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An improved non-convex model for discriminating efficient units in free disposal hull

Rashed Khanjani Shiraz [a], Madjid Tavana [b,c,*], Debora Di Caprio [d,e], Javad Vakili [a]

[a] *Faculty of Mathematics Science, University of Tabriz, Tabriz, Iran*
[b] *Business Systems and Analytics Department, Lindback Distinguished Chair of Information Systems and Decision Sciences, La Salle University, Philadelphia, PA 19141, USA*
[c] *Business Information Systems Department, Faculty of Business Administration and Economics, University of Paderborn, D-33098 Paderborn, Germany*
[d] *Department of Mathematics and Statistics, York University, Toronto M3J 1P3, Canada*
[e] *Polo Tecnologico IISS G. Galilei, Via Cadorna 14, 39100 Bolzano, Italy*

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ABSTRACT

A number of methods, including data envelopment analysis (DEA), have been proposed for evaluating the performance of decision making units (DMUs) converting multiple inputs into multiple outputs. The free disposal hull (FDH) is an alternative non-parametric, deterministic production method useful to evaluate the technical efficiency of DMUs. Unlike DEA, the FDH imposes minimal assumptions on the production technology and it does not require convexity. The conventional FDH model assigns an efficiency score less than one to inefficient DMUs, from which a ranking can be derived. However, all efficient DMUs have an efficiency score of 1 and, hence, equal ranking. Several ranking methods have been proposed to discriminate among the efficient DMUs. These ranking methods are sometimes either break down or produce infeasible solution(s). The Mehrabian, Alirezae, Jahanshahloo (MAJ) (1999) method is usually used to discriminate among the efficient DMUs in convex DEA problems, while the MAJ-FDH method proposed by Sun and Hu (2009) has been used to discriminate among the efficient DMUs in non-convex DEA problems. In this paper: (1) we show that the MAJ-FDH model is not always feasible; (2) we propose a modified MAJ-FDH model, both input and output oriented, and mathematically prove that this model is always feasible; (3) we show that the proposed method can produce an optimal solution without the need to solve the integer programming problem; (4) we prove the validity of the proposed method through theorems; and (5) we provide two numerical examples to demonstrate the applicability and show the efficacy of the proposed method.

1. Introduction

A wide range of methods have been proposed to measure technical efficiency in organizations. These techniques are methodologically divided into two distinct groups. One branch of the existing literature classifies these techniques into a stochastic and a deterministic group with respect to the stochastic nature of the data. A second branch classifies

* Corresponding author at: Business Systems and Analytics Department, Lindback Distinguished Chair of Information Systems and Decision Sciences, La Salle University, Philadelphia, PA 19141, USA. Tel.: +1 215 951 1129; fax: +1 267 295 2854.

E-mail addresses: Rashed.shiraz@gmail.com (R. Khanjani Shiraz), tavana@lasalle.edu (M. Tavana), dicaper@mathstat.yorku.ca (D. Di Caprio), javad_vakili1360@yahoo.com (J. Vakili).

URL: http://tavana.us/ (M. Tavana).

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them into a parametric and a non-parametric group. Parametric methods assume that the boundary of the production possibility set can be represented by a particular functional form with constant parameters while the non-parametric methods focus on the regularity assumptions of the production possibility set itself. The non-parametric methods construct a piecewise linear efficient frontier on the basis of the input–output data. Free disposal hull (FDH) is a deterministic and non-parametric method for evaluating productivity and efficiency in organizations [9,28,29].

Data envelopment analysis (DEA) is a mathematical programming method for evaluating the performance of decision making units (DMUs) that convert multiple inputs into multiple outputs. This method was introduced by Charnes et al. [7] and extended by Banker et al. [6] to allow for the evaluation of the technical efficiencies of a set of comparable DMUs. Compared to DEA, the FDH requires minimal assumptions on the production technology to be satisfied and it does not require convexity [11,23]. As a result, FDH has received a considerable amount of research attention. Tulkens [29] developed the methodological framework and one of the first enumeration algorithms for FDH. He considered computational issues of FDH in relation to the linear programming (LP) techniques used in DEA. Cherchye et al. [8] and Briec et al. [5] continued the research on FDH and developed additional enumeration algorithms. Cherchye et al. [8] extended the FDH efficiency analysis to the general directional distance function framework and the profit interpretation of directional distance functions to the non-convex FDH technologies. Briec et al. [5] showed how linear programs are needed to compute cost and revenue functions under constant returns to scale and a single output or input, respectively, can be replaced with a more efficient enumeration algorithm.

Kerstens and Vanden Eeckaut [16] reviewed the traditional methods for estimating returns to scale on non-parametric deterministic reference technologies and proposed a new and more general method suitable for all reference technologies. They introduced specific returns to scale assumptions in FDH models and illustrated the usefulness of their model by considering different FDH variations. Agrell and Tind [1] presented a linearization of FDH along with dual interpretations. They showed that an input/output-oriented model is equivalent to a maximization of the weighted inputs/outputs, subject to production space feasibility. Keshvari and Dehghan Hardoroudi [17] extended the method for solving four standard technologies (variable returns to scale, constant returns to scale, non-decreasing returns to scale, and non-increasing returns to scale) by employing a numeration algorithm without using linear programming or mixed integer LP. Leleu [18] went one step further and proposed a complete LP framework to deal with all previous FDH models. Paryab et al. [22] modified Leleu's [18] model by considering various returns to scale assumptions involving fewer constraints and variables and could obtain the FDH-cost efficiency directly.

Many researchers have sought to improve the discrimination power of DEA and fully rank both efficient and inefficient DMUs. Adler et al. [3] reviewed these methods and divided them into six groups including: (1) cross-efficiency measures (CEM) (e.g., [10,14,35]); (2) super-efficiency ranking methods (e.g., [2,24,20]); (3) benchmark ranking methods (e.g., [30,19]); (4) multivariate statistics ranking methods (e.g., [27,33]); (5) ranking of inefficient decision-making units (e.g., [4,13]); and (6) multi-criteria decision-making ranking methods (e.g., [31,15,34]).

The conventional FDH model assigns an efficiency score less than one to inefficient DMUs, from which a ranking can be derived. However, efficient DMUs all have an efficiency score of 1 and, hence, are equally ranked. A few researchers have proposed methods for improving the discrimination power of FDH. Andersen and Petersen [2] proposed a modified version of DEA based on a comparative analysis of the efficient DMUs relative to a reference technology spanned by all other units. Their procedure provides a framework for ranking FDH-efficient units and facilitates a comparative analysis among them based on parametric methods. Van Puyenbroeck [32] further modified the standard FDH model by using Andersen and Petersen's [2] modified DEA method and called it A&P FDH model.

Jahanshahloo et al. [12] used 0–1 LP and proposed a method to find the efficient DMUs by putting aside the production possibility set principles of DEA and considering only the observation set. Their one-stage algorithm compares each DMU with the observed DMUs and identifies the efficient ones. Each iteration of their algorithm identifies at least one efficient DMU by solving a 0–1 LP problem. Sun and Hu [25] proposed an alternative method for discriminating among FDH-efficient units. This method is similar to MAJ-FDH model in spirit, however, it often breaks down and it does not always satisfies feasibility.

We propose a simple but still very important modification of the MAJ-FDH model. The modified MAJ-FDH model that we propose is grounded in the DEA ranking method developed by Saati et al. [26]. In particular, our model shares an important feature with the one of Saati et al. [26], that is, the ability to evaluate a DMU in input and output orientation, simultaneously.

We prove mathematically that our modified MAJ-FDH model is always feasible and show that it allows to obtain an optimal solution without the need for solving the integer programming problem. More precisely, we state the validity of the proposed method through theorems and provide two numerical examples to demonstrate its applicability and show its efficacy.

Our model is an improved non-convex FDH model and the proposed method can prove useful also when dealing with the discrimination of DMUs and infeasible solutions in standard A&P super efficiency models. Indeed, our basic setting is the one of a traditional FDH technology under variable returns to scale. Our approach would allow to rank extreme efficient DMUs in A&P-FDH models even in the presence of infeasibility and, again, to obtain an optimal solution without having to solve the integer programming problem.

The current paper is organized as follows. In Section 2, we present some preliminary models and notions. In Section 3, we present the MAJ-FDH model, our modified MAJ-FDH models, the corresponding theorems and an efficiency comparison algorithm. In Section 4, we develop two examples to demonstrate the applicability and exhibit the efficacy of the modified MAJ-FDH model proposed in this study. Finally, we present our conclusion in Section 5.

2. Preliminary models and notations

2.1. FDH

Let DMU_j, with $j = 1, 2, \ldots, n$, denote n DMUs and assume each of them, DMU_j, to be endowed with m inputs and s outputs, that will be denoted by $x_{ij}(i = 1, 2, \ldots, m)$ and $y_{rj}(r = 1, 2, \ldots, s)$, respectively. The input and output vectors of the jth DMU will be denote by $\mathbf{X}_{ij}$ and $\mathbf{Y}_{rj}$, respectively, that is:

$$\mathbf{X}_{ij} = (x_{1j}, x_{2j}, \ldots, x_{mj}) \text{ and } \mathbf{Y}_{rj} = (y_{1j}, y_{2j}, \ldots, y_{sj}).$$

The FDH technology satisfies a free disposability assumption on the production possibility set but does not impose any convexity assumption. The traditional FDH technology is under variable returns to scale and is denoted by $T_{FDH\text{-}VRS}$. Traditionally, the FDH technology is represented by its production possibility set as follows:

$$T_{FDH\text{-}VRS} = \left\{ (x, y) : \sum_{j=1}^{n} \lambda_j y_j \geqslant y, \sum_{j=1}^{n} \lambda_j x_j \leqslant x, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \in \{0, 1\},\ j = 1, \ldots, n \right\} \quad (1)$$

The input orientation of the FDH analysis for the pth DMU is obtained by solving the following mixed integer linear program (LP) [29] for variable returns to scale:

$$\begin{aligned} & E(x_p, y_p) = \min\ \theta \\ & \text{s.t.} \\ & \sum_{j=1}^{n} x_{ij}\lambda_j \leqslant x_{ip}\theta \quad i = 1, \ldots, m, \\ & \sum_{j=1}^{n} y_{rj}\lambda_j \geqslant y_{rp} \quad r = 1, \ldots, s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j \in \{0, 1\}, \quad j = 1, \ldots, n. \end{aligned} \quad (2)$$

where $E(x_p, y_p)$ stands for the efficiency of the pth DMU, DMU_p.

Agrell and Tind [1] introduced a LP problem equivalent to the traditional mixed integer LP problem and compute the FDH efficiency measure under variable returns to scale. The following linear program is derived from Agrell and Tind [1].

$$\begin{aligned} & E(x_p, y_p) = \min \sum_{j=1}^{n} \theta_j \\ & \text{s.t.}: \\ & x_{ij}\lambda_j \leqslant x_{ip}\theta_j, \quad i = 1, \ldots, m;\ j = 1, \ldots, n, \\ & y_{rj}\lambda_j \geqslant \lambda_j y_{rp}, \quad r = 1, \ldots, s;\ j = 1, \ldots, n, \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j \geqslant 0, \quad j = 1, \ldots, n. \end{aligned} \quad (3)$$

Leleu [18] extended the estimation of the FDH using the following LP model with non-decreasing returns to scale, non-increasing returns to scale, and constant returns to scale technologies.

$$\begin{aligned} & E(x_p, y_p) = \min \sum_{j=1}^{n} \theta_j \\ & \text{s.t.}: \\ & x_{ij}(\lambda_j + \omega_j) \leqslant x_{ip}\theta_j \quad i = 1, \ldots, m;\ j = 1, \ldots, n, \\ & y_{rj}(\lambda_j + \omega_j) \geqslant \lambda_j y_{rp} \quad r = 1, \ldots, s;\ j = 1, \ldots, n, \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j \geqslant 0, \quad j = 1, \ldots, n, \end{aligned} \quad (4)$$

where the variable ω_j is the scaling factor of the jth DMU, DMU_j and, hence:

$$\begin{aligned} & \omega_j \in \Gamma_j, \quad \Gamma_j \in \{VRS, NIRS, NDRS, CRS\} \\ & VRS = \{\omega_j : \omega_j = 0\}, \quad NIRS = \{\omega_j : \omega_j \leqslant 0\} \\ & NDRS = \{\omega_j : \omega_j \geqslant 0\}, \quad CRS = \{\omega_j : \omega_j \ unconstrained\} \end{aligned} \quad (5)$$

2.2. A&P FDH model

Van Puyenbroeck [32] proposed the following A&P FDH model to discriminate among FDH-efficient units by modifying the standard FDH model based on the Andersen and Petersen's [2] modified DEA method.

$$\begin{aligned} & E^{AP\ FDH}(x_p, y_p) = \min\ \theta_p \\ & \text{s.t.} \\ & \sum_{\substack{j=1 \\ j\neq p}}^{n} x_{ij}\lambda_j \leqslant x_{ip}\theta_p, \quad i = 1, \ldots, m, \\ & \sum_{\substack{j=1 \\ j\neq p}}^{n} y_{rj}\lambda_j \geqslant y_{rp}, \quad r = 1, \ldots, s, \\ & \sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j = 1, \\ & \lambda_j \in \{0, 1\}, \quad j = 1, \ldots, n. \end{aligned} \quad (6)$$

2.3. MAJ-DEA model

Mehrabian et al. [21] suggested a new measure of efficiency and proposed the following model (Model (7)) to remove the difficulties and guarantee feasibility in the A&P FDH model. This model is now known to be an alternative to the A&P FDH model and it is usually referred to as the MAJ-DEA model.

$$\begin{aligned} & E^{MAJ\text{-}DEA}(x_p, y_p) = \min(1 + w_p) \\ & \text{s.t.} \\ & \sum_{\substack{j=1 \\ j \neq p}}^{n} x_{ij}\lambda_j \leqslant x_{ip} + w_p, \quad i = 1, \ldots, m, \\ & \sum_{\substack{j=1 \\ j \neq p}}^{n} y_{rj}\lambda_j \geqslant y_{rp}, \quad r = 1, \ldots, s, \\ & \lambda_j \geqslant 0, \quad j = 1, \ldots, n. \end{aligned} \tag{7}$$

Mehrabian et al. [21] also provide the following characterization of feasibility for the MAJ-DEA model.

Proposition 1 (*Necessary and sufficient conditions for feasibility in the MAJ-DEA model*). *The MAJ-DEA model (Model (7)) is feasible for evaluation of DMU_p with output vector $\mathbf{Y}_{rp} \geqslant 0$ if and only if, for every $r = 1, \ldots, s$, either $y_{rp} = 0$ or there exists a DMU_j, with $j \neq p$, such that $y_{rj} \neq 0$.*

Proof. See Mehrabian et al. [21]. □

3. Modifying the MAJ-FDH model

3.1. MAJ-FDH model

Sun and Hu [25] proposed the following model known as the MAJ-FDH model by relaxing the convexity assumption in Model (7):

$$\begin{aligned} & E^{MAJ\text{-}FDH}(x_p, y_p) = \min(1 + w_p) \\ & \text{s.t.} \\ & \sum_{\substack{j=1 \\ j \neq p}}^{n} x_{ij}\lambda_j \leqslant x_{ip} + w_p, \quad i = 1, \ldots, m, \\ & \sum_{\substack{j=1 \\ j \neq p}}^{n} y_{rj}\lambda_j \geqslant y_{rp}, \quad r = 1, \ldots, s, \\ & \sum_{\substack{j=1 \\ j \neq p}}^{n} \lambda_j = 1, \\ & \lambda_j \in \{0, 1\}, \quad j = 1, \ldots, n. \end{aligned} \tag{8}$$

The following theorem states the necessary and sufficient conditions for the MAJ-FDH model (that is, Model (8)) to be feasible.

Theorem 1. *The MAJ-FDH model (Model (8)) is feasible for the evaluation of DMU_p if and only if, there exists $j \in \{1, \ldots, n\}$, with $j \neq p$, such that for every $r = 1, 2, \ldots, s$, $y_{rj} \geqslant y_{rp}$.*

Proof. Suppose that there exists $j \in \{1, \ldots, n\}$, with $j \neq p$, such that for every $r = 1, 2, \ldots, s$, $y_{rj} \geqslant y_{rp}$. The second group of constraints and the convexity condition yield:

$$\exists j \in \{1, \ldots, n\},\ j \neq p, \text{ such that } \lambda_j = 1 \text{ and } \lambda_k = 0,\ \forall k \neq j.$$

Using the first group of constraints, we have:

$$\exists j \in \{1, \ldots, n\}, j \neq p, \text{ such that } \forall i \in \{1, \ldots, m\},\ x_{ij} \leqslant x_{ip} + w_p.$$

Then, the values:

$$w_p^{(j)} = \max_{i \in \{1, \ldots, m\}} (x_{ij} - x_{ip}), \quad \lambda_j = 1 \text{ and } \lambda_k = 0,\ \forall k \neq j$$

make the MAJ-FDH model feasible for the evaluation of DMU_p. Conversely, the feasibility of the MAJ-FDH model trivially implies the existence of DMU_j, with $j \neq p$, such that for every $r = 1, 2, \ldots, s$, $y_{rj} \geqslant y_{rp}$. □

Example. Suppose, for instance, that there are three DMUs, each of them producing two outputs (i.e. $s = 2$). If $y_{11} \geqslant y_{12}$ and $y_{21} \geqslant y_{22}$, then, by Theorem 1, DMU_2 is feasible with respect to the MAJ-FDH model, since for $r = 1, 2$ we have $y_{r1} \geqslant y_{r2}$. Similarly, if it happens that $y_{13} \geqslant y_{12}$ and $y_{23} \geqslant y_{22}$.

Suppose now that the outputs are such that $y_{11} < y_{12}$ and $y_{13} < y_{12}$. Then, again by Theorem 1, DMU_2 is infeasible with respect to the MAJ-FDH model. Indeed, it is not possible for DMU_1 to be such that for $r = 1, 2$ we have $y_{r1} \geqslant y_{r2}$, nor it is possible for DMU_3 to be such that for $r = 1, 2$ we have $y_{r3} \geqslant y_{r2}$.

The example above also yields a sufficient condition for infeasibility stronger than that expressed by Theorem 1. This condition is described by the following corollary.

Corollary 1. *The MAJ-FDH model is infeasible for the evaluation of DMU_p if there exists $r \in \{1, 2, \ldots, s\}$ such that $y_{rp} > \max \{y : j \neq p\}$.*

Proof. Suppose that there exists $r \in \{1, 2, \ldots, s\}$ such that $y_{rp} > \max \{y_{rj} : j \neq p\}$. Then, $\forall j \neq p$, $y_{rj} < y_{rp}$. Thus, the condition "$\exists j \neq 1$ such that $\forall r = 1, \ldots, s$, $y_{rj} \geqslant y_{rp}$" cannot be satisfied. By Theorem 1, it follows that DMU_p is infeasible. □

By Theorem 1, if MAJ-FDH model is feasible for the evaluation of DMU_p, the MAJ-FDH efficiency for DMU_p must be calculated using the following formula:

$$1 + w_p^* = \min_{j \neq p} \left\{ \max_{i \in \{1, \ldots, m\}} (x_{ij} - x_{ip}) \right\} + 1.$$

Theorem 2. *Let $p \in \{1, 2, \ldots, n\}$ and suppose that the MAJ-FDH model is feasible for the evaluation of DMU. Then, the efficiency score of DMU_p with respect to the model corresponds to the optimal solution of the model.*

Proof. By Theorem 1, there exists $j \in \{1, \ldots, n\}$, with $j \neq p$, such that for every $r = 1, 2, \ldots, s$, $y_{rj} \neq 0$ and $y_{rj} \geqslant y_{rp}$. As in the proof of Theorem 1, we can show that $\exists j \in \{1, \ldots, n\}$, $j \neq p$, such that: $\forall r \in \{1, \ldots, s\}$, $y_{rj} \geqslant y_{rp}$ and $\forall i \in \{1, \ldots, m\}$, $x_{ij} \leqslant x_{ip} + w_p$.

It follows that:

$$\forall i \in \{1,\ldots,m\}, \quad w_p \geqslant x_{ij} - x_{ip}.$$

Thus, we have:

$$w_p \geqslant \max_{i\in\{1,\ldots,m\}} (x_{ij} - x_{ip})$$

which is equivalent to:

$$w_p^* = \min_{j\neq p}\left\{ \max_{i\in\{1,\ldots,m\}} (x_{ij} - x_{ip})\right\}.$$

Therefore, the optimal solution of model (8) is:

$$1 + w_p^* = \min_{j\neq p}\left\{ \max_{i\in\{1,\ldots,m\}} (x_{ij} - x_{ip})\right\} + 1 \qquad \square.$$

3.2. Modified MAJ-FDH model

In this section, we present our modified MAJ-FDH model which is designed to eliminate the infeasibility difficulties in the A&P FDH and MAJ-FDH models. The proposed modified MAJ-FDH model is grounded in the Saati et al.'s [26] DEA model presented next:

$$\begin{aligned} & E(x_p, y_p) = \min(1 + w_p) \\ & \text{s.t.} \\ & \sum_{\substack{j=1\\ j\neq p}}^{n} x_{ij}\lambda_j \leqslant x_{ip} + w_p, \quad i = 1,\ldots,m, \\ & \sum_{\substack{j=1\\ j\neq p}}^{n} y_{rj}\lambda_j \geqslant y_{rp} - w, \quad r = 1,\ldots,s, \\ & \sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j = 1, \\ & \lambda_j \geqslant 0, \quad j = 1,\ldots,n. \end{aligned} \tag{9}$$

More precisely, we construct the modified MAJ-FDH model by relaxing the convexity assumption in model (9). The modified MAJ-FDH model, which we show to be always feasible, can be formulated as follows:

$$\begin{aligned} & E^{ModifiedMAJ\text{-}FDH}(x_p, y_p) = \min(1 + w_p) \\ & \text{s.t.} \\ & \sum_{\substack{j=1\\ j\neq p}}^{n} x_{ij}\lambda_j \leqslant x_{ip} + w_p, \quad i = 1,\ldots,m, \\ & \sum_{\substack{j=1\\ j\neq p}}^{n} y_{rj}\lambda_j \geqslant y_{rp} - w_p, \quad r = 1,\ldots,s, \\ & \sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j = 1, \\ & \lambda_j \in \{0,1\}, \quad j = 1,\ldots,n. \end{aligned} \tag{10}$$

The following theorem shows that the modified MAJ-FDH model, just introduced, can be used to discriminate among FDH-efficient units and to examine the tie-breaking abilities of the 0–1 LP FDH, A&P FDH, and MAJ-FDH models.

Theorem 3. *The modified MAJ-FDH model (Model 10) is always feasible for the evaluation of all DMUs.*

Proof. Fix a generic pth DMU, DMU_p, where $p \in \{1,2,\ldots,n\}$. By the convexity condition of the modified MAJ-FDH model, we have:

$$\sum_{j=1}^{n} \lambda_j = 1, \quad \lambda_j \in \{0,1\}, \quad j \in \{1,\ldots,n\}.$$

Then,

$$\exists j \in \{1,\ldots,n\},\ j\neq p, \text{ such that } \lambda_j = 1 \text{ and } \lambda_k = 0,\ \forall k\neq j.$$

From the first and second group of constraints it follows that $\exists j \in \{1,\ldots,n\}$, $j\neq p$, such that: $\forall r \in \{1,\ldots,s\}$, $y_{rj} \geqslant y_{rp} - w_p$ and $\forall i \in \{1,\ldots,m\},\ x_{ij} \leqslant x_{ip} + w_p$.

Therefore, the values:

$$\begin{aligned} & w_p^{(j)} = \max\left\{ \max_{i\in\{1,2,\ldots,m\}} (x_{ij} - x_{ip}), \max_{r\in\{1,2,\ldots,s\}} (y_{rp} - y_{rj})\right\}, \\ & \lambda_j = 1 \text{ and } \lambda_k = 0,\ \forall k\neq j, \end{aligned}$$

make the modified MAJ-FDH model feasible for the evaluation of *DMUp*. □

By Theorem 3, all DMUs can be evaluated in the modified MAJ-FDH model, i.e., all DMUs are always feasible. At the same time, the modified MAJ-FDH efficiency for the generic DMU, DMU_p, can be calculated using the following formula:

$$1 + w_p^* = \min_{j\neq p}\left\{ \max\left\{ \max_{i\in\{1,2,\ldots,m\}} (x_{ij} - x_{ip}), \max_{r\in\{1,2,\ldots,s\}} (y_{rp} - y_{rj})\right\}\right\} + 1.$$

The evaluation and comparison of DMUs' efficiency scores in the MAJ-FDH and the modified MAJ-FDH models follow an algorithmic structure, whose graphical representation is provided in Fig. 1. Being based on a model that removes infeasibility for all the DMUs, this algorithm allows us to operate a full comparison among all the most known FDH models presented in the existing literature and their corresponding DMUs-ranking criteria.

Note that the inputs and outputs in the MAJ-FDH and the modified MAJ-FDH models are not homogeneous, while the scale of the objective function in these models is dependent on the units of measurement for the input and output data. Consequently, we can obtain a unitary dependence through a standard normalization process where each input and output of each DMU_j is divided by the largest of them as follows:

$$\begin{aligned} & \forall i = 1,2,\ldots,m,\ \bar{x}_{ij} = \frac{x_{ij}}{\max_{j\in\{1,2,\ldots,n\}}(x_{ij})} \text{ and} \\ & \forall r = 1,2,\ldots,s,\ \bar{y}_{rj} = \frac{y_{rj}}{\max_{j\in\{1,2,\ldots,n\}}(y_{rj})}. \end{aligned}$$

The proposed efficiency comparison algorithm can be summarized as follows:

Efficiency Comparison Algorithm: MAJ-FDH vs Modified MAJ-FDH

Start: Normalize m inputs and s outputs of each DMU_j, $\forall j = 1, 2, \ldots, n$.

$$\text{Compute } \bar{x}_{ij} = \frac{x_{ij}}{\max_{j \in \{1,2,\ldots,n\}}(x_{ij})} \quad \forall i = 1, 2, \ldots, m$$

$$\text{Compute } \bar{y}_{rj} = \frac{y_{rj}}{\max_{j \in \{1,2,\ldots,n\}}(y_{rj})} \quad \forall r = 1, 2, \ldots, s$$

Step 1: Fix the pth DMU, DMU_p.

Step 2: Check feasibility of DMU_p with respect to the MAJ-FDH model.

Find

$$D_p^{MAJ\text{-}FDH} = \{j \neq p : \forall r = 1, 2, \ldots, s,\ y_{rj} \geqslant y_{rp}\} \tag{11}$$

If $D_p^{MAJ-FDH} \neq \varnothing$ then

DMU_p is feasible with respect to the MAJ-FDH model.

Else

DMU_p is infeasible with respect to the MAJ-FDH model.

Step 3: Compute model efficiency of DMU_p.

If feasible, then:

$$\forall j \in D_p^{MAJ\text{-}FDH} \quad \forall i \in \{1, 2, \ldots, m\} : x_{ij} \geqslant x_{ip} \tag{12}$$

Compute the model efficiency for DMU_p in the MAJ-FDH model:

$$1 + w_p^* = \min_{j \in D_p^{MAJ-FDH}} \left\{ \max_{i \in \{1,2,\ldots,m\}} (\bar{x}_{ij} - \bar{x}_{ip}) \right\} + 1$$

Compute the model efficiency for DMU_p in the modified MAJ-FDH model:

$$1 + w_p^* = \min_{j \neq p} \left\{ \max \left\{ \max_{i \in \{1,2,\ldots,m\}} (\bar{x}_{ij} - \bar{x}_{ip}), \max_{r \in \{1,2,\ldots,s\}} (\bar{y}_{rp} - \bar{y}_{rj}) \right\} \right\} + 1$$

If infeasible, then:

$$\forall j \in \{1, 2, \ldots, n\},\ j \neq p,\ \exists r \in \{1, 2, \ldots, s\} \text{ such that } y_{rp} > y_{rj} \tag{13}$$

Compute the model efficiency for DMU_p in the modified MAJ-FDH model:

$$1 + w_p^* = \min_{j \neq p} \left\{ \max \left\{ \max_{i \in \{1,2,\ldots,m\}} (\bar{x}_{ij} - \bar{x}_{ip}), \max_{r \in \{1,2,\ldots,s\}} (\bar{y}_{rp} - \bar{y}_{rj}) \right\} \right\} + 1$$

Step 4: Repeat Step 2 and 3 for all $p' \in \{1, 2, \ldots, n\},\ p' \neq p$.

Step 5: Rank DMUs.

Rank efficiency results only of feasible DMUs in the MAJ-FDH model.

Rank efficiency results of all DMUs in the modified MAJ-FDH model.

The proposed algorithm allows us to determine the efficiency values of each DMU in the MAJ-FDH and the modified MAJ-FDH models. The formulas derived from Theorems 1 and 3 (see Algorithm) also yield the following result.

Theorem 4. *The proposed algorithm generates the MAJ-FDH and modified MAJ-FDH scores in polynomial time.*

4. A comparative analysis of various FDH models

In this section, we use two examples introduced by Sun and Hu [25] to demonstrate the applicability and show the efficacy of the modified MAJ-FDH proposed in this study.

Let us first consider the small problem introduced by Sun and Hu [25] with three DMUs, two inputs, and two outputs. The input and output data for this problem are shown in Table 1. We solved this problem first with the FDH and A&P FDH methods. As shown in Table 2, all DMUs are extremely efficient in FDH, while A&P FDH cannot fully rank all three DMUs.

Afterward, we used the MAJ-FDH and the modified MAJ-FDH methods to solve the same small problem. First, we investigated the feasibility of the MAJ-FDH model and obtained the following results:

$$D_1^{MAJ\text{-}FDH} = \{j \neq 1 : y_{rj} \geqslant y_{r1}, r = 1, 2\} = \{2, 3\}$$

$$D_2^{MAJ\text{-}FDH} = \{j \neq 2 : y_{rj} \geqslant y_{r2}, r = 1, 2\} \Rightarrow D_2^{MAJ\text{-}FDH} = \varnothing. \quad \text{Then DMU}_2 \text{ is infeasible.}$$

$$D_3^{MAJ\text{-}FDH} = \{j \neq 3 : y_{rj} \geqslant y_{r3}, r = 1, 2\} = \{1, 2\}$$

As shown here, and also acknowledged by Sun and Hu [25], DMU2 is infeasible according to the MAJ-FDH method. Hence, we computed the MAJ-FDH efficiency for DMU_1 and DMU_3 as follows:

MAJ-FDH for DMU_1

$$\begin{aligned} D_1^{MAJ\text{-}FDH} &= \{j \neq 1 : y_{rj} \geqslant y_{r1}, r = 1, 2\} = \{2, 3\} \Rightarrow 1 + w_1^* \\ &= \min_{j \in D_1^{MAJ\text{-}FDH}} \left\{ \max_{i \in \{1,2\}} (\bar{x}_{ij} - \bar{x}_{i1}) \right\} + 1 = 1 \\ &+ \min_{j \in \{2,3\}} \left\{ \max_{i \in \{1,2\}} (\bar{x}_{12} - 0.5, \bar{x}_{22} - 0.4), \max_{i \in \{1,2\}} (\bar{x}_{13} - 0.5, \bar{x}_{23} - 0.4) \right\} = 1 \\ &+ \min_{j \in \{2,3\}} \left\{ \max_{i \in \{1,2\}} (0.125 - 0.5, 1 - 0.4), \max_{i \in \{1,2\}} (1 - 0.5, 0.2 - 0.4) \right\} = 1.5 \end{aligned}$$

MAJ-FDH for DMU_3

$$\begin{aligned} D_3^{MAJ\ FDH} &= \{j \neq 3 : y_{rj} \geqslant y_{r3}, r = 1, 2\} = \{1, 2\} \Rightarrow 1 + w_3^* \\ &= \min_{j \in D_3^{MAJ\text{-}FDH}} \left\{ \max_{i \in \{1,2\}} (\bar{x}_{ij} - \bar{x}_{i3}) \right\} + 1 = 1 \\ &+ \min_{j \in \{1,2\}} \left\{ \max_{i \in \{1,2\}} (\bar{x}_{11} - 1, \bar{x}_{21} - 0.2), \max_{i \in \{1,2\}} (\bar{x}_{12} - 1, \bar{x}_{22} - 0.2) \right\} = 1 \\ &+ \min_{j \in \{1,2\}} \left\{ \max_{i \in \{1,2\}} (0.5 - 1, 0.4 - 0.2), \max_{i \in \{1,2\}} (0.125 - 1, 1 - 0.2) \right\} = 1.2 \end{aligned}$$

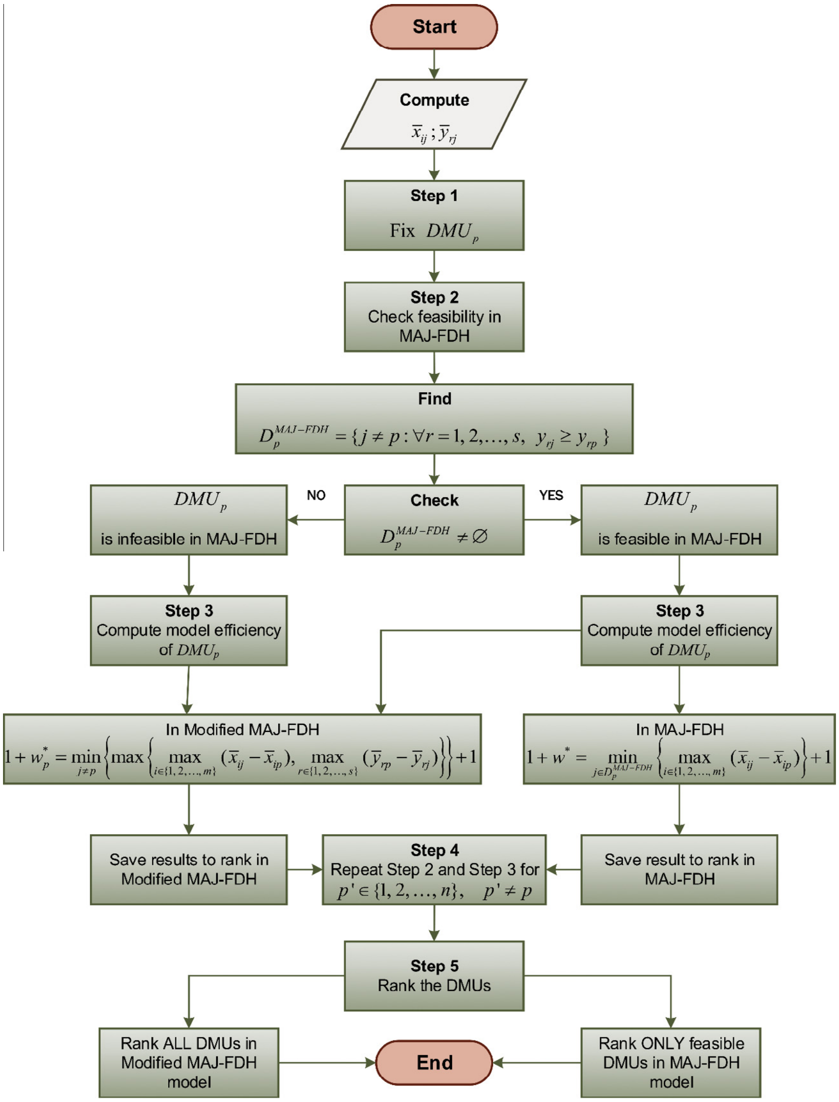

Fig. 1. Efficiency comparison algorithm flowchart: MAJ-FDH vs Modified MAJ-FDH.

On the other hand, we computed the efficiency for the infeasible DMU_2 by using the modified MAJ-FDH model as follows:

$$1 + w_2^* = \min_{j\neq 2}\left\{\max\left\{\max_{i\in\{1,2\}}(\bar{x}_{ij} - \bar{x}_{i2}), \max_{r\in\{1,2\}}(\bar{y}_{r2} - \bar{y}_{rj})\right\}\right\} + 1$$
$$= \min_{j\neq 2}\{\max\{\max(0.5 - 0.125, 0.4 - 1),$$
$$\max(1 - 0, 1 - 1)\}, \max\{\max(1 - 0.125, 0.2 - 1),$$
$$\max(1 - 0, 1 - 1)\}\} + 1 = 2$$

Similarly, we also obtained the efficiency scores of DMU_1 and DMU_2 based on the modified MAJ FDH model.

The efficiencies of the three DMUs according to the four methods, FDH, A&P FDH, MAJ-FDH, and Modified MAJ-FDH, are summarized in Table 2.

We now consider the large problem introduced by Sun and Hu [25]. The problem presents 32 DMUs, six inputs (that is, $m = 6$), and five outputs (that is, $s = 5$). The analysis we performed shows the effectiveness of our modified MAJ-FDH method when compared with the FDH, the A&P FDH, and the MAJ-FDH methods. Table 3 presents the input and output data for this problem.

We computed the FDH, the A&P FDH, the MAJ-FDH, and the modified MAJ-FDH efficiencies of each feasible DMU based on the corresponding algorithms. In particular, we applied the proposed modified MAJ-FDH method. The results obtained in the large problem under analysis show the modified Maj-FDH method to be more efficient from the computational standpoint than any mathematical programming model. More precisely, applying Theorem 1 and Corollary 1, we can check that DMU_1, DMU_{16}, DMU_{27}, and DMU_{28} are infeasible with respect to the MAJ-FDH method. Indeed, using the output data in Table 3, we have:

Table 1
Input–output data for Sun and Hu's [25] small example.

DMU	Input		Output	
	1	2	1	2
1	4.0	2.0	0.0	1.0
2	1.0	5.0	1.0	1.0
3	8.0	1.0	0.0	1.0

Table 2
Comparative results for Sun and Hu's [25] small example.

DMU	Evaluation method			
	FDH	A&P FDH	MAJ-FDH	Modified MAJ-FDH
1	1.00	4	1.5	1.5
2	1.00	Infeasible	Infeasible	2.0
3	1.00	1.2	1.2	1.2

$$\left.\begin{array}{lll} DMU_1 & \rightarrow & y_{3;1} > \max\{y_{3j} : j = 1,2,\ldots,32,\ j \neq 1\} \\ DMU_{16} & \rightarrow & y_{4;16} > \max\{y_{4j} : j = 1,2,\ldots,32,\ j \neq 16\} \\ DMU_{27} & \rightarrow & y_{3;27} > \max\{y_{3j} : j = 1,2,\ldots,32,\ j \neq 27\} \end{array}\right\}$$

$$\rightarrow \text{Apply Corollary 1} \rightarrow \left\{\begin{array}{l} DMU_1 \text{ is infeasible} \\ DMU_{16} \text{ is infeasible} \\ DMU_{27} \text{ is infeasible} \end{array}\right.$$

$$\begin{array}{lll} DMU_{28} & \rightarrow & y_{3;28} = \max\{y_{r28} : r = 1,2,3,4,5\} \text{ and} \\ & & y_{3;28} > \max\{y_{3j} : j = 1,2,\ldots,32, \\ & & j \neq 1,\ j \neq 27,\ j \neq 28,\ j \neq 29\} \end{array}$$

$$\rightarrow \left\{\begin{array}{l} \text{If } j \notin \{1,27,28,29\} \rightarrow y_{3j} < y_{3;28} \\ \text{If } j = 1 \rightarrow y_{4;1} < y_{4;28} \\ \text{If } j = 27 \rightarrow y_{4;27} < y_{4;28} \\ \text{If } j = 29 \rightarrow y_{4;29} < y_{4;28} \end{array}\right\} \rightarrow$$

$\rightarrow$ the condition "$\exists j \neq 28$ such that $\forall r = 1,\ldots,5$, $y_{rj} \geqslant y_{r28}$" is not satisfied $\rightarrow$

$\rightarrow$ Apply Theorem 1 $\rightarrow$ DMU_{28} is infeasible

Therefore, the MAJ-FDH method cannot fully rank these DMUs. However, all 32 DMUs are feasible according to the modified MAJ-FDH method proposed in this study.

The DMUs-rankings, obtained using the Efficiency Comparison Algorithm introduced above, are presented in Table 4. In particular, Table 4 shows that DMU_{28} is ranked first by the modified MAJ-FDH model while DMU_{11} is ranked last. The MAJ-FDH and A&P FDH models cannot evaluate DMU_1, DMU_{16}, DMU_{27}, and DMU_{28}, since their corresponding problems become infeasible. Note that Sun and Hu [25] had incorrectly reported that all 32 DMUs are feasible with respect to the MAJ-FDH method. As

Table 3
Input–output data for Sun and Hu's [25] large example.

DMU	Input						Output				
	1	2	3	4	5	6	1	2	3	4	5
1	55	28	80	43	57	821.6297	392.3953	62.0127	471.1391	19.19460	51.9546
2	12	4.0	3.0	2.0	3.0	69.08080	66.30040	10.9712	6.134400	0.247700	0.17820
3	2.0	1.0	2.0	1.0	1.0	12.92460	4.453400	3.76580	7.487000	0.375400	0.09040
4	7.0	4.0	30	5.0	9.0	218.0060	33.78260	11.8225	188.9325	4.637000	3.11620
5	19	4.0	45	3.0	9.0	211.7006	120.0685	11.2600	77.13200	1.622600	35.3613
6	10	5.0	32	14	12	219.3919	61.82370	13.6316	150.3821	4.832800	19.4298
7	9.0	6.0	13	7.0	10	106.9018	35.45120	13.7407	73.54230	3.384300	0.65420
8	9.0	5.0	8.0	4.0	5.0	103.0289	61.29490	13.1706	45.28470	1.457800	1.39176
9	5.0	3.0	9.0	6.0	5.0	109.9591	55.51730	7.05570	61.65720	4.477500	0.52660
10	17	9.0	52	22	37	363.3056	120.7935	26.1731	257.0555	17.31590	6.72860
11	2.0	2.0	5.0	2.0	3.0	31.57610	13.43340	4.60680	16.38760	1.539300	0.25030
12	8.0	4.0	31	7.0	9.0	180.0819	40.08700	9.77880	139.1302	11.04810	2.66730
13	11	3.0	25	14	11	238.2405	59.03920	12.8907	189.5601	9.628400	3.31840
14	11	4.0	16	5.0	10	109.6939	34.76420	10.4361	79.25850	2.559200	0.46480
15	11	4.0	25	10	10	165.6027	37.16570	12.5500	134.2577	8.957800	0.94860
16	11	2.0	1.0	2.0	14	169.0845	82.00910	11.6801	6.926600	103.2209	5.14230
17	5.0	2.0	17.0	4.0	4.0	121.8967	30.32360	6.67830	103.4982	3.40920	0.79090
18	6.0	2.0	5.00	3.0	4.0	61.81800	35.52250	6.07140	31.53460	2.90750	1.80220
19	4.0	1.0	6.00	2.0	2.0	56.91650	17.43080	4.28540	42.61730	2.51450	0.76610
20	9.0	5.0	24.0	9.0	16	177.5242	49.94510	14.3873	122.1197	4.54030	9.69910
21	5.0	2.0	10.0	4.0	4.0	111.2907	38.59340	6.78440	72.48290	6.41780	1.08900
22	5.0	2.0	16.0	5.0	5.0	111.4448	28.34050	6.92810	86.36610	3.83950	3.25090
23	15	7.0	43.0	13	21	254.6729	61.27040	18.6946	198.2142	12.2306	8.16100
24	8.0	3.0	15.0	4.0	9.0	146.5831	56.92400	8.11210	98.49520	2.45880	2.28930
25	7.0	3.0	27.0	8.0	9.0	143.4506	36.80970	8.91710	109.8258	5.69800	3.50250
26	36	13	60.0	30	31	372.6933	130.9099	30.8536	268.4286	11.7677	5.36550
27	46	14	124	45	55	765.9635	188.4216	42.5586	570.2482	27.8539	57.5849
28	23	6.0	37.0	23	43	497.7266	80.81110	20.4679	309.5052	86.8802	49.6043
29	20	14	87.0	14	32	445.9886	93.25980	30.4755	357.1496	6.70550	16.0935
30	15	6.0	24.0	10	18	165.6646	50.35370	14.6470	114.4559	4.54430	12.1747
31	10	4.0	24.0	9.0	18	205.9418	60.35210	14.3967	139.6452	3.95100	7.56000
32	27	7.0	51.0	26	33	268.7520	72.78580	28.2966	204.6602	7.71470	12.0520

Table 4
Comparative results for Sun and Hu's [25] large example.

DMU	FDH	A&P FDH	Rank	MAJ-FDH	Rank	Modified MAJ-FDH	Rank
1	1.00	Infeasible	–	Infeasible	–	1.55	2
2	1.00	4.666667	3	1.192982	17	1.0625	15
3	1.00	3	9	1.0375	25	1.0375	26
4	1.00	2.6	14	1.209302	15	1.087334	11
5	1.00	1.43E+01	1	1.930233	1	1.276661	5
6	1.00	3.583333	5	1.54386	6	1.12599	8
7	1.00	1.846154	19	1.1375	19	1.049553	19
8	1.00	4	4	1.3	10	1.068655	13
9	1.00	2.777778	12	1.2	16	1.04313	23
10	1.00	2.705882	13	1.690909	3	1.151692	7
11	1.00	3	9	1.072727	23	1.022885	32
12	1.00	2.333333	15	1.210526	14	1.051832	18
13	1.00	2.333333	15	1.225	13	1.064365	14
14	1.00	1.8	20	1.105263	21	1.054511	17
15	1.00	1.438627	22	1.093023	22	1.058583	16
16	1.00	Infeasible	–	Infeasible	–	1.508772	3
17	1.00	2.25	16	1.125	20	1.030043	31
18	1.00	4.8	2	1.2375	11	1.046106	22
19	1.00	2	18	1.066178	24	1.036364	28
20	1.00	2.803711	11	1.395349	8	1.037147	27
21	1.00	3.1	6	1.232558	12	1.046512	21
22	1.00	1.8	20	1.1375	19	1.037543	25
23	1.00	2.047619	18	1.385965	9	1.083879	12
24	1.00	2.25	16	1.157895	18	1.046715	20
25	1.00	2.333333	15	1.210526	14	1.031381	30
26	1.00	2.066667	17	1.546397	5	1.105263	10
27	1.00	Infeasible	–	Infeasible	–	1.457245	4
28	1.00	Infeasible	–	Infeasible	–	1.723202	1
29	1.00	3.071429	7	1.674419	4	1.186047	6
30	1.00	3.004423	8	1.438596	7	1.04299	24
31	1.00	1.791667	21	1.2375	11	1.035714	29
32	1.00	2.850073	10	1.75	2	1.115081	9

shown above, DMU_1, DMU_{16}, DMU_{27}, and DMU_{28} are infeasible by Theorem 1. To further support this claim, we also provide the GAMS software models for the MAJ-FDH and the modified MAJ-FDH models along with their associated results. See Appendices A and B.

5. Conclusion

A framework for ranking FDH-efficient units was proposed by Andersen and Petersen [2]. Van Puyenbroeck [32] showed that FDH method generates a relatively large number of efficient DMUs and modified Andersen and Petersen's [2] model into the so-called A&P FDH model. Adler et al. [3] showed that the A&P FDH method cannot provide a complete ranking of all DMUs. In response to this, Mehrabian et al. [21] suggested a MAJ-DEA method for modifying the dual formulation and ensuring feasibility. Subsequently, Sun and Hu [25] proposed a modified version of Mehrabian et al.'s [21] method, known as MAJ-FDH model, to strengthen the need for discriminating among efficient units.

In this paper, we have suggested an improved non-convex FDH model with the ability to evaluate a DMU in input and output orientation, simultaneously. Grounded in the DEA ranking method developed by Saati et al. [26], this method is a simple but still very important modification of the MAJ-FDH method proposed by Sun and Hu [25]. We showed that the MAJ-FDH model is not always feasible, while the proposed modified MAJ-FDH model is. We mathematically proved that the modified MAJ-FDH model we propose is always feasible and showed that it can be used to obtain an optimal solution without the need for solving the integer programming problem. We also proved the validity of the proposed method through theorems and provided two numerical examples to demonstrate its applicability and show its efficacy with respect to existing methods.

Finally, the new and alternative efficiency measure defined in this paper exploits the characteristics of a traditional FDH technology under variable returns to scale. This makes our results flexible enough to be applicable also when dealing with the discrimination of DMUs and infeasible solutions in standard A&P super efficiency models.

Acknowledgement

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Appendix A.

A.1. GAMS models

MAJ-FDH GAMS Model

```
  OPTIONS Decimals = 8;
Sets
  j/1*32/
    i/1*6/
    r/1*5/
K(j);
alias (j,t,l);

Acronyms Optimal, Infeasible, Unbounded
Parameters
    Xprime(i)
    Yprime(r)
    status
    Rank(j);
Table X(j,i)
          1             2             3        4             5             6
1         1             1             1        1             1             1
2         0.218181818   0.142857143   0.0375   0.046511628   0.052631579   0.084077779
3         0.036363636   0.035714286   0.025    0.023255814   0.01754386    0.015730444
4         0.127272727   0.142857143   0.375    0.11627907    0.157894737   0.265333641
5         0.345454545   0.142857143   0.5625   0.069767442   0.157894737   0.257659381
6         0.181818182   0.178571429   0.4      0.325581395   0.210526316   0.267020411
7         0.163636364   0.214285714   0.1625   0.162790698   0.175438596   0.130109464
8         0.163636364   0.178571429   0.1      0.093023256   0.087719298   0.125395784
9         0.090909091   0.107142857   0.1125   0.139534884   0.087719298   0.133830483
10        0.309090909   0.321428571   0.65     0.511627907   0.649122807   0.44217681
11        0.036363636   0.071428571   0.0625   0.046511628   0.052631579   0.03843106
12        0.145454545   0.142857143   0.3875   0.162790698   0.157894737   0.219176473
13        0.2           0.107142857   0.3125   0.325581395   0.192982456   0.289960915
14        0.090909091   0.142857143   0.2      0.11627907    0.175438596   0.13350771
15        0.163636364   0.142857143   0.3125   0.23255814    0.175438596   0.201553936
16        0.163636364   0.071428571   0.0125   0.046511628   0.245614035   0.205791611
17        0.090909091   0.071428571   0.2125   0.093023256   0.070175439   0.148359656
18        0.109090909   0.071428571   0.0625   0.069767442   0.070175439   0.075238273
19        0.072727273   0.035714286   0.075    0.046511628   0.035087719   0.069272691
20        0.163636364   0.178571429   0.3      0.209302326   0.280701754   0.216063514
21        0.090909091   0.071428571   0.125    0.093023256   0.070175439   0.135451165
22        0.090909091   0.071428571   0.2      0.11627907    0.087719298   0.135638719
23        0.272727273   0.25          0.5375   0.302325581   0.368421053   0.309960679
24        0.145454545   0.107142857   0.1875   0.093023256   0.157894737   0.178405308
25        0.127272727   0.107142857   0.3375   0.186046512   0.157894737   0.174592764
26        0.654545455   0.464285714   0.75     0.697674419   0.543859649   0.453602517
27        0.836363636   0.5           1.55     1.046511628   0.964912281   0.932249041
28        0.418181818   0.214285714   0.4625   0.534883721   0.754385965   0.605779708
29        0.363636364   0.5           1.0875   0.325581395   0.561403509   0.542809735
30        0.272727273   0.214285714   0.3      0.23255814    0.315789474   0.201629274
31        0.181818182   0.142857143   0.3      0.209302326   0.315789474   0.250650384
32        0.490909091   0.25          0.6375   0.604651163   0.578947368   0.327096258;

Table Y(j,r)
          1          2         3          4          5
1         392.3953   62.0127   471.1391   19.1946    51.9546
2         66.3004    10.9712   6.1344     0.2477     0.1782
3         4.4534     3.7658    7.487      0.3754     0.0904
4         33.7826    11.8225   188.9325   4.637      3.1162
5         120.0685   11.26     77.132     1.6226     35.3613
6         61.8237    13.6316   150.3821   4.8328     19.4298
7         35.4512    13.7407   73.5423    3.3843     0.6542
8         61.2949    13.1706   45.2847    1.4578     1.39176
9         55.5173    7.0557    61.6572    4.4775     0.5266
10        120.7935   26.1731   257.0555   17.3159    6.7286
11        13.4334    4.6068    16.3876    1.5393     0.2503
12        40.087     9.7788    139.1302   11.0481    2.6673
13        59.0392    12.8907   189.5601   9.6284     3.3184
14        34.7642    10.4361   79.2585    2.5592     0.4648
15        37.1657    12.55     134.2577   8.9578     0.9486
16        82.0091    11.6801   6.9266     103.2209   5.1423
17        30.3236    6.6783    103.4982   3.4092     0.7909
18        35.5225    6.0714    31.5346    2.9075     1.8022
19        17.4308    4.2854    42.6173    2.5145     0.7661
20        49.9451    14.3873   122.1197   4.5403     9.6991
21        38.5934    6.7844    72.4829    6.4178     1.089
```

(continued on next page)

```
22    28.3405    6.9281    86.3661    3.8395    3.2509
23    61.2704    18.6946   198.2142   12.2306   8.161
24    56.924     8.1121    98.4952    2.4588    2.2893
25    36.8097    8.9171    109.8258   5.698     3.5025
26    130.9099   30.8536   268.4286   11.7677   5.3655
27    188.4216   42.5586   570.2482   27.8539   57.5849
28    80.8111    20.4679   309.5052   86.8802   49.6043
29    93.2598    30.4755   357.1496   6.7055    16.0935
30    50.3537    14.647    114.4559   4.5443    12.1747
31    60.3521    14.3967   139.6452   3.951     7.56
32    72.7858    28.2966   204.6602   7.7147    12.0527;
binary Variables
    Lambda(j)
    ;
Free Variable
    teta
    z;
Equations
    Obj
    Outputs(r)
    Inputs(i)
    Convexity;
Obj .. z = e = 1 + Teta;
Outputs(r) ..sum(j$k(j), Y(j,r)*Lambda(j))-Yprime(r)=g = 0;
Inputs(i) ..sum(j$k(j), X(j,i)*Lambda(j))-Xprime(i)-Teta = l = 0;
Convexity ..sum(j$k(j), Lambda(j))=e = 1;
Model MAJFDH /all/;
*________ F I L E D E C E L E R A T I O N ________
file My /"d:\res.txt"/;
put My;
My.nd = 10;
loop(t,
    Xprime(i)=X(t,i);
    Yprime(r)=Y(t,r);
Loop(j,
k(j)=Yes;
k(t)=No);
  Solve MAJFDH using MIP minimize Z;
  put MAJFDH.modelstat, "———————"/;
  put "DMU ", t.tl:3," = ", z.L:6 /;
   );
putclose My;
```

Modified MAJ-FDH GAMS Model

```
OPTIONS Decimals = 8;
Sets
  j/1*32/
  i/1*6/
  r/1*5/
K(j);
alias (j,t,l);

Acronyms Optimal, Infeasible, Unbounded
Parameters
  Xprime(i)
  Yprime(r)
  status
  Rank(j);
Table X(j,i)
              1              2              3         4              5              6
1             1              1              1         1              1              1
2             0.218181818    0.142857143    0.0375    0.046511628    0.052631579    0.084077779
3             0.036363636    0.035714286    0.025     0.023255814    0.01754386     0.015730444
4             0.127272727    0.142857143    0.375     0.11627907     0.157894737    0.265333641
5             0.345454545    0.142857143    0.5625    0.069767442    0.157894737    0.257659381
6             0.181818182    0.178571429    0.4       0.325581395    0.210526316    0.267020411
7             0.163636364    0.214285714    0.1625    0.162790698    0.175438596    0.130109464
8             0.163636364    0.178571429    0.1       0.093023256    0.087719298    0.125395784
9             0.090909091    0.107142857    0.1125    0.139534884    0.087719298    0.133830483
10            0.309090909    0.321428571    0.65      0.511627907    0.649122807    0.44217681
11            0.036363636    0.071428571    0.0625    0.046511628    0.052631579    0.03843106
12            0.145454545    0.142857143    0.3875    0.162790698    0.157894737    0.219176473
13            0.2            0.107142857    0.3125    0.325581395    0.192982456    0.289960915
```

```
14    0.090909091  0.142857143  0.2     0.11627907   0.175438596  0.13350771
15    0.163636364  0.142857143  0.3125  0.23255814   0.175438596  0.201553936
16    0.163636364  0.071428571  0.0125  0.046511628  0.245614035  0.205791611
17    0.090909091  0.071428571  0.2125  0.093023256  0.070175439  0.148359656
18    0.109090909  0.071428571  0.0625  0.069767442  0.070175439  0.075238273
19    0.072727273  0.035714286  0.075   0.046511628  0.035087719  0.069272691
20    0.163636364  0.178571429  0.3     0.209302326  0.280701754  0.216063514
21    0.090909091  0.071428571  0.125   0.093023256  0.070175439  .135451165
22    0.090909091  0.071428571  0.2     0.11627907   0.087719298  0.135638719
23    0.272727273  0.25         0.5375  0.302325581  0.368421053  0.309960679
24    0.145454545  0.107142857  0.1875  0.093023256  0.157894737  0.178405308
25    0.127272727  0.107142857  0.3375  0.186046512  0.157894737  0.174592764
26    0.654545455  0.464285714  0.75    0.697674419  0.543859649  0.453602517
27    0.836363636  0.5          1.55    1.046511628  0.964912281  0.932249041
28    0.418181818  0.214285714  0.4625  0.534883721  0.754385965  0.605779708
29    0.363636364  0.5          1.0875  0.325581395  0.561403509  0.542809735
30    0.272727273  0.214285714  0.3     0.23255814   0.315789474  0.201629274
31    0.181818182  0.142857143  0.3     0.209302326  0.315789474  0.250650384
32    0.490909091  0.25         0.6375  0.604651163  0.578947368  0.327096258;
Table Y(j,r)
      1            2            3            4            5
1     1            1            0.826200065  0.185956526  0.902226104
2     0.168963288  0.176918599  0.010757421  0.002399708  0.003094561
3     0.011349269  0.060726271  0.013129371  0.00363686   0.001569856
4     0.086093284  0.190646432  0.331316258  0.044923073  0.054114881
5     0.305988629  0.181575709  0.135260401  0.015719685  0.614072439
6     0.157554639  0.219819489  0.263713415  0.046819975  0.33741137
7     0.090345629  0.221578806  0.128965422  0.032786965  0.011360617
8     0.156207019  0.212385527  0.079412263  0.014123109  0.024168836
9     0.141483091  0.113778307  0.108123445  0.043377843  0.009144758
10    0.307836256  0.42206032   0.450778275  0.167755755  0.116846604
11    0.034234355  0.074288009  0.028737662  0.014912678  0.004346626
12    0.102159735  0.15769028   0.243981831  0.107033556  0.046319434
13    0.150458479  0.207871936  0.332416832  0.093279559  0.057626218
14    0.088594843  0.168289721  0.138989479  0.024793428  0.00807156
15    0.094714947  0.2023779    0.235437306  0.086782812  0.016473068
16    0.208996132  0.188350128  0.012146641  1            0.089299452
17    0.077278194  0.107692457  0.181496759  0.033028195  0.013734503
18    0.090527333  0.097905752  0.05529978   0.028167745  0.031296399
19    0.044421531  0.069105199  0.074734651  0.024360377  0.013303835
20    0.127282615  0.232005702  0.214151838  0.043986247  0.168431308
21    0.09835337   0.109403396  0.127107635  0.062175393  0.018911208
22    0.072224362  0.111720664  0.151453525  0.037196924  0.056454036
23    0.156144582  0.301464055  0.347592855  0.118489569  0.14172118
24    0.145067996  0.130813527  0.172723386  0.023820757  0.039755214
25    0.093807699  0.143794739  0.19259298   0.055201999  0.060823237
26    0.333617401  0.497536795  0.470722398  0.114005013  0.093175468
27    0.480183121  0.686288454  1            0.269847482  1
28    0.205943089  0.33005981   0.542755242  0.841691944  0.861411585
29    0.237667984  0.491439657  0.626305528  0.064962619  0.279474307
30    0.128323912  0.236193554  0.200712427  0.044024999  0.211421744
31    0.15380434   0.232157284  0.244884947  0.038277132  0.131284417
32    0.185491009  0.456303306  0.358896705  0.074739709  0.209290977;
binary Variables
   Lambda(j)
   ;
Free Variable
   Teta
   z;
Equations
   Obj
   Outputs(r)
   Inputs(i)
   Convexity
;
Obj ..z = e = 1 + Teta;
Outputs(r) ..sum(j$K(j), Y(j,r)*Lambda(j))-Yprime(r)+teta = g = 0;
Inputs(i) ..sum(j$K(j), X(j,i)*Lambda(j))-Xprime(i)-Teta = l = 0;
Convexity .. sum(j$K(j), Lambda(j))=e = 1;
Model MMAJFDH /all/;
*________ F I L E D E C E L E R A T I O N ________
```

(*continued on next page*)

```
file My /"d:\res.txt"/;
put My;
My.nd = 10;
loop(t,
Xprime(i)=X(t,i);
Yprime(r)=Y(t,r);
Loop(j,
    k(j)=Yes;
    k(t)=No);
Solve MMAJFDH using MIP minimize Z;
put MMAJFDH.modelstat, "——————————————————————————————————————————"/;
put "DMU ", t.tl:3," = ", z.L:10 /;
);
putclose My;
```

Appendix B.

B.1. GAMS results

MAJ-FDH GAMS Results	Modified MAJ-FDH GAMS Results
1.0000000E+1	1.0000000000
DMU 1 = 0.0000	DMU 1 = 1.55000000
1.0000000000	1.0000000000
DMU 2 = 1.1930	DMU 2 = 1.06250000
1.0000000000	1.0000000000
DMU 3 = 1.0375	DMU 3 = 1.03750000
1.0000000000	1.0000000000
DMU 4 = 1.2093	DMU 4 = 1.08733443
1.0000000000	1.0000000000
DMU 5 = 1.9302	DMU 5 = 1.27666107
1.0000000000	1.0000000000
DMU 6 = 1.5439	DMU 6 = 1.16898006
1.0000000000	1.0000000000
DMU 7 = 1.1375	DMU 7 = 1.11820800
1.0000000000	1.0000000000
DMU 8 = 1.3000	DMU 8 = 1.06865484
1.0000000000	1.0000000000
DMU 9 = 1.2000	DMU 9 = 1.12727273
1.0000000000	1.0000000000
DMU 10 = 1.6909	DMU 10 = 1.20224083
1.0000000000	1.0000000000
DMU 11 = 1.0970	DMU 11 = 1.02288509
1.0000000000	1.0000000000
DMU 12 = 1.2105	DMU 12 = 1.07400536
1.0000000000	1.0000000000
DMU 13 = 1.2250	DMU 13 = 1.06436519
1.0000000000	1.0000000000
DMU 14 = 1.1053	DMU 14 = 1.06059726
1.0000000000	1.0000000000
DMU 15 = 1.0930	DMU 15 = 1.10647188
1.0000000E+1	1.0000000000
DMU 16 = 1.0930	DMU 16 = 1.50877193
1.0000000000	1.0000000000
DMU 17 = 1.1250	DMU 17 = 1.05438912
1.0000000000	1.0000000000
DMU 18 = 1.2375	DMU 18 = 1.10909091
1.0000000000	1.0000000000
DMU 19 = 1.0662	DMU 19 = 1.06160528
1.0000000000	1.0000000000
DMU 20 = 1.3953	DMU 20 = 1.03714689
1.0000000000	1.0000000000
DMU 21 = 1.2326	DMU 21 = 1.12727273
1.0000000000	1.0000000000
DMU 22 = 1.1930	DMU 22 = 1.07671887
1.0000000000	1.0000000000
DMU 23 = 1.3860	DMU 23 = 1.14377378
1.0000000000	1.0000000000
DMU 24 = 1.1579	DMU 24 = 1.06778980

B.1 (*continued*)

MAJ-FDH GAMS Results	Modified MAJ-FDH GAMS Results
1.0000000000	1.0000000000
DMU 25 = 1.2105	DMU 25 = 1.04113945
1.0000000000	1.0000000000
DMU 26 = 1.5464	DMU 26 = 1.10526316
1.0000000E+1	1.0000000000
DMU 27 = 1.5464	DMU 27 = 1.45724476
1.0000000E+1	1.0000000000
DMU 28 = 1.5464	DMU 28 = 1.77211213
1.0000000000	1.0000000000
DMU 29 = 1.6744	DMU 29 = 1.18604651
1.0000000000	1.0000000000
DMU 30 = 1.4386	DMU 30 = 1.10000000
1.0000000000	1.0000000000
DMU 31 = 1.2375	DMU 31 = 1.03571429
1.0000000000	1.0000000000
DMU 32 = 1.7500.	DMU 32 = 1.11508055

A 1.0000000E+1 value indicates infeasibility and a 1.0000000000 value indicates feasibility. MAJ-FDH GAMS Results show that DMU1, DMU16, DMU27, and DMU28 are infeasible. Modified MAJ-FDH GAMS Results show that all DMUs, including DMU1, DMU16, DMU27, and DMU28, are feasible.

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