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An overall profit Malmquist productivity index with fuzzy and interval data

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ABSTRACT

Although crisp data are fundamentally indispensable for determining the profit Malmquist productivity index (MPI), the observed values in real-world problems are often imprecise or vague. These imprecise or vague data can be suitably characterized with fuzzy and interval methods. In this paper, we reformulate the conventional profit MPI problem as an imprecise data envelopment analysis (DEA) problem, and propose two novel methods for measuring the overall profit MPI when the inputs, outputs, and price vectors are fuzzy or vary in intervals. We develop a fuzzy version of the conventional MPI model by using a ranking method, and solve the model with a commercial off-the-shelf DEA software package. In addition, we define an interval for the overall profit MPI of each decision-making unit (DMU) and divide the DMUs into six groups according to the intervals obtained for their overall profit efficiency and MPIs. We also present two numerical examples to demonstrate the applicability of the two proposed models and exhibit the efficacy of the procedures and algorithms.

1. Introduction

Efficiency and productivity measurement in organizations has enjoyed a great deal of interest among researchers studying performance analysis. Data envelopment analysis (DEA) is a popular method for comparing the inputs and outputs of a set of homogenous decision-making units (DMUs) by evaluating their relative efficiency. Charnes et al. [1] originally proposed the first DEA model, known as the CCR model (see also [2]). DEA is non-parametric, and it utilizes linear programming (LP) to measure the relative efficiency of the DMUs without a priori specification of input and output weights (or multipliers). A score of 1 is assigned to the frontier (efficient) units. The frontier units in DEA are those with maximum output levels for given input levels or with minimum input levels for given output levels. While DEA does not provide

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a precise mechanism for achieving efficiency, it does help in quantifying the magnitude of change required to make the inefficient DMUs efficient and hence contribute to productivity growth.

In addition to comparing the relative performance of a set of DMUs at a specific period, conventional DEA can also calculate the productivity change of a DMU over time. Caves et al. [3,4] proposed a Malmquist productivity index, hereafter referred to as the MPI, which calculated the relative performance of a DMU for different time periods. While the efficiency measures are calculated by DEA, the productivity is measured by the MPI and defined as the ratio between efficiency, as calculated by the DEA, for the same DMU in two different time periods. Several decompositions for calculating the MPI have been proposed in the literature. The most popular method is the one proposed by Färe et al. [5], which uses the geometric mean of the MPIs calculated from two base periods. Färe et al. [5] also proposed a non-parametric Malmquist index for productivity analysis that relied on constructing a best practice frontier and computing the distance of individual observations from the frontier. Ray and Desli [6] proposed a variable returns to scale (VRS) decomposition for the MPI to calculate scale efficiency changes based on a technology defined by DMUs of two periods.

Many applications have been reported in the literature [7] using the MPI to calculate a productivity change (e.g., [8–11]). More specifically, Asmild et al. [12] have presented a framework where DEA was used to measure the overall efficiency by considering more general behavioral goals. They clarified the relationships between models used to measure overall efficiency and various cone-ratio DEA models. They showed that as the multiplier cones tighten, the cone-ratio DEA models converge to the measures of overall efficiency. Furthermore, they argued that multiplier cone and cone-ratio model selection must be consistent with the behavioral goals. Consistent with this reasoning, they introduced two new models for measuring effectiveness when value measures were represented by separable or linked cones.

The conventional MPI requires precise measurement of the inputs and outputs. However, one of the main challenges associated with the application of the MPI is the difficulty in quantifying some of the input and output data in real-world problems where the observed values are often imprecise or vague. Imprecise or vague data may be the result of unquantifiable, incomplete, and non-obtainable information. It is also generally very difficult to accurately evaluate the data for DMUs when the system's complexity increases. Furthermore, decision makers generally prefer using linguistic phrases and expressions such as "good" profit or "low" inventory in their day-to-day communication. In order to use vague and uncertain data in the MPI, it is necessary to represent them with the interval approach or fuzzy set theory.

The imprecise data representation with interval, ordinal, and ratio interval data was initially proposed by Cooper et al. [13–15] to study the uncertainty in DEA. Soon after, many researchers adopted the concept and proposed different DEA models with interval data in the DEA literature [16–20].

The alternative imprecise method, which is called "fuzzy DEA", can be classified into four general classes of methods, as follows: (1) the fuzzy ranking approach (e.g., [21,22]); (2) the possibility approach (e.g., [23,24]); (3) the tolerance approach (e.g., [25]); and (4) the α-level-based approach (e.g., [26–29]).

Computing the MPI using DEA with fuzzy data or interval data has not been studied extensively in the literature, as most DEA methods using ambiguous and vague data have been based on the simple CCR or BCC models, where only the input and output data are represented in fuzzy and/or interval forms (e.g., [30,20,28,31,32,22–24,26,16,19]). In this paper, we reformulate the conventional profit MPI model as an imprecise DEA model and propose two novel methods for measuring the overall profit MPI when the inputs, outputs, and price vectors are fuzzy or vary in intervals. We develop a fuzzy version of the conventional MPI model by using a ranking method and solve the model with a commercial off-the-shelf DEA software package. In addition, we define an interval for the overall profit MPI of each DMU and divide the DMUs into six groups according to the intervals obtained for their overall profit efficiency and Malmquist indices.

The remainder of this paper is organized as follows. The next section presents an overview of the overall profit efficiency and Malmquist indices followed by some basic definitions of fuzzy set theory. We then present overall profit Malmquist with fuzzy data followed by overall profit Malmquist with interval data. Next we present two numerical examples to demonstrate the applicability of the proposed frameworks and exhibit the efficacy of the procedures and algorithms. Finally, we present our conclusions and future research directions.

2. Overall profit efficiency and MPIs

The MPI is the ratio of the efficiency measures for the same production unit in two different time periods or between two different observations for the same period. Färe et al. [33,5] developed a non-parametric Malmquist index using DEA. The DEA-based Malmquist productivity is a combined index that can be extended to measure the productivity change of DMUs over time. In this section, we discuss the overall profit efficiency and the overall profit MPIs.

2.1. Overall profit efficiency

Suppose that we have n DMUs, DMU$_j$ ($j = 1, \ldots, n$), each consuming various amounts of m inputs to produce s outputs. Let $x_j = (x_{1j}, \ldots, x_{mj})$ and $y_j = (y_{1j}, \ldots, y_{sj})$ be the input and output vectors, respectively, for DMU$_j$ ($j = 1, \ldots, n$), where $x_j \geq 0$, $y_j \geq 0$, $x_j \neq 0$, and $y_j \neq 0$. We also consider X and Y as the $m \times n$ matrix of the inputs and the $s \times n$ as the matrix of the outputs, respectively. In addition, c and r are the input and output price vectors, respectively, for DMU$_j$ ($j = 1, \ldots, n$), where $c \geq 0$, $r \geq 0$, $c \neq 0$, and $r \neq 0$.

Asmild et al. [12] have proposed the following model (see also [34]) for measuring the overall profit efficiency:

$$\begin{aligned}\max_{\varphi,\theta,\lambda}\quad & \varphi-\theta\\ \text{s.t}\quad & \varphi[r_o^T y_o]\le r_o^T Y\lambda\\ & \theta[c_o^T x_o]\ge c_o^T X\lambda\\ & \lambda\ge 0,\end{aligned}\tag{1}$$

where the index o signifies the particular DMU under consideration: DMU_o ($o = 1, \ldots, n$). The objective function of this linear program is to maximize the profit by maximizing the total revenue (φ) and minimizing total cost (θ) for a given price vector, $p_o^T = (r_o^T, c_o^T)$.

Toloo et al. [35] extended this model for measuring the overall profit efficiency of DMU_o with n different price vectors:

$$\begin{aligned}\max_{\varphi,\theta,\lambda}\quad & \varphi-\theta\\ \text{s.t}\quad & \varphi[r_j^T y_o]\le r_j^T Y\lambda,\quad \forall j,\\ & \theta[c_j^T x_o]\ge c_j^T X\lambda,\quad \forall j,\\ & \lambda\ge 0.\end{aligned}\tag{2}$$

Although the objective of model (2) is similar to that of model (1), this model considers the input and output price vectors, $p_j^T = (r_j^T, c_j^T)$. The following definition and theorem are referred to [35].

Definition 1. DMU_o is overall efficient if, in model (2), $\varphi - \theta = 0$.

Theorem 1. *For every optimal solution* $(\varphi^*, \theta^*, \lambda^*)$ *of* (2)*, we have* $\varphi - \theta \ge 0$.

It should be noted that since CRS models implicitly assume zero maximum profit it is more suitable for profit efficiency evaluation to consider VRS by adding convexity constraint ($1\lambda = 1$) to the above models.

2.2. Overall profit MPIs

The constant returns to scale overall profit MPI can be decomposed using four output distance functions, each defined on a benchmark technology using DEA in two different time periods.

By using the following models, similar to model (2), the within-period distance functions can be estimated as

$$\begin{aligned}& D_o^p(x_o^p, y_o^p | p = t, t+1) = \max_{\varphi,\theta,\lambda}\quad \varphi-\theta\\ & \text{s.t.}\quad \varphi[(r_j^p)^T y_o^p]\le (r_j^p)^T Y^p\lambda,\quad \forall j,\\ & \qquad \theta[(c_j^p)^T x_o^p]\ge (c_j^p)^T X^p\lambda,\quad \forall j,\\ & \qquad \lambda\ge 0.\end{aligned}\tag{3a}$$

X^p and Y^p are respectively the input and output matrices of the observed data for period p. Therefore, we solve model (3a) for $p = t$ and $t + 1$. The adjacent-period distance functions can then be estimated using model (3b), similar to model (2), where model (3b) is solved for $p, q = t, t + 1$ when $p \ne q$:

$$\begin{aligned}& D_o^p(x_o^q, y_o^q | p, q = t, t+1, p\ne q) = \max_{\varphi,\theta,\lambda}\quad \varphi-\theta\\ & \text{s.t.}\quad \varphi[(r_j^p)^T y_o^p]\le (r_j^q)^T Y^q\lambda,\quad \forall j,\\ & \qquad \theta[(c_j^p)^T x_o^p]\ge (c_j^q)^T X^q\lambda,\quad \forall j,\\ & \qquad \lambda\ge 0.\end{aligned}\tag{3b}$$

Theorem 2. *For every optimal solution* $(\varphi^*, \theta^*, \lambda^*)$ *of* (3a)*, we have*

$$\varphi^* - \theta^* \ge 0.$$

Proof. Model (3a) has a feasible solution $\varphi = 1, \theta = 1, \lambda_o = 1, \lambda_j = 0$ ($j \ne o$). Hence, the optimal solution $\varphi - \theta$, denoted by $\varphi^* - \theta^*$, is greater than or equal to 0, i.e., $\varphi^* - \theta^* \ge 0$. □

Remark 1. The objective function values for model (3b) can be less than or equal to zero.

Definition 2. DMU_o is overall efficient if $D_o^{t+1}(x_o^{t+1}, y_o^{t+1}) = 0$ and $D_o^t(x_o^t, y_o^t) = 0$.

Definition 3. The efficiency scores of models (3a) and (3b) are computed as follows

(i) If $\varphi - \theta \ge 0$, then $\rho = \dfrac{1}{1 + \varphi - \theta}$.

(ii) If $\varphi - \theta \leq 0$, then $\rho = 1 + \theta - \varphi$.

Obviously, if $\rho = 1$ DMU_o is efficient, and if $\rho < 1$, DMU_o is inefficient.

Definition 4. The relative overall profit efficiency change for DMU_o is defined as

$$\mathrm{TEC}_o = \frac{\rho_o^{t+1}(x_o^{t+1}, y_o^{t+1})}{\rho_o^t(x_o^t, y_o^t)}.$$

Definition 5. The change between t and $t+1$ is calculated as follows:

$$\mathrm{FS}_o = \sqrt{\frac{\rho_o^t(x_o^{t+1}, y_o^{t+1})}{\rho_o^{t+1}(x_o^{t+1}, y_o^{t+1})} \times \frac{\rho_o^t(x_o^t, y_o^t)}{\rho_o^{t+1}(x_o^t, y_o^t)}}.$$

The overall profit MPI for DMU_o is calculated by multiplying the overall profit efficiency change and technology change at periods t and $t+1$ as follows:

$$M_o = \frac{\rho_o^{t+1}(x_o^{t+1}, y_o^{t+1})}{\rho_o^t(x_o^t, y_o^t)} \cdot \sqrt{\frac{\rho_o^t(x_o^{t+1}, y_o^{t+1})}{\rho_o^{t+1}(x_o^{t+1}, y_o^{t+1})} \times \frac{\rho_o^t(x_o^t, y_o^t)}{\rho_o^{t+1}(x_o^t, y_o^t)}}. \tag{4}$$

The above relation can be simplified as

$$M_o = \sqrt{\frac{\rho_o^t(x_o^{t+1}, y_o^{t+1})}{\rho_o^t(x_o^t, y_o^t)} \times \frac{\rho_o^{t+1}(x_o^{t+1}, y_o^{t+1})}{\rho_o^{t+1}(x_o^t, y_o^t)}}. \tag{5}$$

Therefore, we have three conditions:

(i) $M_o > 1$, increase productivity and observe progress;
(ii) $M_o < 1$, decrease productivity and observe regress; and
(iii) $M_o = 1$, no change in productivity at time $t+1$ in comparison with t.

3. Background of fuzzy sets

In this section, we review some basic definitions for fuzzy sets [36–39].

Definition 6. Let U be a universe set. A fuzzy set $\tilde{A}$ of U is defined by a membership function $\mu_{\tilde{A}}(x) \rightarrow [0, 1]$, where $\mu_{\tilde{A}}(x), \forall x \in U$, indicates the degree of membership of $\tilde{A}$ to U.

Definition 7. A fuzzy subset $\tilde{A}$ of real numbers R is convex iff

$$\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq (\mu_{\tilde{A}}(x) \wedge \mu_{\tilde{A}}(y)), \quad \forall x, y \in R, \forall \lambda \in [0, 1],$$

where "$\wedge$" denotes the minimum operator.

Definition 8. $\tilde{A}$ is a fuzzy number iff $\tilde{A}$ is a normal (see Definition 10) and convex fuzzy subset of R.

Definition 9. A fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4)$ is called a generalized trapezoidal fuzzy number with membership function $\mu_{\tilde{A}}$, and it has the following properties:

(a) $\mu_{\tilde{A}}$ is a continuous mapping from R to the closed interval $[0, 1]$,
(b) $\mu_{\tilde{A}}(x) = 0$ for all $x \in (-\infty, a_1]$,
(c) μ_A is strictly increasing on $[a_1, a_2]$,
(d) $\mu_{\tilde{A}}(x) = 1$ for all $x \in [a_2, a_3]$,
(e) $\mu_{\tilde{A}}$ is strictly decreasing on $[a_3, a_4]$, and
(f) $\mu_{\tilde{A}}(x) = 0$ for all $x \in [a_4, +\infty)$.

The membership function $\mu_{\tilde{A}}$ of $\tilde{A}$ can be defined as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} f_a(x), & a_1 \leq x \leq a_2, \\ 1, & a_2 \leq x \leq a_3, \\ g_a(x), & a_3 \leq x \leq a_4, \\ 0, & \text{otherwise}, \end{cases} \tag{6}$$

where $f_a : [a_1, a_2] \rightarrow [0, 1]$ and $g_a : [a_3, a_4] \rightarrow [0, 1]$.

The inverse functions of f_a and g_a, denoted as f_a^{-1} and g_a^{-1}, exist. Since $f_a : [a_1, a_2] \to [0, 1]$ is continuous and strictly increasing, $f_a^{-1} : [0, 1] \to [a_1, a_2]$ is also continuous and strictly increasing. Similarly, since $g_a : [a_3, a_4] \to [0, 1]$ is continuous and strictly decreasing, $g_a^{-1} : [0, 1] \to [a_3, a_4]$ is also continuous and strictly increasing. That is, both $\int_0^1 f_a^{-1}\mathrm{d}y$ and $\int_0^1 g_a^{-1}\mathrm{d}y$ exist [40].

In particular, we are working with a special type of trapezoidal fuzzy number with a membership function $\mu_{\tilde{A}}$ expressed by

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1}, & a_1 \le x \le a_2, \\ 1, & a_2 \le x \le a_3, \\ \dfrac{a_4-x}{a_4-a_3}, & a_3 \le x \le a_4, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

The trapezoidal fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4)$ is reduced to a real number A if $a_1 = a_2 = a_3 = a_4$. Conversely, a real number A can be written as a trapezoidal fuzzy number $\tilde{A} = (a, a, a, a)$. If $a_2 = a_3$, then $\tilde{A} = (a_1, a_2, a_4)$ is called a triangular fuzzy number. Also, let $F(R)$ be the family of fuzzy numbers on R.

Definition 10. The α-level of fuzzy set $\tilde{A}$, $\tilde{A}_\alpha$, is the crisp set $\tilde{A}_\alpha = \{x|\mu_{\tilde{A}}(x) \ge \alpha\}$. The support of $\tilde{A}$ is the crisp set $\mathrm{Sup}(\tilde{A}) = \{x|\mu_{\tilde{A}}(x) > 0\}$. $\tilde{A}$ is normal iff $\mathrm{Sup}_{x\in U}\mu_{\tilde{A}}(x) = 1$, where U is the universal set. The lower and upper endpoints of any α-cut set, $[\tilde{A}]_\alpha$, are represented by $[\tilde{A}]_\alpha^L$ and $[\tilde{A}]_\alpha^U$, respectively. Note that $[\tilde{A}]_\alpha^L$ and $[\tilde{A}]_\alpha^U$ of each triangular fuzzy number (a_1, a_2, a_3) are computed by $a_1 + \alpha(a_2 - a_1)$ and $a_3 - \alpha(a_3 - a_2)$, respectively.

Definition 11. A fuzzy number $\tilde{A}$ is named a non-negative fuzzy number if $[\tilde{A}]_0^L \ge 0$.

Ranking fuzzy numbers is an important part of the decision-making process in a fuzzy environment because of imprecise measurements. Many fuzzy number ranking approaches have been developed in the literature. Chen and Hwang [41] have surveyed the existing methods and classified them into four categories: (1) preference relation methods including degree of optimality, Hamming distance, α-cut and comparison function, (2) fuzzy mean and spread methods based on probability distributions, (3) fuzzy scoring methods including proportional optimal, left/right scores, centroid index and area measurement, and (4) linguistic methods. To examine the methods of each category, see [41].

Definition 12. We use the following distance measure for two arbitrary fuzzy numbers, $\tilde{A}$ and $\tilde{B}$:

$$d(\tilde{A}, \tilde{B}) = \int_0^1 s(\alpha)([\tilde{A}]_\alpha^L + [\tilde{A}]_\alpha^U - [\tilde{B}]_\alpha^L - [\tilde{B}]_\alpha^U)\mathrm{d}\alpha, \tag{8}$$

where $s(\alpha)$ is an increasing function, $s(0) = 0$, $s(1) = 1$, and $\int_0^1 s(\alpha)\mathrm{d}\alpha = \frac{1}{2}$. See [42] for further details.

Definition 13. We also use the following ranking system proposed by Yao and Wu [43] for two arbitrary fuzzy numbers, $\tilde{A}$ and $\tilde{B}$:

$$\begin{aligned} &d(\tilde{A}, \tilde{B}) > 0 \quad \text{iff } \tilde{A} \succ \tilde{B}, \\ &d(\tilde{A}, \tilde{B}) < 0 \quad \text{iff } \tilde{A} \prec \tilde{B}, \\ &d(\tilde{A}, \tilde{B}) = 0 \quad \text{iff } \tilde{A} \approx \tilde{B}. \end{aligned}$$

4. Overall profit Malmquist with fuzzy data

Although several conventional DEA models have explicitly addressed the effectiveness analysis, all the data assume the form of specific numerical values. However, the observed values of the input and output data in real-world situations are sometimes inexact, incomplete, vague, ambiguous, or imprecise. These imprecise or ambiguous data can be represented as linguistic variables characterized by fuzzy numbers for reflecting a kind of general sense or experience of experts. The fuzzy set theory developed by Zadeh [36] is the formal body of theory that allows the treatment of imprecise estimates in uncertain environments. In this section, we propose a fuzzy overall profit Malmquist of a set of DMUs with fuzzy inputs and outputs. Hence, we consider model (2) in crisp form and extend it to a fuzzy model using a fuzzy signed distance.

Let us assume that we have a set of DMUs consisting of DMU_j $(j = 1, \ldots, n)$, with fuzzy input–output vectors $(\tilde{x}_j, \tilde{y}_j)$, where $\tilde{x}_j \in (F(R) \ge 0)^m$, $\tilde{y}_j \in (F(R) \ge 0)^s$, and $F(R) \ge 0$ is the family of all non-negative fuzzy numbers. We also assume that $\tilde{X}$ and $\tilde{Y}$ represent the $m \times n$ matrix of fuzzy inputs and the $s \times n$ matrix of fuzzy outputs, respectively. Hence, the overall profit MPI at times t and $t + 1$, in which time $t + 1$ with the frontier of time t (the within-period time) and time t with the frontier of time $t + 1$ (the adjacent-period time), can be obtained as follows.

Within-period time

$$\begin{aligned} & D_o^p(x_o^p, y_o^p \mid p = t, t+1) = \max_{\varphi,\theta,\lambda} \quad \varphi - \theta \\ & \text{s.t.} \quad \varphi[(\tilde{r}^p)^T \tilde{y}_o^p] \le (\tilde{r}^p)^T \tilde{Y}^p \lambda, \\ & \qquad \theta[(\tilde{c}^p)^T \tilde{x}_o^p] \ge (\tilde{c}^p)^T \tilde{X}^p \lambda, \\ & \qquad \lambda \ge 0. \end{aligned} \tag{9a}$$

Adjacent-period time

$$\begin{aligned} & D_o^p(x_o^q, y_o^q \mid p, q = t, t+1, p \ne q) = \max_{\varphi,\theta,\lambda} \quad \varphi - \theta \\ & \text{s.t.} \quad \varphi[(\tilde{r}^p)^T \tilde{y}_o^p] \le (\tilde{r}^q)^T \tilde{Y}^q \lambda, \\ & \qquad \theta[(\tilde{c}^p)^T \tilde{x}_o^p] \ge (\tilde{c}^q)^T \tilde{X}^q \lambda, \\ & \qquad \lambda \ge 0. \end{aligned} \tag{9b}$$

Here, $\tilde{c}^p(\tilde{c}^q)$ and $\tilde{r}^p(\tilde{r}^q)$ are the fuzzy input and fuzzy output price vectors, respectively, for DMU$_j$ $(j = 1, \ldots, n)$ at times $p(q)$, where they are non-negative fuzzy numbers. It is important to note that all DMUs in (9a) and (9b) have the same fuzzy price vectors $p^T = (\tilde{c}^p, \tilde{r}^p)$.

In this paper, we use the fuzzy ranking approach to defuzzify models (9a) and (9b) into crisp models (see also [21,44,45,22,32]). Hence, we use Definition 12 and transform models (9a) and (9b) into models (10a) and (10b), respectively:

$$\begin{aligned} & D_o^p(x_o^p, y_o^p \mid p = t, t+1) = \max_{\varphi,\theta,\lambda} \quad \varphi - \theta \\ & \text{s.t.} \quad d((\tilde{r}^p)^T \tilde{Y}^p \lambda, \varphi[(\tilde{r}^p)^T \tilde{y}_o^p]) \ge 0 \\ & \qquad d((\tilde{c}^p)^T \tilde{X}^p \lambda, \theta[(\tilde{c}^p)^T \tilde{x}_o^p]) \le 0, \\ & \qquad \lambda \ge 0. \end{aligned} \tag{10a}$$

$$\begin{aligned} & D_o^p(x_o^q, y_o^q \mid p, q = t, t+1, p \ne q) = \max_{\varphi,\theta,\lambda} \quad \varphi - \theta \\ & \text{s.t.} \quad d((\tilde{r}^q)^T \tilde{Y}^q \lambda, \varphi[(\tilde{r}^p)^T \tilde{y}_o^p]) \ge 0, \\ & \qquad d((\tilde{c}^q)^T \tilde{X}^q \lambda, \theta[(\tilde{c}^p)^T \tilde{x}_o^p]) \le 0, \\ & \qquad \lambda \ge 0. \end{aligned} \tag{10b}$$

Next, we transform models (10a) and (10b) into crisp models (11a) and (11b), respectively:

$$\begin{aligned} & D_o^p(x_o^p, y_o^p \mid p = t, t+1) = \max_{\varphi,\theta,\lambda} \quad \varphi - \theta \\ & \text{s.t.} \quad \int_0^1 s(\alpha)[[(\tilde{r}^p)^T Y^p \lambda]_\alpha^L + [(\tilde{r}^p)^T Y^p \lambda]_\alpha^U - [\varphi[(\tilde{r}^p)^T y_o^p]]_\alpha^L - [\varphi[(\tilde{r}^p)^T y_o^p]]_\alpha^U] \mathrm{d}\alpha \ge 0 \\ & \qquad \int_0^1 s(\alpha)[[(\tilde{c}^p)^T X^p \lambda]_\alpha^L + [(\tilde{c}^p)^T X^p \lambda]_\alpha^U - [\theta[(\tilde{c}^p)^T x_o^p]]_\alpha^L - [\theta[(\tilde{c}^p)^T x_o^p]]_\alpha^U] \mathrm{d}\alpha \le 0 \\ & \qquad \lambda \ge 0. \end{aligned} \tag{11a}$$

$$\begin{aligned} & D_o^p(x_o^p, y_o^p \mid p, q = t, t+1, p \ne q) = \max_{\varphi,\theta,\lambda} \quad \varphi - \theta \\ & \text{s.t.} \quad \int_0^1 s(\alpha)[[(\tilde{r}^q)^T Y^q \lambda]_\alpha^L + [(\tilde{r}^q)^T Y^q \lambda]_\alpha^U - [\varphi[(\tilde{r}^p)^T y_o^p]]_\alpha^L - [\varphi[(\tilde{r}^p)^T y_o^p]]_\alpha^U] \mathrm{d}\alpha \ge 0 \\ & \qquad \int_0^1 s(\alpha)[[(\tilde{c}^q)^T X^q \lambda]_\alpha^L + [(\tilde{c}^q)^T X^q \lambda]_\alpha^U - [\theta[(\tilde{c}^p)^T x_o^p]]_\alpha^L - [\theta[(\tilde{c}^p)^T x_o^p]]_\alpha^U] \mathrm{d}\alpha \le 0 \\ & \qquad \lambda \ge 0. \end{aligned} \tag{11b}$$

Next, we assume that $s(\alpha) = \alpha$, and introduce the following variables to simplify the above models:

$$\begin{aligned} & \overline{Y}^p = \int_0^1 \alpha[(\tilde{r}^p)^T Y^p]_\alpha^U \mathrm{d}\alpha \qquad \overline{Y}^q = \int_0^1 \alpha[(\tilde{r}^q)^T Y^q]_\alpha^U \mathrm{d}\alpha \\ & \underline{Y}^p = \int_0^1 \alpha[(\tilde{r}^p)^T Y^p]_\alpha^L \mathrm{d}\alpha \qquad \underline{Y}^q = \int_0^1 \alpha[(\tilde{r}^q)^T Y^q]_\alpha^L \mathrm{d}\alpha \\ & \overline{X}^p = \int_0^1 \alpha[(\tilde{c})^T X^p]_\alpha^U \mathrm{d}\alpha \qquad \overline{X}^q = \int_0^1 \alpha[(\tilde{c})^T X^q]_\alpha^U \mathrm{d}\alpha \\ & \underline{X}^p = \int_0^1 \alpha[(\tilde{c}^p)^T X^p]_\alpha^L \mathrm{d}\alpha \qquad \underline{X}^q = \int_0^1 \alpha[(\tilde{c}^q)^T X^q]_\alpha^L \mathrm{d}\alpha. \end{aligned} \tag{12}$$

Finally, we use the above variables and transform models (11a) and (11b) to the linear programming models (13a) and (13b) given below:

$$
\begin{aligned}
& D_o^p(x_o^p, y_o^p | p = t, t+1) = \max_{\varphi,\theta,\lambda} \quad \varphi - \theta \\
& \text{s.t.} \quad \varphi(\underline{Y}_o^p + \overline{Y}_o^p) \le \lambda(\underline{Y}^p + \overline{Y}^p), \\
& \qquad \theta(\underline{X}_o^p + \overline{X}_o^p) \ge \lambda(\underline{X}^p + \overline{X}^p), \\
& \qquad \lambda \ge 0.
\end{aligned} \tag{13a}
$$

$$
\begin{aligned}
& D_o^p(x_o^q, y_o^q | p, q = t, t+1, p \neq q) = \max_{\varphi,\theta,\lambda} \quad \varphi - \theta \\
& \text{s.t.} \quad \varphi(\underline{Y}_o^p + \overline{Y}_o^p) \le \lambda(\underline{Y}^q + \overline{Y}^q), \\
& \qquad \theta(\underline{X}_o^p + \overline{X}_o^p) \ge \lambda(\underline{X}^q + \overline{X}^q), \\
& \qquad \lambda \ge 0.
\end{aligned} \tag{13b}
$$

Definition 14. Using Definitions 3, 4 and models (13a) and (13b), the overall profit MPI for DMU_o is calculated as

$$
M_o = \sqrt{\frac{\rho_o^t(x_o^{t+1}, y_o^{t+1})}{\rho_o^t(x_o^t, y_o^t)} \times \frac{\rho_o^{t+1}(x_o^{t+1}, y_o^{t+1})}{\rho_o^{t+1}(x_o^t, y_o^t)}}.
$$

(i) $M_o > 1$, increase productivity and observe progress.
(ii) $M_o < 1$, decrease productivity and observe regress.
(iii) $M_o = 1$, no change in productivity at time $t + 1$ in comparison to t.

Note that models (3a) and (3b) with crisp data are special cases of models (13a) and (13b), respectively.

5. Overall profit Malmquist with interval data

Let us proceed with our earlier assumption that there are n DMUs under consideration. For each DMU_j, $j = 1, \ldots, n$, we introduce the following models to measure the overall profit MPI and capture the uncertainty in models (3a) and (3b) with interval data.

Within-period time

$$
\begin{aligned}
& B_o^p(x_o^p, y_o^p | p = t, t+1) = \max_{\varphi,\theta,\lambda} \quad \varphi - \theta \\
& \text{s.t.} \quad \varphi[(\tilde{r}_j^p)^T \tilde{y}_o^p] \le (\tilde{r}_j^p)^T \tilde{Y}^p \lambda, \quad \forall j, \\
& \qquad \theta[(\tilde{c}_j^p)^T \tilde{x}_o^p] \ge (\tilde{c}_j^p)^T \tilde{X}^p \lambda, \quad \forall j, \\
& \qquad \lambda \ge 0.
\end{aligned} \tag{14a}
$$

Adjacent-period time

$$
\begin{aligned}
& B_o^p(x_o^q, y_o^q | p, q = t, t+1, p \neq q) = \max_{\varphi,\theta,\lambda} \quad \varphi - \theta \\
& \text{s.t.} \quad \varphi[(\tilde{r}_j^p)^T \tilde{y}_o^p] \le (\tilde{r}_j^q)^T \tilde{Y}^q \lambda, \quad \forall j, \\
& \qquad \theta[(\tilde{c}_j^p)^T \tilde{x}_o^p] \ge (\tilde{c}_j^q)^T \tilde{X}^q \lambda, \quad \forall j, \\
& \qquad \lambda \ge 0.
\end{aligned} \tag{14b}
$$

Here, $\tilde{X} \in [\underline{X}, \overline{X}]$, $\tilde{Y} \in [\underline{Y}, \overline{Y}]$, $\tilde{r}_j^T \in [\underline{r}_j^T, \overline{r}_j^T]$ and $\tilde{c}_j^T \in [\underline{c}_j^T, \overline{c}_j^T]$ are the matrix of interval inputs, the matrix of outputs, the input price vector, and the output price vector, respectively. In this case, the efficiency of each DMU can be an interval denoted as $[\overline{B}_j^p, \underline{B}_j^p]$. The upper and lower bounds of the interval overall profit efficiency of the DMU_o are obtained from the pessimistic and optimistic viewpoints, respectively, using the following pair of LP models [(15a), (15b) and (16a), (16b)].

Optimistic viewpoint in the within-period time (upper bound)

$$
\begin{aligned}
& \overline{B}_o^p(x_o^p, y_o^p | p = t, t+1) = \max_{\varphi,\theta,\lambda} \quad \varphi - \theta \\
& \text{s.t.} \quad \varphi[(\overline{r}_j^p)^T \overline{y}_o^p] \le (\underline{r}_j^p)^T \underline{Y}^p \lambda, \quad \forall j, \\
& \qquad \theta[(\underline{c}_j^p)^T \underline{x}_o^p] \ge (\overline{c}_j^p)^T \overline{X}^p \lambda, \quad \forall j, \\
& \qquad \lambda \ge 0.
\end{aligned} \tag{15a}
$$

Pessimistic viewpoint in the within-period time (lower bound)

$$\begin{aligned} \underline{B}_o^p(x_o^p, y_o^p | p = t, t+1) = \max_{\varphi,\theta,\lambda} \quad & \varphi - \theta \\ \text{s.t.} \quad & \varphi[(\underline{r}_j^p)^T \underline{y}_o^p] \le (\overline{r}_j^p)^T \overline{Y}^p \lambda, \quad \forall j, \\ & \theta[(\overline{c}_j^p)^T \overline{x}_o^p] \ge (\underline{c}_j^p)^T \underline{X}^p \lambda, \quad \forall j, \\ & \lambda \ge 0. \end{aligned} \tag{15b}$$

Optimistic viewpoint in the adjacent-period time (upper bound)

$$\begin{aligned} \overline{B}_o^p(x_o^q, y_o^q | p, q = t, t+1, p \ne q) = \max_{\varphi,\theta,\lambda} \quad & \varphi - \theta \\ \text{s.t.} \quad & \varphi[(\overline{r}_j^p)^T \overline{y}_o^p] \le (\underline{r}_j^q)^T \underline{Y}^q \lambda, \quad \forall j, \\ & \theta[(\underline{c}_j^p)^T \underline{x}_o^p] \ge (\overline{c}_j^q)^T \overline{X}^q \lambda, \quad \forall j, \\ & \lambda \ge 0. \end{aligned} \tag{16a}$$

Pessimistic viewpoint in the adjacent-period time (lower bound)

$$\begin{aligned} \underline{B}_o^p(x_o^q, y_o^q | p, q = t, t+1, p \ne q) = \max_{\varphi,\theta,\lambda} \quad & \varphi - \theta \\ \text{s.t.} \quad & \varphi[(\underline{r}_j^p)^T \underline{y}_o^p] \le (\overline{r}_j^q)^T \overline{Y}^q \lambda, \quad \forall j, \\ & \theta[(\overline{c}_j^p)^T \overline{x}_o^p] \ge (\underline{c}_j^q)^T \underline{X}^q \lambda, \quad \forall j, \\ & \lambda \ge 0. \end{aligned} \tag{16b}$$

Theorem 3. a: *Let $(\overline{\varphi}, \overline{\theta}, \overline{\lambda})$, $(\hat{\varphi}, \hat{\theta}, \hat{\lambda})$, and $(\varphi^*, \theta^*, \lambda^*)$ be the optimal solutions for (14a), (15a) and (15b), respectively, when $p = t$. Then, $\overline{B}_o^t \le B_o^t \le \underline{B}_o^t$.*

b: *Let $(\overline{\varphi}, \overline{\theta}, \overline{\lambda})$, $(\hat{\varphi}, \hat{\theta}, \hat{\lambda})$, and $(\varphi^*, \theta^*, \lambda^*)$ be the optimal solutions for (14a), (15a) and (15b), respectively, when $p = t+1$. Then, $\overline{B}_o^{t+1} \le B_o^{t+1} \le \underline{B}_o^{t+1}$.*

c: *Let $(\overline{\varphi}, \overline{\theta}, \overline{\lambda})$, $(\hat{\varphi}, \hat{\theta}, \hat{\lambda})$, and $(\varphi^*, \theta^*, \lambda^*)$ be the optimal solutions for (14b), (16a) and (16b), respectively, when $p = t+1$ and $q = t$. Then, $\overline{B}_o^{t+1} \le B_o^{t+1} \le \underline{B}_o^{t+1}$.*

d: *Let $(\overline{\varphi}, \overline{\theta}, \overline{\lambda})$, $(\hat{\varphi}, \hat{\theta}, \hat{\lambda})$, and $(\varphi^*, \theta^*, \lambda^*)$ be the optimal solutions for (14b), (16a) and (16b), respectively, when $p = t$ and $q = t+1$. Then, $\overline{B}_o^t \le B_o^t \le \underline{B}_o^t$.*

Proof. a: $(\overline{\varphi}, \overline{\theta}, \overline{\lambda})$ is a feasible solution for model (15b) when $p = t$, since

$$\overline{\varphi}[(\underline{r}_j^t)^T \underline{y}_o^t] \le \overline{\varphi}[(\tilde{r}_j^t)^T \underline{y}_o^t] \le \overline{\varphi}[(\tilde{r}_j^t)^T \tilde{y}_o^t] \le (\tilde{r}_j^t)^T \tilde{Y}^t \overline{\lambda} \le (\overline{r}_j^t)^T \tilde{Y}^t \overline{\lambda} \le (\overline{r}_j^t)^T \overline{Y}^t \overline{\lambda},$$
$$\overline{\theta}[(\overline{c}_j^t)^T \overline{x}_o^t] \ge \overline{\theta}[(\tilde{c}_j^t)^T \overline{x}_o^t] \ge \overline{\theta}[(\tilde{c}_j^t)^T \tilde{x}_o^t] \ge (\tilde{c}_j^t)^T \tilde{X}^t \overline{\lambda} \ge (\underline{c}_j^t)^T \tilde{X}^t \overline{\lambda} \ge (\underline{c}_j^t)^T \underline{X}^t \overline{\lambda}.$$

Thus, we obtain $B_o^t \le \underline{B}_o^t$.

Furthermore, $(\hat{\varphi}, \hat{\theta}, \hat{\lambda})$ is the feasible solution for model (14a) when $p = t$, since

$$\hat{\varphi}[(\tilde{r}_j^t)^T \tilde{y}_o^t] \le \hat{\varphi}[(\overline{r}_j^t)^T \tilde{y}_o^t] \le \hat{\varphi}[(\overline{r}_j^t)^T \overline{y}_o^t] \le (\underline{r}_j^t)^T \underline{Y}^t \hat{\lambda} \le (\tilde{r}_j^t)^T \underline{Y}^t \hat{\lambda} \le (\tilde{r}_j^t)^T \tilde{Y}^t \hat{\lambda},$$
$$\hat{\theta}[(\tilde{c}_j^t)^T \tilde{x}_o^t] \ge \hat{\theta}[(\underline{c}_j^t)^T \tilde{x}_o^t] \ge \hat{\theta}[(\underline{c}_j^t)^T \underline{x}_o^t] \ge (\overline{c}_j^p)^T \overline{X}^p \hat{\lambda} \ge (\tilde{c}_j^p)^T \overline{X}^p \hat{\lambda} \ge (\tilde{c}_j^p)^T \tilde{X}^p \hat{\lambda}.$$

Hence, $\overline{B}_t^t \le B_t^t$.

The proofs of b, c, and d are similar to the proof of a. □

Definition 15. The lower and upper bounds of the overall profit MPI are obtained as follows

$$\overline{M} = \sqrt{\frac{\overline{\rho}_t^{t+1}}{\underline{\rho}_t^t} \times \frac{\overline{\rho}_{t+1}^{t+1}}{\underline{\rho}_{t+1}^t}},$$
$$\underline{M} = \sqrt{\frac{\underline{\rho}_t^{t+1}}{\overline{\rho}_t^t} \times \frac{\underline{\rho}_{t+1}^{t+1}}{\overline{\rho}_{t+1}^t}}.$$

Note that $\overline{\rho}_p^p(\underline{\rho}_p^p)$, $p = t, t+1$, represents the optimistic (pessimistic) efficiency in the within-period time, and $\overline{\rho}_q^p(\underline{\rho}_q^p)$, $p, q = t, t+1, p \ne q$, represents the optimistic (pessimistic) efficiency in the adjacent-period time.

Table 1
The input and output data for the five DMUs in Example 1 at time t and time $t+1$.

DMU_j	(x_{1j}^L, x_{1j}^U)		(x_{2j}^L, x_{2j}^U)		(y_{1j}^L, y_{1j}^U)		(y_{2j}^L, y_{2j}^U)	
	t	$t+1$	t	$t+1$	t	$t+1$	t	$t+1$
1	(12, 15)	(10, 14)	(0.21, 0.48)	(0.32, 0.5)	(138, 144)	(130, 140)	(21, 22)	(20, 23)
2	(10, 17)	(11, 15)	(0.1, 0.7)	(0.21, 0.4)	(143, 159)	(137, 150)	(28, 35)	(24, 30)
3	(4, 5)	(3, 7)	(0.16, 0.35)	(0.22, 0.42)	(157, 198)	(146, 160)	(21, 29)	(20, 30)
4	(19, 22)	(14, 23)	(0.12, 0.19)	(0.31, 0.39)	(158, 181)	(159, 170)	(21, 25)	(25, 32)
5	(14, 15)	(17, 18)	(0.06, 0.09)	(0.1, 0.17)	(157, 180)	(160, 189)	(28, 40)	(18, 35)

Theorem 4. *Any $\underline{M} \le M \le \overline{M}$ can be considered as the overall profit Malmquist for* DMU_o.

Proof. Since $\overline{\rho}_{t+1}^{t+1} \ge \rho_{t+1}^{t+1}$ and $\rho_t^t \ge \underline{\rho}_t^t$, $\frac{\overline{\rho}_{t+1}^{t+1}}{\underline{\rho}_t^t} \ge \frac{\rho_{t+1}^{t+1}}{\rho_t^t}$, and since $\overline{\rho}_t^{t+1} \ge \rho_t^{t+1}$ and $\rho_{t+1}^t \ge \underline{\rho}_{t+1}^t$, $\frac{\overline{\rho}_{t+1}^{t+1}}{\underline{\rho}_t^t} \ge \frac{\rho_t^{t+1}}{\rho_{t+1}^t}$. Hence, we have $\underline{M} \le M$.

Similarly, we can show that $M \le \overline{M}$. This completes the proof. □

Considering that the overall PMI of any DMU lies in an interval, the DMUs can be divided into one of the six following classes.

- *The no change in productivity class*. This class includes all the DMUs with constant productivity; that is, $E^0 = \{\text{DMU}_j : \underline{M}_j = \overline{M}_j = 1\}$.
- *The fully increasing productivity class*. This class includes all the DMUs with increasing productivity and observed progress in the pessimistic viewpoint; that is, $E^{++} = \{\text{DMU}_j : 1 < \underline{M}_j \le \overline{M}_j\}$.
- *The fully decreasing productivity class*. This class includes all the DMUs with decreasing productivity and observed regress in the optimistic viewpoint; that is, $E^{--} = \{\text{DMU}_j : \underline{M}_j \le \overline{M}_j < 1\}$.
- *The partially increasing productivity class*. This class includes all the DMUs with increasing productivity in the optimistic viewpoint and no change in productivity in the pessimistic viewpoint; that is, $E^{+} = \{\text{DMU}_j : \underline{M}_j = 1, \overline{M}_j > 1\}$.
- *The partially decreasing productivity class*. This class includes all the DMUs with decreasing productivity in the pessimistic viewpoint and no change in productivity in the optimistic viewpoint; that is, $E^{-} = \{\text{DMU}_j : \underline{M}_j < 1, \overline{M}_j = 1\}$.
- *The partially increasing–decreasing productivity class*. This class includes all the DMUs with increasing productivity in the optimistic viewpoint and decreasing productivity in the pessimistic viewpoint; that is, $E = \{\text{DMU}_j : \underline{M}_j < 1 < \overline{M}_j\}$. Assuming that $a_j = \overline{M}_j - 1$ and $b_j = 1 - \underline{M}_j$, this class can be divided into three subclasses, as follows.
  1. $E^{00} = \{\text{DMU}_j : a_j = b_j\}$. This subclass is an equilibrium between the optimistic and pessimistic viewpoints.
  2. $E^{+-} = \{\text{DMU}_j : a_j > b_j\}$. In this subclass, the productivity increase in the optimistic viewpoint is stronger than the productivity decrease in the pessimistic viewpoint.
  3. $E^{-+} = \{\text{DMU}_j : b_j > a_j\}$. In this subclass, the productivity decrease in the pessimistic viewpoint is stronger than the productivity increase in the optimistic viewpoint.

Next, we present two numerical examples to demonstrate the applicability of the proposed methods.

6. Numerical examples

In the following two examples we solved above model under VRS by adding convexity constraint ($1\lambda = 1$) to all models, however we did not report scale efficiency changes, as it is not the aim in this paper.

6.1. Example 1: A case of overall profit MPI with interval data

We consider five DMUs with two interval input variables and two interval output variables, as shown in Table 1. Table 2 shows the interval price vectors at time t and time $t+1$. Using models 15 and 16, the efficiencies and the overall profit MPI are as given in Table 3. As shown in Table 3, all DMUs are classified in the partially increasing–decreasing productivity class, and in all cases the productivity increase in the optimistic viewpoint is stronger than the productivity decrease in the pessimistic viewpoint. DMU_3 has the highest productivity progress of 2.306 in the optimistic viewpoint, and DMU_3 has the highest productivity decrease of 0.237. According to the optimistic viewpoint, we can rank the DMUs by their progress in productivity in the order $\text{DMU}_3 \succ \text{DMU}_1 \succ \text{DMU}_2 \succ \text{DMU}_4 \succ \text{DMU}_5$. However, according to the pessimistic viewpoint, the productivity regress is in the order $\text{DMU}_3 \succ \text{DMU}_2 \succ \text{DMU}_5 \succ \text{DMU}_1 \succ \text{DMU}_4$. Obviously the order in productivity increase may be different than the order in productivity decrease.

Table 2
The price vector data for the five DMUs in Example 1 at time t and time $t+1$.

DMU_j	$(\underline{r}_{j1}, \overline{r}_{j1})$		$(\underline{r}_{j2}, \overline{r}_{j2})$		$(\underline{c}_{j1}, \overline{c}_{j1})$		$(\underline{c}_{j2}, \overline{c}_{j2})$	
	t	$t+1$	t	$t+1$	t	$t+1$	t	$t+1$
1	(10, 12)	(11, 15)	(30, 35)	(23, 30)	(100, 110)	(110, 115)	(50, 55)	(40, 50)
2	(9, 10)	(8, 9)	(27, 28)	(24, 29)	(110, 115)	(114, 117)	(40, 44)	(42, 45)
3	(8, 9)	(5, 9)	(25, 27)	(25, 26)	(105, 110)	(102, 109)	(42, 45)	(42, 50)
4	(9, 11)	(7, 12)	(29, 31)	(23, 26)	(107, 115)	(114, 115)	(50, 57)	(49, 52)
5	(10, 11)	(10, 14)	(28, 31)	(24, 30)	(111, 117)	(110, 114)	(47, 62)	(42, 52)

Table 3
The efficiencies and overall profit Malmquist of the five DMUs in Example 1.

DMU_j	$(\overline{B}_j^t(x_j^t, y_j^t), \underline{B}_j^t(x_j^t, y_j^t))$	$(\overline{B}_j^{t+1}(x_j^{t+1}, y_j^{t+1}), \underline{B}_j^{t+1}(x_j^{t+1}, y_j^{t+1}))$	$(\overline{B}_j^t(x_j^{t+1}, y_j^{t+1}), \underline{B}_j^t(x_j^{t+1}, y_j^{t+1}))$	$(\overline{B}_j^{t+1}(x_j^t, y_j^t), \underline{B}_j^{t+1}(x_j^t, y_j^t))$	$\underline{M}_j$	$\overline{M}_j$
1	(0.4104, 1.2877)	(0.1081, 1.3041)	(0.2411, 1.5158)	(0.0316, 1.0748)	0.4855	2.0588
2	(0.1358, 1.1418)	(0.1189, 1.1795)	(0.1981, 1.3183)	(0.3042, 0.9532)	0.4150	1.9770
3	(0.7464, 0.1271)	(1.9381, 0.9742)	(1.1280, 1.0333)	(1.5450, 0.0428)	0.2366	2.3064
4	(0.4234, 1.2279)	(0.0280, 1.1016)	(0.2342, 1.2323)	(0.1005, 1.0220)	0.5778	1.9372
5	(0.2122, 1.0290)	(0.0153, 1.1796)	(0.2408, 1.2932)	(0.1332, 0.8603)	0.4624	1.7312

Table 4
The fuzzy input and output data for the six DMUs in Example 2 at time t and time $t+1$.

DMU_j	Period t			Period $t+1$		
	I_1	I_2	O_1	I_1	I_2	O_1
1	(5, 6, 7)	(100, 110, 115)	(20, 21, 23)	(4, 6, 8)	(99, 101, 103)	(18, 20, 23)
2	(4, 5, 7)	(121, 123, 129)	(19, 20, 24)	(2, 4, 5)	(123, 125, 127)	(20, 21, 23)
3	(3, 5, 9)	(140, 141, 143)	(17, 20, 21)	(10, 11, 12)	(135, 142, 143)	(16, 19, 22)
4	(8, 10, 11)	(90, 93, 97)	(19, 23, 25)	(8, 9, 10)	(92, 93, 95)	(20, 23, 24)
5	(11, 12, 13)	(95, 99, 100)	(18, 25, 27)	(7, 8, 10)	(93, 95, 101)	(17, 22, 27)
6	(7, 9, 10)	(80, 90, 100)	(14, 19, 27)	(9, 11, 13)	(82, 85, 90)	(13, 17, 24)

Table 5
The efficiencies and the overall profit Malmquist for the six DMUs in Example 2.

DMU_j	$D_j^t(x_j^t, y_j^t)$	$D_j^{t+1}(x_j^{t+1}, y_j^{t+1})$	$D_j^t(x_j^{t+1}, y_j^{t+1})$	$D_j^{t+1}(x_j^t, y_j^t)$	Overall profit Malmquist
1	10.8297	2.6555	3.8689	0.2917	0.9261
2	8.8324	0.1438	0.7570	0.7358	1.6788
3	0.8000	0.4666	0.9150	0.2616	0.8991
4	10.0937	0.3314	0.7534	4.5893	5.1571
5	9.6349	0	0.6943	0.0167	2.4859
6	0	0.6335	1.0219	1.8441	0.3260

6.2. *Example 2: A case of overall profit MPI with fuzzy data*

Consider six DMUs with two fuzzy inputs and a single fuzzy output, as shown in Table 4. Further, assume that the prices are fuzzy vectors at times t and $t+1$, as follows:

$$
\begin{aligned}
r_1^t &= (50, 51, 52), & r_1^{t+1} &= (45, 49, 52),\\
c_1^t &= (120, 121, 123), & c_1^{t+1} &= (112, 115, 116),\\
c_2^t &= (30, 31, 32), & c_2^{t+1} &= (32, 33, 35).
\end{aligned}
$$

Using Definition 10, we first calculate the lower and upper bounds of input/output variables at times t and $t+1$ for each DMU. We then use models (3a) and (3b) to obtain the efficiencies and the overall profit MPI. The results are given in Table 5.

As shown in Table 5, the overall profit MPI for DMU_2, DMU_4, and DMU_5 are greater than 1, so productivity progress is observed. In addition, DMU_4 has the best progress, and the overall profit MPI values for DMU_1, DMU_3, and DMU_6 are less than 1, so there is decrease in productivity and observed regress. DMU_6 has the highest regress.

7. Conclusions and future research directions

In addition to comparing the relative performance of a set of DMUs at a specific period, conventional DEA can also be used to calculate the productivity change of a DMU over time with the profit MPI model. While precise data are often used in conventional DEA, real-world data are sometimes imprecise and vague. Furthermore, it is very expensive for organizations to collect precise data for efficiency analysis. Consequently, there is a strong impetus for developing cost-effective methodologies to capture imprecision in productivity and efficiency analysis.

The conventional profit MPI model lacks the flexibility to deal with imprecise or vague data. These imprecise or vague data can be suitably characterized with fuzzy and interval methods. In this paper, we reformulated the conventional profit MPI model as an imprecise DEA model and proposed two novel methods for measuring the overall profit MPI when the inputs, outputs, and price vectors are fuzzy or vary in intervals. We developed a fuzzy version of the conventional MPI model by using a ranking method and solved the model with a commercial off-the-shelf DEA software package. In addition, we defined an interval for the overall profit MPI of each DMU and classified the DMUs into six groups according to the intervals obtained for their overall profit efficiency and MPIs. We also presented two numerical examples to demonstrate the applicability of the proposed frameworks.

The contribution of this paper is fourfold: (1) we considered ambiguous, uncertain, and imprecise data in the MPI model; (2) we characterized these imprecise and vague data with fuzzy and interval methods; (3) we proposed two novel methods for measuring the overall profit MPI when the inputs, outputs, and price vectors are fuzzy or vary in intervals; and (4) we demonstrated the practical aspects of our model with two numerical examples.

From a future research point of view, we suggest that researchers develop similar models for measuring the overall profit MPI with fuzzy and interval data for non-radial DEA models such as additive and slack-based measures. We also suggest to those who are interested to extend the proposed methods in this paper for variable returns to scale model that would allow us to measure the scale efficiency change with fuzzy and interval data. Finally, we plan to implement the proposed method in the real world and write a follow-up paper demonstrating the practical implications of our models in real-life problems.

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