 4x_2 + x_4 = 10\\&x_1,\ \ x_2,\ \ x_3, x_4 \ge 0\end{aligned} \tag{12}$$

Now, we use Algorithm 4.1 for solving the FAP (12).

**Iteration 1:**

The identity matrix $B = [a_3, a_4] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the initial feasible basis. In this case, the non-basis matrix $N$ will be $N = [a_1, a_2] = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$.

**Step 1:** We solve system $Bx_B = b$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}.$$

The solution to this system is given as $(x_{B_1} = x_3 = 6, x_{B_2} = x_4 = 10)$. Thus, $x_1 = 0$ and $x_2 = 0$ are the values of the non-basic variables. The fuzzy objective function is obtained as

$$\tilde{z} \approx \tilde{c}_B x_B = ((0,0,0,0),(0,0,0,0))\begin{bmatrix} 6 \\ 10 \end{bmatrix} = (0,0,0,0).$$

*Step 2:* Now, we solve the fuzzy system $\tilde{y}B = \tilde{c}_B$:

$$(\tilde{y}_1, \tilde{y}_2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (0,0,0,0) \\ (0,0,0,0) \end{bmatrix}.$$

By solving this fuzzy system, we obtain $\tilde{y}_1 = \tilde{y}_2 = (0,0,0,0)$.

*Step 3:* We compute the fuzzy values of $\tilde{\bar{c}}_j = \tilde{c}_j - \tilde{y}a_j$ for all non-basic variables:

$$\begin{aligned}\tilde{\bar{c}}_1 &= \tilde{c}_1 - \tilde{y}a_1 = (5,8,2,5) - ((0,0,0,0),(0,0,0,0))\begin{bmatrix} 2 \\ 5 \end{bmatrix}\\ &= (5,8,2,5);\end{aligned}$$

$$\begin{aligned}\tilde{\bar{c}}_2 &= \tilde{c}_2 - \tilde{y}a_2 = (6,10,2,6) - ((0,0,0,0),(0,0,0,0))\begin{bmatrix} 3 \\ 4 \end{bmatrix}\\ &= (6,10,2,6).\end{aligned}$$

Therefore,

$$\max\left\{\Re(\tilde{\bar{c}}_2) = \frac{29}{2}, \Re(\tilde{\bar{c}}_2) = 9\right\} = \Re(\tilde{\bar{c}}_2) = 9.$$

*Step 4:* Now, we solve system $By_2 = a_2$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

This means that $y_2 = \begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

*Step 5:* The following test gives the index of the leaving variable $x_{Br}$:

$$\min\left\{\frac{\overline{b}_1}{y_{12}}, \frac{\overline{b}_2}{y_{22}}\right\} = \min\left\{\frac{6}{3}, \frac{10}{4}\right\} = \frac{6}{3} = 2.$$

Hence, the basic variable $x_{B_1} = x_3$ is selected as the leaving variable.

*Step 6:* The updated basis $B$ is given as follows, where $a_2$ replaces $a_{B_1} = a_3$:

$$B = [a_2, a_4] = \begin{bmatrix} 3 & 0 \\ 4 & 1 \end{bmatrix}.$$

Also, we update the set of indexes of the non-basis variable $J_N = \{1,2\}$ as $J_N = \{1,3\}$.

**Iteration 2:**

*Step 1:* The new BFS is found by solving system $Bx_B = b$:

$$\begin{aligned} 3x_2 &= 6\\ 4x_2 + x_4 &= 10\end{aligned}$$

The solution to this system is $(x_{B_1} = x_2 = 2, x_{B_2} = x_4 = 2)$. The non-basic variables are $x_1 = 0$ and $x_3 = 0$. Also, the fuzzy objective value is obtained as

$$\tilde{z} \approx \tilde{c}_B x_B = ((6,10,2,6),(0,0,0,0))\begin{bmatrix} 2 \\ 2 \end{bmatrix} = (12,20,4,12)$$

.

*Step 2:* Now, we solve the fuzzy system $\tilde{y}B = \tilde{c}_B$:

$$\begin{aligned} 3\tilde{y}_1 + 4\tilde{y}_2 &= (6,10,2,6)\\ \tilde{y}_2 &= (0,0,0,0)\end{aligned}$$

The fuzzy solution to this system is $\tilde{y}_1 = \left(2, \frac{10}{3}, \frac{2}{3}, 2\right),\ \ \tilde{y}_2 = (0,0,0,0)$.

*Step 3:* Now, we obtain the fuzzy values of $\tilde{\bar{c}}_j = \tilde{c}_j - \tilde{y}a_j$ for all current non-basic variables:

$$\begin{aligned}\tilde{\bar{c}}_1 &= \tilde{c}_1 - \tilde{y}a_1 = (5,8,2,5)\\ &\quad - \left(\left(2, \frac{10}{3}, \frac{2}{3}, 2\right),(0,0,0,0)\right)\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \left(\frac{5}{3}, 4, 6, \frac{19}{3}\right);\end{aligned}$$

$$\begin{aligned}\tilde{\bar{c}}_3 &= \tilde{c}_3 - \tilde{y}a_3 = (0,0,0,0)\\ &\quad - \left(\left(2, \frac{10}{3}, \frac{2}{3}, 2\right),(0,0,0,0)\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left(-\frac{10}{3}, -2, 2, \frac{2}{3}\right).\end{aligned}$$

Therefore,

$$\max\left\{\Re(\tilde{\tilde{c}}_1)=\frac{5}{4},\Re(\tilde{\tilde{c}}_3)=-3\right\}=\Re(\tilde{\tilde{c}}_1)=\frac{5}{4}\not\leq 0.$$

*Step 4:* Now, system $By_1=a_1$ is solved as follows:

$$3y_{11}=2$$
$$4y_{11}+y_{21}=5$$

By solving this system, we obtain

$$y_1=\begin{bmatrix}y_{11}\\ y_{21}\end{bmatrix}=\begin{bmatrix}\frac{2}{3}\\ \frac{7}{3}\end{bmatrix}$$

.

*Step 5:* Now, the following test is used to determine the index $r$ of the leaving variable $x_{Br}$:

$$\min\left\{\frac{\overline{b}_1}{y_{11}},\frac{\overline{b}_2}{y_{21}}\right\}=\min\left\{\frac{2}{\frac{2}{3}},\frac{2}{\frac{7}{3}}\right\}=\frac{6}{7}.$$

Hence, the basic variable $x_{B_2}=x_4$ is selected as the leaving variable.

*Step 6:* The new basis $B$ is given as follows, where $a_1$ replaces $a_{B_2}=a_4$:

$$B=[a_2,a_1]=\begin{bmatrix}3&2\\4&5\end{bmatrix}.$$

Also, we update the set of indexes of the non-basis variable $J_N=\{1,3\}$ as $J_N=\{3,4\}$.

**Iteration 3:**

*Step 1:* The new BFS is given by solving system $Bx_B=b$:

$$3x_2+2x_1=6$$
$$4x_2+5x_1=10$$

This leads to $x_{B_1}=x_2=\frac{10}{7}$ and $x_{B_2}=x_1=\frac{6}{7}$. The value of a fuzzy objective function is

$$\tilde{z}=\tilde{c}_Bx_B=((6,10,2,6),(5,8,2,5))\begin{bmatrix}\frac{10}{7}\\ \frac{6}{7}\end{bmatrix}$$
$$=\left(\frac{90}{7},\frac{148}{7},\frac{32}{7},\frac{96}{7}\right).$$

*Step 2:* Now, the fuzzy system $\tilde{y}B=\tilde{c}_B$ is solved as follows:

$$3\tilde{y}_1+4\tilde{y}_4=(6,10,2,6)$$
$$2\tilde{y}_1+5\tilde{y}_4=(5,8,2,5)$$

The fuzzy solution to this system is given as

$$\tilde{y}_1=\left(\frac{-2}{7},\frac{30}{7},\frac{30}{7},\frac{38}{7}\right),\ \tilde{y}_2=\left(\frac{-5}{7},\frac{12}{7},\frac{18}{7},\frac{19}{7}\right)$$

.

*Step 3:* The values of $\tilde{\tilde{c}}_j=\tilde{c}_j-\tilde{y}a_j$ are calculated for all current non-basic variables as follows:

$$\begin{aligned}\tilde{\tilde{c}}_3=&\tilde{c}_3-\tilde{y}a_3=(0,0,0,0)\\&-\left(\left(\frac{-2}{7},\frac{30}{7},\frac{30}{7},\frac{38}{7}\right),\left(\frac{-5}{7},\frac{12}{7},\frac{18}{7},\frac{19}{7}\right)\right)\begin{bmatrix}1\\0\end{bmatrix}\\&=\left(\frac{-30}{7},\frac{2}{7},\frac{38}{7},\frac{30}{7}\right);\\ \tilde{\tilde{c}}_4=&\tilde{c}_4-\tilde{y}a_4=(0,0,0,0)\\&-\left(\left(\frac{-2}{7},\frac{30}{7},\frac{30}{7},\frac{38}{7}\right),\left(\frac{-5}{7},\frac{12}{7},\frac{18}{7},\frac{19}{7}\right)\right)\begin{bmatrix}0\\1\end{bmatrix}\\&=\left(\frac{-12}{7},\frac{5}{7},\frac{19}{7},\frac{18}{7}\right).\end{aligned}$$

Now, the ranks of $\tilde{\tilde{c}}_3$ and $\tilde{\tilde{c}}_4$ are computed as follows:

$$\max\left\{\Re(\tilde{\tilde{c}}_3)=-\frac{16}{7},\Re(\tilde{\tilde{c}}_4)=-\frac{15}{28}\right\}=-\frac{15}{28}\leq 0.$$

This means that the current basis $B=[a_2,a_1]=\begin{bmatrix}3&2\\4&5\end{bmatrix}$ is optimal. The optimal solution to the FAP (11), therefore, is given by $(x_1,x_2,x_3,x_4)=\left(\frac{6}{7},\frac{10}{7},0,0\right)$. Thus, as mentioned in Sect. 4, we can obtain the fuzzy optimal solution to the FVLP problem (10) as follows:

$$\begin{aligned}\tilde{y}&=(\tilde{y}_1,\tilde{y}_2)=\tilde{c}_BB^{-1}\\&=((6,10,2,6),(5,8,2,5))\begin{bmatrix}\frac{5}{7}&\frac{-2}{7}\\ \frac{-4}{7}&\frac{3}{7}\end{bmatrix}\\&=\left(\left(\frac{-2}{7},\frac{30}{7},\frac{30}{7},\frac{38}{7}\right),\left(\frac{-5}{7},\frac{12}{7},\frac{18}{7},\frac{19}{7}\right)\right)\end{aligned}\tag{13}$$

**Solution based on the second existing approach (Mahdavi-Amiri and Nasseri, 2007)**

The fuzzy dual approach proposed by Mahdavi-Amiri and Nasseri (2007) is used to obtain the fuzzy optimal solution to the FVLP problem (10). This problem can be reformulated as follows for the change of all the parameters and variables:

$$\begin{aligned} &\text{Min} \;\; \tilde{z} \approx 6\tilde{x}_1 + 10\tilde{x}_2 \\ &s.t. \\ &2\tilde{x}_1 + 5\tilde{x}_2 \succeq (5, 8, 2, 5) \\ &3\tilde{x}_1 + 4\tilde{x}_2 \succeq (6, 10, 2, 6) \\ &\tilde{x}_1, \;\; \tilde{x}_2 \succeq \tilde{0} \end{aligned} \tag{14}$$

The FVLP problem (14) is reformulated as follows:

$$\begin{aligned} &\text{Min} \;\; \tilde{z} \approx 6\tilde{x}_1 + 10\tilde{x}_2 \\ &s.t. \\ &-2\tilde{x}_1 - 5\tilde{x}_2 + \tilde{x}_3 \approx (-8, -5, 5, 2) \\ &-3\tilde{x}_1 - 4\tilde{x}_2 + \tilde{x}_4 \approx (-10, -6, 6, 2) \\ &\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4 \succeq \tilde{0} \end{aligned} \tag{15}$$

Now, we apply Algorithm 3.2 to solve this problem.

**Iteration 1:**

The identity matrix $B = [a_3, a_4] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the initial starting dual feasible basis. Thus, the non-basis matrix $N$ is $N = [a_1, a_2] = \begin{bmatrix} -2 & -5 \\ -3 & -4 \end{bmatrix}$.

*Step 1:* Now, we solve the fuzzy system $B\tilde{x}_B = \tilde{b}$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} = \begin{bmatrix} (-8, -5, 5, 2) \\ (-10, -6, 6, 2) \end{bmatrix}.$$

The fuzzy solution to this fuzzy system is

$$\begin{bmatrix} \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} = \begin{bmatrix} (-8, -5, 5, 2) \\ (-10, -6, 6, 2) \end{bmatrix} = \begin{bmatrix} \tilde{\bar{b}}_1 \\ \tilde{\bar{b}}_2 \end{bmatrix}$$

. The fuzzy value of the objective function value is given as follows:

$$\tilde{z} = c_B \tilde{x}_B = (0, 0) \begin{bmatrix} (-8, -5, 5, 2) \\ (-10, -6, 6, 2) \end{bmatrix} = (0, 0, 0, 0).$$

*Step 2:* Therefore, we have:

$$\min\left\{ \Re(\tilde{\bar{b}}_1) = -\frac{29}{4}, \Re(\tilde{\bar{b}}_2) = -9 \right\} = \Re(\bar{b}_2) = -9 \not\succeq 0.$$

Hence, the variable $\tilde{x}_{B_2} = \tilde{x}_4$ is selected as the leaving variable.

*Step 3:* Now, the system $By_j = a_j$ is solved for $j \in J_N = \{1, 2\}$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix} = \begin{bmatrix} -5 \\ -4 \end{bmatrix}.$$

By solving the above systems, we obtain

$$y_1 = \begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}, y_2 = \begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix} = \begin{bmatrix} -5 \\ -4 \end{bmatrix}.$$

*Step 4:* Now, we solve system $yB = c_B$:

$$(y_1, y_2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

By solving this system, we obtain $y_1 = y_2 = 0$.

Now, $z_j - c_j = ya_j - c_j$ for all $j \in J_N = \{1, 2\}$ is calculated as follows:

$$z_1 - c_1 = ya_1 - c_1 = (0, 0) \begin{bmatrix} -2 \\ -3 \end{bmatrix} - 6 = -6;$$

$$z_2 - c_2 = ya_2 - c_2 = (0, 0) \begin{bmatrix} -5 \\ -4 \end{bmatrix} - 10 = -10.$$

Hence, the following test is used to determine the index $k$ of the entering variable $x_k$:

$$\min\left\{ \frac{z_1 - c_1}{y_{21}}, \frac{z_2 - c_2}{y_{22}} \right\} = \min\left\{ \frac{-6}{-3}, \frac{-10}{-4} \right\} = \frac{-6}{-3} = 2.$$

Hence, $x_k = x_1$ is selected as the entering variable.

*Step 5:* The updated basis $B$ is given as follows, where $a_1$ replaces $a_{B_2} = a_4$:

$$B = [a_3, a_1] = \begin{bmatrix} 1 & -2 \\ 0 & -3 \end{bmatrix}.$$

The new set of indices of non-basic variables is $J_N = \{2, 4\}$.

**Iteration 2:**

*Step 1:* Now, the fuzzy system $B\tilde{x}_B = \tilde{b}$ is solved as follows:

$$\begin{aligned} \tilde{x}_3 - 2\tilde{x}_1 &= (-8, -5, 5, 2), \\ -3\tilde{x}_1 &= (-10, -6, 6, 2) \end{aligned}$$

The fuzzy solution to this system is given as follows:

$$\begin{bmatrix} \tilde{x}_3 \\ \tilde{x}_1 \end{bmatrix} = \begin{bmatrix} \left(-4, \frac{5}{3}, \frac{19}{3}, 6\right) \\ \left(2, \frac{10}{3}, \frac{2}{3}, 2\right) \end{bmatrix} = \begin{bmatrix} \tilde{\bar{b}}_1 \\ \tilde{\bar{b}}_2 \end{bmatrix}.$$

By substituting the fuzzy values of $\tilde{x}_3$ and $\tilde{x}_1$ in $\tilde{z} \approx c_B \tilde{x}_B$, we obtain the fuzzy objective function value as follows:

$$\tilde{z} \approx (0, 6) \begin{bmatrix} \left(-4, \frac{5}{3}, \frac{19}{3}, 6\right) \\ \left(2, \frac{10}{3}, \frac{2}{3}, 2\right) \end{bmatrix} = (12, 20, 4, 12).$$

*Step 2:* The following test is used to obtain the index of the leaving variable:

$$\min\left\{ \Re(\tilde{\bar{b}}_1) = -\frac{15}{12}, \Re(\tilde{\bar{b}}_2) = 3 \right\} = \Re(\bar{b}_1) = -\frac{15}{12} \not\succeq 0.$$

This means that the variable $\tilde{x}_{B_1} = \tilde{x}_3$ is selected as the leaving variable.

*Step 3:* Now, system $By_j = a_j$, for $j \in J_N = \{2,4\}$, is solved as follows:

$$\begin{bmatrix} 1 & -2 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix} = \begin{bmatrix} -5 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} y_{14} \\ y_{24} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Thus, we obtain

$$y_2 = \begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix} = \begin{bmatrix} \frac{-7}{3} \\ \frac{4}{3} \end{bmatrix}, y_4 = \begin{bmatrix} y_{14} \\ y_{24} \end{bmatrix} = \begin{bmatrix} \frac{-2}{3} \\ \frac{-1}{3} \end{bmatrix}$$

.

*Step 4:* Now, system $yB = c_B$ is solved as follows:

$$(y_1, y_2)\begin{bmatrix} 1 & -2 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix} \Rightarrow y_1 = 0, \; y_2 = -2.$$

Now, we calculate the values of $z_j - c_j$, for all $j \in J_N = \{2,4\}$:

$$z_2 - c_2 = (0,-2)\begin{bmatrix} -5 \\ -4 \end{bmatrix} - 10 = -2;$$

$$z_4 - c_4 = (0,-2)\begin{bmatrix} 0 \\ 1 \end{bmatrix} - 0 = -2.$$

Now, the following test is used to determine the index of the entering variable $x_k$:

$$\min\left\{\frac{z_2 - c_2}{y_{12}}, \frac{z_4 - c_4}{y_{14}}\right\} = \min\left\{\frac{-2}{\frac{-7}{3}}, \frac{-2}{\frac{-2}{3}}\right\} = \frac{-2}{\frac{-7}{3}} = \frac{6}{7}.$$

Hence, the entering variable will be $x_k = x_2$.

*Step 5:* The updated basis $B$ is given as follows, where $a_2$ replaces $a_{B_1} = a_3$:

$$B = [a_2, a_1] = \begin{bmatrix} -5 & -2 \\ -4 & -3 \end{bmatrix}.$$

Also, the updated set of the index of non-basic variables is $J_N = \{3,4\}$.

**Iteration 3:**

*Step 1:* We solve the fuzzy system $B\tilde{x}_B = \tilde{b}$:

$$-5\tilde{x}_2 - 2\tilde{x}_1 = (-8, \; -5, \; 5, \; 2)$$
$$-4\tilde{x}_2 - 3\tilde{x}_1 = (-10, \; -6, \; 6, \; 2)$$

The fuzzy solution to this fuzzy system is given as follows:

$$\begin{bmatrix} \tilde{x}_2 \\ \tilde{x}_1 \end{bmatrix} = \begin{bmatrix} \left(\frac{-5}{7}, \frac{12}{7}, \frac{18}{7}, \frac{19}{7}\right) \\ \left(\frac{-2}{7}, \frac{30}{7}, \frac{30}{7}, \frac{38}{7}\right) \end{bmatrix} = \begin{bmatrix} \tilde{\bar{b}}_1 \\ \tilde{\bar{b}}_2 \end{bmatrix}.$$

Hence, by substituting the fuzzy values of $\tilde{x}_2$ and $\tilde{x}_1$ in the objective function, we have:

$$\tilde{z} \approx (10,6)\begin{bmatrix} \left(\frac{-5}{7}, \frac{12}{7}, \frac{18}{7}, \frac{19}{7}\right) \\ \left(\frac{-2}{7}, \frac{30}{7}, \frac{30}{7}, \frac{38}{7}\right) \end{bmatrix}$$
$$= \left(\frac{-62}{7}, \frac{300}{7}, \frac{360}{7}, \frac{418}{7}\right).$$

*Step 2:* To check the feasibility conditions of the current fuzzy solution, we use the following test:

$$\min\left\{\Re(\tilde{\bar{b}}_1) = \frac{15}{28}, \Re(\tilde{\bar{b}}_2) = \frac{16}{7}\right\} = \Re(\bar{b}_1) = \frac{15}{28} \geq 0.$$

Thus, the basis $B = [a_2, a_1]$ is the optimal basis. Now, the optimal solution to the fuzzy FVLP problem (14) is given by:

$$\begin{bmatrix} \tilde{x}_2 \\ \tilde{x}_1 \end{bmatrix} = \begin{bmatrix} \left(\frac{-5}{7}, \frac{12}{7}, \frac{18}{7}, \frac{19}{7}\right) \\ \left(\frac{-2}{7}, \frac{30}{7}, \frac{30}{7}, \frac{38}{7}\right) \end{bmatrix}. \tag{16}$$

As we can see, the solution given in Eq. (16) obtained by the fuzzy dual method matches the solution given in Eq. (13) obtained by the methods proposed by Maleki et al. (2000) and Maleki (2002).

**Solution based on our proposed approach:**

Finally, we solve the FVLP (14) based on our proposed approach. We first obtain the rank of all the fuzzy parameters of the FVLP problem (14), leading to the following equivalent crisp problem:

$$\begin{aligned} &\text{Min } z = 6x_1 + 10x_2 \\ &s.t. \\ &2x_1 + 5x_2 \geq (5, \; 8, \; 2, \; 5) \\ &3x_1 + 4x_2 \geq (6, \; 10, \; 2, \; 6) \\ &x_1, \; x_2 \geq 0 \end{aligned} \tag{17}$$

The LP problem (17) is reformulated as follows by introducing the surplus variables $x_3$ and $x_4$:

$$\begin{aligned} &\text{Min } z = 6x_1 + 10x_2 \\ &s.t. \\ &2x_1 + 5x_2 - x_3 = \frac{29}{4} \\ &3x_1 + 4x_2 - x_4 = 9 \\ &x_1, \; x_2, \; x_3, \; x_4 \geq 0 \end{aligned} \tag{18}$$

The crisp problem (18) is solved using the standard dual simplex algorithm. Based on this approach, we obtain the basis $B = [a_1, a_2] = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$ as the optimal basis for both the LP problem (17) and the FVLP problem (14) or (10). Thus, as mentioned in Sect. 5, the fuzzy optimal solution to the FVLP problem (14) or (10) is computed as follows:

$$\tilde{x}_B = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = B^{-1}\tilde{b} = \begin{bmatrix} \frac{-4}{7} & \frac{5}{7} \\ \frac{3}{7} & \frac{-2}{7} \end{bmatrix} \begin{bmatrix} (5,8,2,5) \\ (6,10,2,6) \end{bmatrix} = \begin{bmatrix} \left(\frac{-2}{7},\frac{30}{7},\frac{30}{7},\frac{38}{7}\right) \\ \left(\frac{-5}{7},\frac{12}{7},\frac{18}{7},\frac{19}{7}\right) \end{bmatrix}. \tag{19}$$

As shown here, the solution given in Eq. (19) obtained by our proposed approach matches the solution given in Eq. (13) derived from Maleki et al.'s (2000) method and the solution given in Eq. (16) derived from Mahdavi-Amiri and Nasseri's (2007) method. However, in contrast to existing methods (Mahdavi-Amiri & Nasseri, 2007; Maleki, 2002; Maleki et al., 2000), our approach greatly reduces the computational complexity.

***Example 6.2.*** Consider the following FVLP problem:

$$\begin{aligned} &\text{Max } \tilde{z} \approx 4\tilde{y}_1 + 6\tilde{y}_2 \\ &s.t. \\ &3\tilde{y}_1 + 4\tilde{y}_2 \preceq (4, 6, 2, 6) \\ &2\tilde{y}_1 + \tilde{y}_2 \preceq (2, 3, 1, 3) \\ &\tilde{y}_1, \tilde{y}_2 \succeq \tilde{0} \end{aligned} \tag{20}$$

**Solution based on the first existing approach** (Maleki, 2002; Maleki et al., 2000)

To solve the FVLP problem (20) using Maleki et al.'s (2000) method, we should solve the following FAP:

$$\begin{aligned} &\text{Min } \tilde{u} \approx (4, 6, 2, 6)x_1 + (2, 3, 1, 3)x_2 \\ &s.t. \\ &3x_1 + 2x_2 \geq 4 \\ &4x_1 + x_2 \geq 6 \\ &x_1, x_2 \geq 0 \end{aligned} \tag{21}$$

For solving the FAP (21) by Maleki et al.'s (2000) method, we need to solve the following LP problem by introducing the surplus variables $x_3$ and $x_4$, and the artificial variables $x_5$ and $x_6$. This problem minimizes the sum of the artificial variables over the feasible space of the LP problem (19):

$$\begin{aligned} &\text{Min } \tilde{u} \approx (1, 1, 0, 0)x_5 + (1, 1, 0, 0)x_6 \\ &s.t. \\ &3x_1 + 2x_2 - x_3 + x_5 = 4 \\ &4x_1 + x_2 - x_4 + x_6 = 6 \\ &x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned} \tag{22}$$

At first, this problem is solved by using Algorithm 4.1 to find an initial fuzzy BFS. After that, we must minimize the original objective function of problem (21). Finally, concerning the relationship between the FAP (21) and the FVLP problem (20), we can solve problem (20). So, this process is time-consuming and computationally inefficient.

However, this approach gives the basis $B = [a_1, a_2] = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ as the optimal basis for problem (21). In this case, the fuzzy optimal solution to the FVLP problem (20) is obtained as follows:

$$\tilde{y} = \tilde{c}_B B^{-1} = (\tilde{y}_1, \tilde{y}_2) = ((4, 6, 2, 6), (2, 3, 1, 3)) \begin{bmatrix} \frac{-1}{5} & \frac{2}{5} \\ \frac{4}{5} & \frac{-3}{5} \end{bmatrix} = \left(\left(\frac{2}{5},\frac{8}{5},\frac{10}{5},\frac{14}{5}\right), \left(\frac{-1}{5},\frac{6}{5},\frac{13}{5},\frac{15}{5}\right)\right) \tag{23}$$

It is worth noting that the FVLP problem (20) cannot be solved by the fuzzy dual approach (Mahdavi-Amiri & Nasseri, 2007), because an initial feasible dual solution with an identity basis is not at hand.

**Solution based on our proposed approach:**

To solve the FVLP (20) based on our proposed approach, we first reformulate it by changing all the parameters and variables, as follows:

$$\begin{aligned} &\text{Max } \tilde{z} \approx 4\tilde{x}_1 + 6\tilde{x}_2 \\ &s.t. \\ &3\tilde{x}_1 + 4\tilde{x}_2 \preceq (4, 6, 2, 6) \\ &2\tilde{x}_1 + \tilde{x}_2 \preceq (2, 3, 1, 3) \\ &\tilde{x}_1, \tilde{x}_2 \succeq \tilde{0} \end{aligned} \tag{24}$$

We then obtain the rank order of all the fuzzy data in the FVLP problem (24), leading to the following crisp problem:

$$\begin{aligned} &\text{Max } z = 4x_1 + 6x_2 \\ &s.t \\ &3x_1 + 4x_2 \leq 6 \\ &2x_1 + x_2 \leq 3 \\ &x_1, x_2 \geq 0 \end{aligned} \tag{25}$$

The LP problem (25) is reformulated as follows:

$$\begin{aligned} &\text{Max } z = 4x_1 + 6x_2 \\ &s.t. \\ &3x_1 + 4x_2 + x_3 = 6 \\ &2x_1 + x_2 + x_4 = 3 \\ &x_1, x_2, x_3, x_4 \geq 0 \end{aligned} \tag{26}$$

The classical primal simplex algorithm solves the crisp problem (26). This gives the basis $B = [a_1, a_2] = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$ as the optimal basis for both the LP problem (26) and the

FVLP problem (24) or (20). Thus, as mentioned in Sect. 5, we obtain the optimal solution to the FVLP problem (24) or (20) as follows:

$$\tilde{x}_B = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = B^{-1}\tilde{b} = \begin{bmatrix} \frac{-1}{5} & \frac{4}{5} \\ \frac{2}{5} & \frac{-3}{5} \end{bmatrix} \begin{bmatrix} (4,6,2,6) \\ (2,3,1,3) \end{bmatrix} = \begin{bmatrix} \left(\frac{2}{5},\frac{8}{5},\frac{10}{5},\frac{14}{5}\right) \\ \left(\frac{-1}{5},\frac{6}{5},\frac{13}{5},\frac{15}{5}\right) \end{bmatrix}. \quad (27)$$

As shown here, the solution given in Eq. (27) is equivalent to the optimal solution (23) derived from the method proposed by Maleki et al. (2000).

***Example 6.3.*** Consider the following FVLP problem (Ebrahimnejad et al., 2010):

$$\begin{aligned} &\text{Min } \tilde{z} \approx -\tilde{y}_1 - 6\tilde{y}_2 \\ &s.t. \\ &\tilde{y}_1 + \tilde{y}_2 - \tilde{y}_3 \cong (1, 3, 1, 1) \\ &\tilde{y}_1 + 2\tilde{y}_2 + \tilde{y}_4 \cong (2, 4, 1, 1) \\ &\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4 \succeq \tilde{0} \end{aligned} \quad (28)$$

For solving the FVLP problem (28), Ebrahimnejad et al. (2010) used the following FAP as the dual of problem (28):

$$\begin{aligned} &\text{Max } \tilde{u} \approx (1, 3, 1, 1)x_1 + (2, 4, 1, 1)x_2 \\ &s.t. \\ &w_1 + w_2 \leq -1 \\ &w_1 + 2w_2 \leq -6 \\ &w_1, \quad w_2 \geq 0 \end{aligned} \quad (29)$$

Now, we use Algorithm 4.3 for solving the FVLP problem (28) and simultaneously explore our approach.

**Iteration 1:**

*Step 1:* An initial dual feasible solution is given by $w = (0, -3)$. By substituting $w$ in each dual constraint, we get $\Omega = \{j : wa_j - c_j = 0\} = \{2, 3\}$.

*Step 2:* We solve the following restricted FVLP problem:

$$\begin{aligned} &\text{Min } \tilde{x}_0 \approx \tilde{y}_5 + \tilde{y}_6 \\ &s.t. \\ &y_2 - \tilde{y}_3 + \tilde{y}_5 \cong (1, 3, 1, 1) \\ &2\tilde{y}_2 \quad + \tilde{y}_6 \cong (2, 4, 1, 1) \\ &\tilde{y}_2, \tilde{y}_3, \tilde{y}_5, \tilde{y}_6 \succeq \tilde{0} \end{aligned} \quad (30)$$

Similar to Example 6.1 and Example 6.2, problem (30) can be solved using the method proposed by Maleki (2002) or Mahdavi-Amiri and Nasseri (2007). But, these approaches require a large number of fuzzy additions, subtractions, and comparisons on the fuzzy numbers for all the iterations. Thus, based on our proposed approach, we get the following crisp problem:

$$\begin{aligned} &\text{Min } x_0 \approx y_5 + y_6 \\ &s.t. \\ &y_2 - y_3 + y_5 = 2 \\ &2y_2 + y_6 = 3 \\ &y_2, y_3, y_5, y_6 \geq 0 \end{aligned} \quad (31)$$

The classical simplex algorithm solves the crisp problem (31). This gives the basis $B = [a_2, a_5] = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ as the optimal basis for both the LP problem (31) and the restricted FVLP problem (30). Thus, as mentioned in Sect. 5, we obtain the optimal solution to the restricted FVLP problem (30) as follows:

$$\tilde{y}_B = \begin{pmatrix} \tilde{y}_2 \\ \tilde{y}_5 \end{pmatrix} = B^{-1}\tilde{b} = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} (1,3,1,1) \\ (2,4,1,1) \end{bmatrix} = \begin{bmatrix} \left(1,2,\frac{1}{2},\frac{1}{2}\right) \\ \left(-1,2,\frac{3}{2},\frac{3}{2}\right) \end{bmatrix}. \quad (32)$$

The non-basic basic variable is $\tilde{y}_6 \cong \tilde{0}$. Thus, the fuzzy optimal value of the objective function (30) is $\tilde{y}_0 \cong \left(-3, 4, \frac{7}{2}, \frac{7}{2}\right)$. Now, we need the optimal solution to the dual of the restricted FVLP problem (30). The dual problem is given as follows:

$$\begin{aligned} &\text{Max } (1, 3, 1, 1)v_1 + (2, 4, 1, 1)v_2 \\ &s.t. \\ &v_1 + 2v_2 \leq 0 \\ &-v_1 \leq 0 \\ &v_1 \leq 1 \\ &v_2 \leq 1 \end{aligned} \quad (33)$$

The fuzzy problem (33) plays the role of FAP for the restricted FVLP problem (30) and can be solved by using Algorithm 4.1 that requires solving fuzzy systems with all the iterations. Thus, according to our approach, we first obtain the rank of all the fuzzy parameters of the FAP (32), leading to the following equivalent crisp problem:

$$\begin{aligned} &\text{Max } 2v_1 + 3v_2 \\ &s.t. \\ &v_1 + 2v_2 \leq 0 \\ &-v_1 \leq 0 \\ &v_1 \leq 1 \\ &v_2 \leq 1 \end{aligned} \quad (34)$$

The classical simplex algorithm solves the crisp problem (34). This gives the optimal solution $v^* = \left(1, \frac{-1}{2}\right)$ as the optimal solution to both the LP problem (34) and the FAP (33).

*Step 3:* By computing $v^* a_j$ for all $j \in \Omega$, we conclude that $v^* a_1 = \frac{1}{2}$ and $v^* a_4 = \frac{-1}{2}$. Thus, $\theta$ and $w$ are determined as $\theta = \min\left\{\frac{-(-2)}{\frac{1}{2}}\right\} = 4$ and $w = (0, -3) + 4\left(1, \frac{-1}{2}\right) = (4, -5)$, respectively.

**Iteration 2:**

*Step 1:* An initial dual feasible solution $w = (4, -5)$ is given by $w = (0, -3)$. By substituting $w$ in each dual constraint, we get $\Omega = \{j : wa_j - c_j = 0\} = \{2, 3\}$.

With the new dual fuzzy solution $w = (4, -5)$, we recompute Q and obtain $\Omega = \{1, 2\}$.

*Step 2:* We solve the following new restricted FVLP problem:

$$\begin{aligned} &\text{Min } \tilde{x}_0 \approx \tilde{y}_5 + \tilde{y}_6 \\ &s.t. \\ &\tilde{y}_1 + \tilde{y}_2 + \tilde{y}_5 \cong (1, 3, 1, 1) \\ &\tilde{y}_1 + 2\tilde{y}_2 + \tilde{y}_6 \cong (2, 4, 1, 1) \\ &\tilde{y}_1, \tilde{y}_2, \tilde{y}_5, \tilde{y}_6 \succeq \tilde{0} \end{aligned} \tag{35}$$

Again, the FVLP problem (35) can be solved using Algorithms 4.1 and 4.2. However, based on our proposed approach, we get the following crisp problem:

$$\begin{aligned} &\text{Min } x_0 = y_5 + y_6 \\ &s.t. \\ &y_1 + y_2 + y_5 = 2 \\ &y_1 + 2y_2 + y_6 = 3 \\ &y_1, y_2, y_5, y_6 \geq 0 \end{aligned} \tag{36}$$

The classical simplex algorithm gives the basis $B = [a_1, a_2] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ as the optimal basis for both the LP problem (37) and the restricted FVLP problem (36). Thus, we obtain the optimal solution to the restricted FVLP problem (35) as follows:

$$\tilde{y}_B = \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} = B^{-1}\tilde{b} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} (1,3,1,1) \\ (2,4,1,1) \end{bmatrix} = \begin{bmatrix} (-2,4,3,3) \\ (-1,3,2,2) \end{bmatrix}. \tag{37}$$

The non-basic basic variables are $\tilde{y}_5 \cong \tilde{0}$ and $\tilde{y}_6 \cong \tilde{0}$. Thus, the current solution is optimal. The optimal solution to the original FVLP problem (28) with the optimal primal and dual solutions are:

$$\tilde{y}_1 \cong (-2, 4, 3, 3),\ \tilde{y}_2 \cong (-1, 3, 2, 2),\ \tilde{y}_3 = \tilde{y}_4 \cong \tilde{0},\ w = (4, -5)). \tag{38}$$

As shown here, the solution obtained by our proposed approach matches the solution given in Eq. (38) obtained by the method proposed by Ebrahimnejad et al. (2010). However, unlike the fuzzy primal–dual simplex algorithm (Ebrahimnejad et al., 2010), our proposed approach requires fewer fuzzy computations.

## 7 Coal transportation case study

The transportation problem (TP) is one of the most used and tangible applications of LP problems that apply to a variety of practical settings (Charles et al., 2011; Ebrahimnejad & Verdegay, 2016b, 2018c). The main objective of this problem is to find an optimal transfer plan with the minimum cost of shipping the goods so that the demands of the destinations are satisfied using the supplies available at the sources. In this section, a real-life fuzzy TP is used to show the applicability of our proposed approach.

It is worth noting that coal as a crucial energy source plays an important role in the development of an economy. The main aim of the coal transportation problem, which is represented by a complex system of interconnected transport links, is to find an optimal plan to transport the coal from the mines to different areas at the lowest cost (Gupta et al., 2018). Suppose that there are three coal mines that supply coal to four cities. The decision-maker (DM) wants to design the optimal transportation plan for the next month. Initially, he/she needs to determine the value of the supply, demand, and the unit transportation cost of the goods. The DM knows the exact value of the transportation cost of the coal from the different mines to the different cities. However, due to some uncontrollable factors, the values of the supply and demand may be unpredictable in a TP. So, in this case, we are modeling a fuzzy transportation problem (FTP). Table 1 summarizes the supplies available from the three mines and the demands from the four cities for the next month. This problem aims to minimize the total transportation costs.

The DM must determine the amounts of coal transported from the mines to the cities. Thus, we denote the fuzzy amounts of coal in tons transported from mine $i$ to city $j$ (for $i$ = 1, 2, 3, and $j$ = 1, 2, 3, 4) by $\tilde{x}_{ij}$, which are fuzzy decision variables in the FTP under consideration.

In this case, the total fuzzy cost of supplying the coal demands to the cities is given as follows:

**Table 1** Fuzzy coal transportation problem

| Source | Destination | | | | Supply (in tons) |
|---|---|---|---|---|---|
| | City 1 | City 2 | City 3 | City 4 | |
| Mine 1 | 18 | 6 | 8 | 12 | (100, 105, 5, 10) |
| Mine 2 | 9 | 7 | 9 | 15 | (60, 63, 3, 6) |
| Mine 3 | 10 | 5 | 8 | 8 | (20, 21, 1, 2) |
| Demand (in tons) | (80, 84, 4, 8) | (40, 42, 2, 4) | (40, 42, 2, 4) | (20, 21, 1, 2) | |

$$18\tilde{x}_{11} + 6\tilde{x}_{12} + 8\tilde{x}_{13} + 12\tilde{x}_{14} + 9\tilde{x}_{21} + 7\tilde{x}_{22} + 9\tilde{x}_{23} + 15\tilde{x}_{24} + 10\tilde{x}_{31} + 5\tilde{x}_{32} + 8\tilde{x}_{33} + 8\tilde{x}_{34}.$$

There are two types of constraints. First, the total coal supplied by each mine cannot exceed the capacity of the respective mine. We may formulate the supply constraint as follows:

$$\tilde{x}_{11} + \tilde{x}_{12} + \tilde{x}_{13} + \tilde{x}_{14} \tilde{\leq} (100,\ 105,\ 5, 10)$$
$$\tilde{x}_{21} + \tilde{x}_{22} + \tilde{x}_{23} + \tilde{x}_{24} \tilde{\leq} (60, 63, 3, 6)$$
$$\tilde{x}_{31} + \tilde{x}_{32} + \tilde{x}_{33} + \tilde{x}_{34} \tilde{\leq} (20, 21, 1, 2)$$

Second, the following constraints, known as demand constraints, ensure that each city will receive sufficient coal to satisfy its demand:

$$\tilde{x}_{11} + \tilde{x}_{21} + \tilde{x}_{31} \tilde{\geq} (80, 84, 4, 8)$$
$$\tilde{x}_{21} + \tilde{x}_{22} + \tilde{x}_{32} \tilde{\geq} (40, 42, 2, 4)$$
$$\tilde{x}_{31} + \tilde{x}_{31} + \tilde{x}_{33} \tilde{\geq} (40, 42, 2, 4)$$
$$\tilde{x}_{41} + \tilde{x}_{42} + \tilde{x}_{43} \tilde{\geq} (20, 21, 1, 2)$$

Thus, we obtain the following FVLP formulation of the regional coal transportation problem:

$$\begin{aligned}
\text{Min } \tilde{z} \approx\ & 18\tilde{x}_{11} + 6\tilde{x}_{12} + 8\tilde{x}_{13} + 12\tilde{x}_{14} + 9\tilde{x}_{21} + 7\tilde{x}_{22} + \\
& 9\tilde{x}_{23} + 15\tilde{x}_{24} + 10\tilde{x}_{31} + 5\tilde{x}_{32} + 8\tilde{x}_{33} + 8\tilde{x}_{34} \\
s.t.\ & \\
& \tilde{x}_{11} + \tilde{x}_{12} + \tilde{x}_{13} + \tilde{x}_{14} \tilde{\leq}\ (100,\ 105,\ 5, 10) \\
& \tilde{x}_{21} + \tilde{x}_{22} + \tilde{x}_{23} + \tilde{x}_{24} \tilde{\leq}\ (60,\ 63,\ 3, 6) \\
& \tilde{x}_{31} + \tilde{x}_{32} + \tilde{x}_{33} + \tilde{x}_{34} \tilde{\leq}\ (20,\ 21,\ 1, 2) \\
& \tilde{x}_{11} + \tilde{x}_{21} + \tilde{x}_{31} \tilde{\geq}\ (80,\ 84,\ 4, 8) \\
& \tilde{x}_{21} + \tilde{x}_{22} + \tilde{x}_{32} \tilde{\geq}\ (40,\ 42,\ 2, 4) \\
& \tilde{x}_{31} + \tilde{x}_{31} + \tilde{x}_{33} \tilde{\geq}\ (40,\ 42,\ 2, 4) \\
& \tilde{x}_{41} + \tilde{x}_{42} + \tilde{x}_{43} \tilde{\geq}\ (20,\ 21,\ 1, 2) \\
& \tilde{x}_{ij} \tilde{\geq} \tilde{0},\ i = 1, 2, 3,\ j = 1, 2, 3, 4.
\end{aligned} \tag{38}$$

Figure 1 shows a graphical representation of the fuzzy coal transportation problem.

By using the proposed method for solving problem (38), the optimal solution is obtained as follows:

$$\tilde{x}^*_{11} = (20, 21, 1, 2),\ \tilde{x}^*_{12} = (40, 42, 2, 4),$$
$$\tilde{x}^*_{13} = (40, 42, 2, 4),$$
$$\tilde{x}^*_{21} = (60, 63, 3, 6),\ \tilde{x}^*_{34} = (20, 21, 1, 2),$$
$$\tilde{z}^* = (1540,\ 1617,\ 77, 154)$$

These values represent the fuzzy amounts of coal that should be sent from each mine to each city (Table 2). Figure 2 shows the graphical representation of the optimal transportation plan.

Now, we interpret the minimum total fuzzy transportation cost. Figure 3 shows the membership function of the obtained total fuzzy cost. As we can see from Fig. 3, the least and the greatest amounts of minimum total fuzzy cost are 1463 and 1771 units, respectively. Moreover, the greatest possible amount of minimum total fuzzy cost lies between 1540 and 1617 units, *i.e.*, the maximum chances are that it will lie between 1540 and 1617 units.

## 8 Conclusions and future research

In this study, we proposed a solution methodology for solving FVLP problems where only the values of the resource vectors and the decision variables are represented in terms of fuzzy numbers. An equivalent crisp LP problem was proposed to derive the fuzzy optimal solution to the FVLP problem. In the proposed approach, the FVLP problem was transformed into a crisp equivalent LP problem. The obtained results confirmed that our proposed approach requires less fuzzy computations as opposed to the existing fuzzy methods (Ebrahimnejad et al., 2010; Mahdavi-Amiri & Nasseri, 2007; Maleki, 2002; Maleki et al., 2000). In summary, to solve the FVLP problem by using the existing methods, fuzzy arithmetic operations and the solution to fuzzy systems of equations are required. By contrast, only arithmetic operations of real numbers and the solution to crisp systems of equations are required to solve the same problem with the method proposed in this study.

It is worth noting that based on Notation 1, Maleki et al. (2000) defined the rank for each fuzzy number for comparison purposes. From the decision-maker's point of view, the rank of the trapezoidal fuzzy number $\tilde{A} = (m_1, m_2, \alpha_1, \alpha_2)$ has been defined by

**Fig. 1** Graphical representation of the fuzzy coal transportation problem

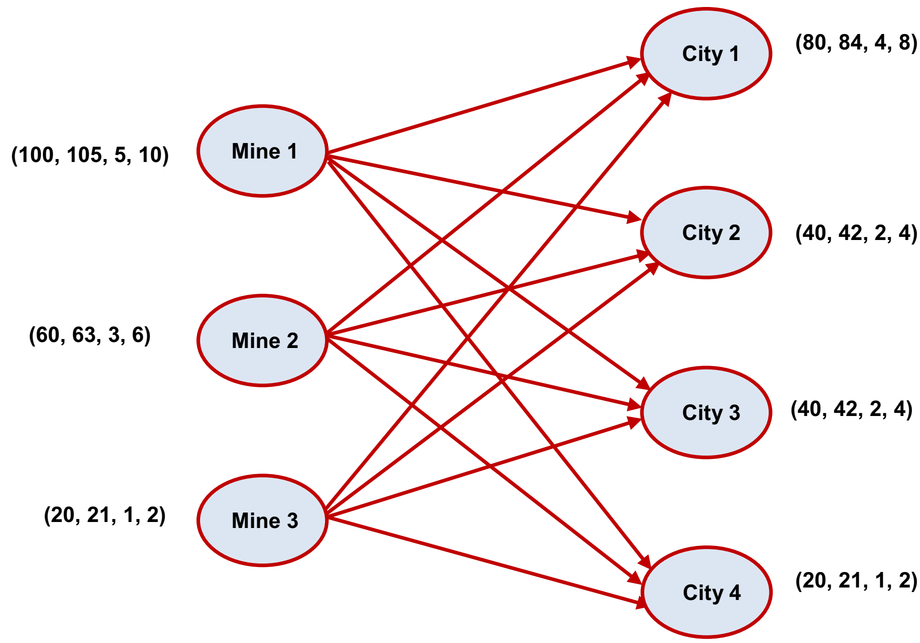


**Table 2** Tabular representation of the optimal solution

| Source | Destination | | | | Supply (in tons) |
|---|---|---|---|---|---|
| | City 1 | City 2 | City 3 | City 4 | |
| Mine 1 | 18 (20, 21, 1, 2) | 6 (40, 42, 2, 4) | 8 (40, 42, 2, 4) | 12 | (100, 105, 5, 10) |
| Mine 2 | 9 (60, 63, 3, 6) | 7 | 9 | 15 | (60, 63, 3, 6) |
| Mine 3 | 10 | 5 | 8 | 8 (20, 21, 1, 2) | (20, 21, 1, 2) |
| Demand (in tons) | (80, 84, 4, 8) | (40, 42, 2, 4) | (40, 42, 2, 4) | (20, 21, 1, 2) | |

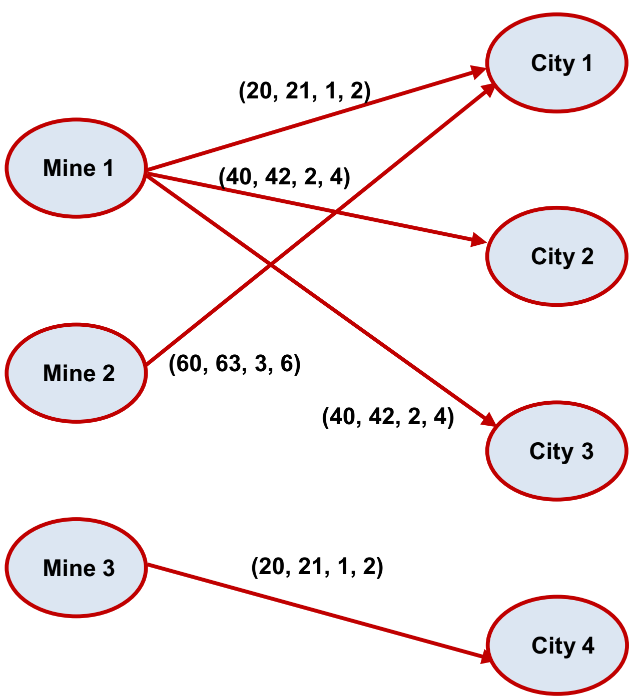


**Fig. 2** Optimal transportation plan

$A = \Re(\tilde{A}) = \frac{1}{2}\left[m_1 + m_2 + \frac{\alpha_2 - \alpha_1}{2}\right]$. However, $\Re(\tilde{A}) = \Re(\tilde{B})$ cannot guarantee equality $\tilde{A} = \tilde{B}$. Therefore, it would be a useful topic for future research to define a new ranking function for solving FVLP problems satisfying this property. Moreover, we point out that the FLP problem considered here does not involve fuzzy numbers as the coefficients of the objective function and the constraint matrix. Future research could focus on developing a new approach to solving FFLP problems to overcome this limitation. Finally, the current model considers the cost objective function or profit objective function as a single objective function, but a multi-objective optimization model that accounts for both cost and profit objective functions is helpful in practice. Thus, by extending the model to a multi-objective optimization model, computational intelligence techniques (Beed et al., 2020) may be more effective for real-world applications.

From a practical perspective, our proposed approach can be employed in a variety of real-world applications. For example, it can be applied to the case of bid evaluations

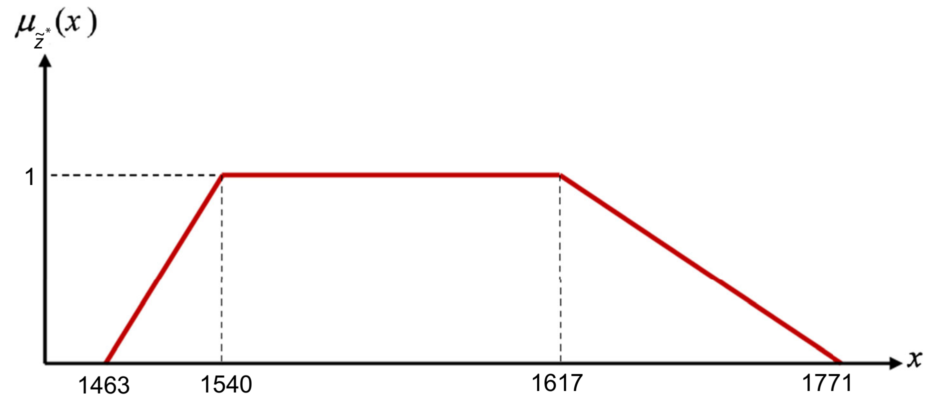


**Fig. 3** Minimum fuzzy transportation cost

since the process of bid evaluation (Chen et al., 2021b) is subject to indetermination, imprecision, and uncertainty in terms of their alternative-criterion decision appraisals. In such a case, fuzzy expressions can be used to evaluate bidder performance. Furthermore, the proposed approach can be used to determine passenger demands and evaluate passenger satisfaction by combining online review analysis and large-scale group decision-making in the high-speed rail industry (Chen et al., 2021a). In such a case, fuzzy numbers can be applied to express satisfaction degrees regarding passenger demands.

**Acknowledgements** Dr. Madjid Tavana is grateful for the partial support he received from the Czech Science Foundation (GCR19-13946S) for this research. The authors are also thankful to the reviewers for their valuable comments on previous drafts of this paper.

**Author contributions** Ali Ebrahimnejad contributed to conceptualization, methodology, formal analysis, validation, data curation, writing, and reviewing. Madjid Tavana contributed to methodology, formal analysis, visualization, writing, reviewing, and Editing. Vincent Charles contributed to methodology, investigation, formal analysis, validation, writing, Reviewing, and editing.

## Declarations

**Conflict of interest** The above authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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