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ORIGINAL ARTICLE

Efficiency measurement in data envelopment analysis in the presence of ordinal and interval data

Bohlool Ebrahimi[1] · Madjid Tavana[2,3] · Morteza Rahmani[1] · Francisco J. Santos-Arteaga[4,5]

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Abstract Several methods have been proposed in data envelopment analysis (DEA) for measuring efficiency in problems with interval or ordinal data. In this study, we review the weaknesses and drawbacks of these methods and show how converting ordinal or interval data into precise data can lead to violations of established DEA axioms. One of the axioms violated by these conversion processes is the inclusion of observations axiom, which requires a consistent definition of the production possibility set. We describe the special properties of ordinal and interval data together with their effect on the DEA-based rankings using a theorem and an example. We also propose a new algorithm and apply random dataset generation to overcome the problems arising from violations of the inclusion of observations axiom in DEA settings with ordinal or internal data. Several numerical examples are presented to demonstrate the applicability and exhibit the efficacy of the proposed method.

Keywords Data envelopment analysis · Efficiency · Ranking · Ordinal data · Interval data

✉ Madjid Tavana
tavana@lasalle.edu;
http://tavana.us/

Bohlool Ebrahimi
b_ebrahimi@jdsharif.ac.ir

Morteza Rahmani
rahmanimr@jdsharif.ac.ir

Francisco J. Santos-Arteaga
fsantosarteaga@unibz.it; fransant@ucm.es

1 Department of Industrial Engineering, Technology Development Institute (ACECR), Sharif University of Technology Branch, Tehran, Iran

2 Business Systems and Analytics Department, Distinguished Chair of Business Analytics, La Salle University, Philadelphia, PA 19141, USA

3 Business Information Systems Department, Faculty of Business Administration and Economics, University of Paderborn, 33098 Paderborn, Germany

4 School of Economics and Management, Free University of Bolzano, Bolzano, Italy

5 Instituto Complutense de Estudios Internacionales, Universidad Complutense de Madrid, Madrid, Spain

1 Introduction

Charnes et al. [2] proposed data envelopment analysis (DEA) to measure the efficiency of several similar decision-making units (DMUs) that use multiple inputs to produce multiple outputs. DEA assumes that the values of inputs and outputs are exactly known. However, in most real-life problems, the exact values of the inputs and outputs are imprecise because they either are not known or cannot be exactly measured. In response to this issue, Cooper et al. [4] proposed using bounded (interval) and weak ordinal data in an imprecise DEA (IDEA) model, which is nonlinear and non-convex.

Imprecise data can be of various types: ordinal (weak and strong), bounded (interval), ratio bound, multiplied order, and so on. In this paper, we focus on two types of imprecise data: weak ordinal and interval.

- Ordinal data express a special relation among the observations when their actual values are unknown. Consider, for example, the specific criteria that need to be evaluated in the supplier selection problem. One important criterion is supplier reputation, which can be measured in ordinal format. For example, for suppliers A and B, the decision maker may consider that the reputation of supplier A is higher than or equal to that of supplier B, which can be expressed as reputation(A) $\geq$ reputation(B).

This type of data has been used for the supplier selection problems studied in Farzipoor Saen [8], Toloo and Nalchigar [20], and Toloo [19] and for the evaluation of information technology (IT) projects described in Asosheh et al. [1]. These latter authors used the balanced scorecard to define the evaluation criteria of the IT projects and an integrated mixed integer IDEA model to obtain the most efficient project.

Similarly, Toloo and Nalchigar [20] proposed an integrated mixed integer IDEA model to find the best supplier, while Toloo [19] described some drawbacks of their model and suggested an improvement. It should be noted that both IDEA models handled the imprecise data using the approach proposed by Zhu [22]. The mathematical representation of weak ordinal data is defined as follows [22]:

$$\begin{aligned} &x_{i1} \leq x_{i2} \leq \cdots \leq x_{in} \quad (i \in OI) \\ &y_{r1} \leq y_{r2} \leq \cdots \leq y_{rn} \quad (r \in OO) \end{aligned} \tag{1}$$

where OI and OO are the associated sets including weak ordinal inputs and outputs, respectively.

- Interval data are an approach designed to treat vague or ambiguous observations, particularly the case where the exact value of the data is unavailable and lies within a bounded interval. Similar to ordinal data, interval data have also been used in many real-world applications such as evaluating a mobile telecommunication company [6] or the efficiency of different bank branches [10].

More recently, Karsak and Dursun [11] developed a supplier selection methodology by incorporating quality function deployment and DEA in the presence of imprecise data. Khalili-Damghani et al. [12] used a DEA model to evaluate the performance of a combined cycle power plant in the presence of undesirable outputs and uncertain data. They modeled the uncertain data using interval data.

The mathematical representation of interval data is defined as follows [22]:

$$\begin{aligned} &\underline{x}_{ij} \leq x_{ij} \leq \bar{x}_{ij} \quad (i \in II) \\ &\underline{y}_{rj} \leq y_{rj} \leq \bar{y}_{rj} \quad (r \in IO) \end{aligned} \tag{2}$$

where $\underline{y}_{rj}$ and $\bar{y}_{rj}$ are the lower and the upper bounds for outputs, $\underline{x}_{ij}$ and $\bar{x}_{ij}$ are the lower and the upper bounds for inputs, and II and IO are the associated sets including interval inputs and outputs, respectively.

Zhu [22, 23], Wang et al. [21], and Park [17] have proposed the most commonly used DEA models with ordinal and interval data. However, as will be illustrated through the paper, the implementation of these methods is subject to several potential drawbacks that can be classified into two main categories:

(a) Replacing ordinal data with specific values assigned ad hoc, such as zero and one, can lead to incorrect efficiency scores and unacceptable results. This problem is triggered by the fact that the probability of occurrence of the specific realizations assigned is zero in practice.

(b) Violating the inclusion of observations axiom implies that in some cases the production possibility set (PPS) cannot be constructed or the production frontier found. If this were the case, the correct values of the relative efficiencies cannot be computed.

In the current paper, we will show that treating the imprecise data as stochastic variables overcomes the above-mentioned drawbacks and provides more reasonable results. Indeed, the main contribution of this paper is twofold:

1. studying and describing the weaknesses and drawbacks of the existing methods used to calculate the relative efficiencies of DMUs in the presence of weak ordinal and interval data;
2. proposing a new method that eliminates the aforementioned drawbacks.

The remainder of this paper is organized as follows. Section 2 explains the existing methods and describes some of their drawbacks. Section 3 studies the properties of ordinal data and proposes a new approach to calculate the relative efficiencies that addresses the previous drawbacks. Section 4 presents a numerical example illustrating the ranking differences that arise among the different methods. Section 5 concludes.

2 Existing methods and their drawbacks

In this section, we describe some drawbacks of the existing models in the literature, which will be specifically addressed in Sect. 3. Throughout this section, we will make use of three examples with ordinal and interval data.

Cooper et al. [4] were the first authors to include interval and weak ordinal data in DEA. They named their model imprecise DEA (IDEA). The discussion in this section will depart from the seminal constant returns to scale model shown in (3)

$$\begin{aligned} \max \quad & \sum_{r=1}^{m} u_r y_{rp} \\ \text{s.t.} \quad & \\ & \sum_{i=1}^{n} v_i x_{ip} = 1 \\ & \sum_{r=1}^{m} u_r y_{rj} - \sum_{i=1}^{n} v_i x_{ij} \le 0, \; j = 1, \ldots, k \\ & x_{ij} \in \theta_i^+, \; y_{rj} \in \theta_r^- \\ & u_r, v_i \ge 0 \end{aligned} \tag{3}$$

where $x_{ij} \in \theta_i^+$ and $y_{rj} \in \theta_r^-$ represent subsets of the imprecise data presented in (1) and (2). Obviously, model (3) is nonlinear and non-convex. Most existing approaches have used model (3) as a starting point. They differ in the way they linearize and replace imprecise data (of the sort described in Examples 1, 2, and 3 below) with precise data. For example, Kim et al. [13] implemented an early variant of IDEA to evaluate the performance of telephone offices with partial data.

Cooper et al. [4] applied the unit-invariant property of DEA and converted the nonlinear model (3) into an equivalent linear model through scale transformation and variable alterations. Their method has three shortcomings:

(a) The high volume of calculations.
(b) The necessity of an exact maximum value for scale transformation in interval data.
(c) Only the upper bound efficiencies are calculated, while the lower bound ones are not considered.

Cooper et al. [5] addressed the second problem by introducing some dummy variables. Lee et al. [14] extended the IDEA concept to the additive DEA model. They used a simple variable alteration to convert the nonlinear IDEA model into an equivalent linear model. Similarly, Despotis and Smirlis [7] proposed a method to calculate the lower and upper bounds of the efficiency scores using an appropriate variable alteration. They developed two linear programming models to estimate the lower and upper bound efficiencies by considering a pessimistic and an optimistic state for each DMU. In fact, the efficiency score for each DMU is an interval in their method.

2.1 Zhu's [22, 23] method

Zhu [22] showed that the scale transformation (normalization of data) in the method of Cooper et al. [4–6] is redundant. He used a simple variable alteration to convert the nonlinear and non-convex IDEA model into an equivalent linear model. The method proposed by Zhu [22] reduced the high volume of calculations of the method of Cooper et al. [4]. Similarly, Park [16] reduced the volume of calculations required by Cooper et al. [5] using a simple variable alteration, which is the same as the variable alteration proposed by Zhu [22].

Zhu [23] converted weak ordinal data into bounded data and the latter into exact data. He then showed that the efficiency score for the exact data is equivalent to the result of solving the nonlinear IDEA model. Zhu [23] used the method for performance evaluation in the Korean mobile telecommunication company analyzed by Cooper et al. [6].

Zhu [22] showed that model (3) can be converted into an equivalent linear model with the following simple variable alterations:

$$\begin{aligned} X_{ij} &= v_i x_{ij} \quad \forall i, j \\ Y_{rj} &= u_r y_{rj} \quad \forall r, j \end{aligned} \tag{4}$$

In fact, Zhu [22, 23] ranked the DMUs based only on their upper bound efficiencies. Some existing methods such as those of Cooper et al. [4–6] and Park [16] also ranked the DMUs based on their upper bound efficiencies, which can yield unacceptable results. To clarify this statement, consider the following simple example.

Example 1 Consider the two DMUs described in Table 1, each of which uses one input to produce one ordinal output.

The Zhu [22] approach (variable alteration (4)) implies that both these DMUs are efficient, which is unacceptable. This is the case since, as can be easily seen, DMU_2 dominates DMU_1 and it is only in one special situation, i.e., $x_{11} = x_{12}$ and $y_{12} = y_{11}$, which occurs with zero probability, when these DMUs are both efficient.

Converting ordinal data into exact data Suppose that DMU_p is under evaluation, model (3) has been solved and the following optimal solution has been obtained for the ordinal data:

$$\begin{aligned} & x_{i1}^* \le x_{i2}^* \le \cdots \le x_{i,p-1}^* \le x_{ip}^* \le x_{i,p+1}^* \le \cdots \le x_{in}^* \\ & y_{r1}^* \le y_{r2}^* \le \cdots \le y_{r,p-1}^* \le y_{rp}^* \le y_{r,p+1}^* \le \cdots \le y_{rn}^* \end{aligned} \tag{5}$$

In this case, by considering the unit-invariant property of DEA, $\rho_i x_{ij}^*$ and $\rho_r y_{rj}^*$ are also optimal solutions $(\rho_i, \rho_r > 0, \forall i, r)$. Thus, we can assume $y_{rp}^* = x_{ip}^* = 1$, and so the optimal solution can be expressed as follows:

Table 1 Input–output data for Example 1 with two DMUs

DMU	Input	Output[a]
1	[1 1000]	y_{11}
2	1	y_{12}

[a] Where $y_{12} \ge y_{11}$

$$0 \le x^*_{i1} \le x^*_{i2} \le \cdots \le x^*_{i,p-1} \le x^*_{ip} = 1 \le x^*_{i,p+1} \le \cdots \le x^*_{in} \le M$$
$$0 \le y^*_{r1} \le y^*_{r2} \le \cdots \le y^*_{r,p-1} \le y^*_{rp} = 1 \le y^*_{r,p+1} \le \cdots \le y^*_{rn} \le M \qquad (6)$$

M is a positive large number [22]. In particular, Zhu [23] proposed that M could be the number of DMUs. Using the relation described in (6), Zhu [22, 23] converted the ordinal data into the following interval data (note that DMU_p is under evaluation):

$$x_{ij} \in [0,1] \;\&\; y_{rj} \in [0,1] \text{ for } DMU_j,\ \forall j \in \{1,2,\ldots,p-1\}$$
$$x_{ij} \in [1,M] \;\&\; y_{rj} \in [1,M] \text{ for } DMU_j,\ \forall j \in \{p+1,\ldots,k\} \qquad (7)$$

Then, the interval data are converted into the following exact data:

$$x_{ij} = 1,\ \forall j \le p \;\&\; x_{ij} = M,\ \forall j \ge p+1$$
$$y_{rj} = 0,\ \forall j \le p-1 \;\&\; y_{rj} = 1,\ \forall j \ge p \qquad (8)$$

Indeed, when DMU_p is under evaluation, its ordinal inputs and ordinal outputs take the value of one, and those of the other DMUs take their worst possible values. Thus, the efficiency score of DMU_p will depend on the value of M. In other words, by increasing the value of M, the efficiency score of DMU_p will be increased and vice versa. In the following example, it will be shown that the efficiency scores are dependent on the value of M. Also, it will be shown that the results of the two methods proposed by Zhu [22, 23], i.e., the variable alteration method and the conversion of imprecise data into precise one, differ.

Example 2 Consider the three DMUs described in Table 2, each of which uses one ordinal input to produce one precise output.

Using the variable alteration method described in (4), we conclude that DMU_1 is efficient. Consider now the second approach when DMU_1 is under evaluation. Zhu [22, 23] uses the exact data presented in Table 3 to evaluate DMU_1.

Table 2 Input–output data for Example 2 with three DMUs

DMU	Input (ordinal)[a]	Output (exact)
1	x_{11}	2
2	x_{12}	3
3	x_{13}	12

[a] Ranking such that $x_{11} \le x_{12} \le x_{13}$

Table 3 Precise input–output data for Example 2 according to Zhu [22, 23]

DMU	Input (ordinal)	Output (exact)
1	1	2
2	M	3
3	M	12

The data presented in Table 3 show that DMU_1 is efficient for $M \ge 6$ and inefficient for $M < 6$. Indeed, if we set $M = 3$, then DMU_1 will be inefficient based on the approach of Zhu [23], which is inconsistent with the result derived from the variable alteration method. In particular, the efficiency score of DMU_1 will vary in $\left[\frac{1}{6}, 1\right]$ when M varies in the interval $[1, +\infty)$. This example shows that the efficiency scores are dependent on the value of M.

Overall, the weaknesses and drawbacks of the approach of Zhu [22, 23] can be summarized as follows:

(a) The approach does not consider the lower bound efficiencies and ranks the DMUs based only on the upper bound ones. As a result, Example 1 has shown that this approach may yield unacceptable results.
(b) The efficiencies obtained are dependent on the values of M.
(c) Considering only the values of 0, 1, and M to generate exact data from the ordinal ones gives place to a substantial amount of zero output values being produced using (strictly) positive inputs. Moreover, the probability of occurrence of these specific data is zero in practice.
(d) As illustrated in Example 2, the results obtained using the variable alteration approach and those following from the conversion of imprecise data into precise one differ.

2.2 Wang et al. [21] method

Wang et al. [21] showed that Despotis and Smirlis [7] used different production frontiers to calculate the efficiencies. They claimed that a unique production frontier should be used to evaluate all of the DMUs. This frontier is obtained by considering all the DMUs in the best potential situation. Then, they defined the two mathematical programming models described in [9] to obtain the lower and upper bound efficiencies (in the presence of interval data) by considering a unique (fixed) production frontier for all the DMUs. Finally, they proposed a minimax regret-based method to rank the interval efficiencies obtained.

The upper bound efficiency for DMU_p

$$\begin{aligned} \text{Max}\quad & \theta_p^U = \frac{\sum_{r=1}^{s} u_r y_{rp}^U}{\sum_{i=1}^{m} v_i x_{ip}^L} \\ \text{s.t.}\quad & \theta_j^U = \frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i x_{ij}^L} \le 1, \quad j = 1, 2, \ldots, n \\ & u_r, v_i \ge \varepsilon \quad \forall i, r \end{aligned}$$

The lower bound efficiency for DMU_p

$$\begin{aligned} \text{Max}\quad & \theta_p^L = \frac{\sum_{r=1}^{s} u_r y_{rp}^L}{\sum_{i=1}^{m} v_i x_{ip}^U} \\ \text{s.t.}\quad & \theta_j^U = \frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i x_{ij}^L} \le 1, \quad j = 1, 2, \ldots, n \\ & u_r, v_i \ge \varepsilon \quad \forall i, r \end{aligned} \tag{9}$$

Obviously, these models can be converted into linear models.

In order to adapt these models to deal with ordinal data, the authors obtained the relation described in (10) by applying the following scale transformation to the output variables (it should be noted that an equivalent transformation can be applied to the input variables):

$$1 \ge \hat{y}_{r1} \ge \hat{y}_{r2} \ge \cdots \ge \hat{y}_{rn} \ge \sigma_r \tag{10}$$

where σ_r is a small positive number, reflecting the ratio of the possible minimum of $\{y_{rj} | j = 1, 2, \ldots, n\}$ to its possible maximum, that should be estimated by the decision maker. Then, the ordinal data are converted into interval data as follows:

$$\hat{y}_{rj} \in [\sigma_r, 1], \ \forall j \tag{11}$$

That is, in the method of Wang et al. [21] the relations among ordinal data are removed, and the ordinal data being considered are regarded as equal, which may lead to incorrect results. To illustrate this problem, consider the following example.

Example 3 Consider the two DMUs described in Table 4, each of which uses one ordinal input to produce one ordinal output.

Suppose that the values of σ_1 for the ordinal input and output estimated by the decision maker are given by 0.1 and 0.05, respectively. In this case, Wang et al. [21] consider the data presented in Table 5 to evaluate these two DMUs.

Obviously, these data imply that the two DMUs are always the same and have the same rank, which is unacceptable given the fact that DMU_2 dominates DMU_1. It should be noted that none of the approaches proposed in Cooper et al. [4–6], Zhu [22, 23], or Park [16] are able to rank these two DMUs correctly. In other words, all these approaches conclude that both DMUs are efficient.

Table 4 Input–output data for Example 3 with two DMUs

DMU	Input 1[a]	Output 1[a]
1	x_{11}	y_{11}
2	x_{12}	y_{12}

[a] Where $x_{11} \ge x_{12}$ and $y_{12} \ge y_{11}$

2.3 Park [17] method

Park [17] applied the concepts of supremum and infimum to propose the mathematical programming model described in Eq. (12) for calculating the lower bound efficiencies of the DMUs. He used different production frontiers to calculate the relative efficiencies and categorized the DMUs into three groups: perfectly efficient, potentially efficient, and inefficient, in a similar way to the Despotis and Smirlis [7] method.

$$\begin{aligned} \max\quad & \sum_{r=1}^{m} u_r \inf\{y_{rp} | y_r \in D_r^+\} \\ \text{s.t.}\quad & \\ & \sum_{i=1}^{n} v_i \sup\{x_{ip} | x_i \in D_i^-\} = 1 \\ & \sum_{r=1}^{m} u_r \inf\{y_{rp} | y_r \in D_r^+\} - \sum_{i=1}^{n} v_i \sup\{x_{ip} | x_i \in D_i^-\} \le 0 \\ & \sum_{r=1}^{m} u_r \sup\{y_{rj} | y_r \in D_r^+\} - \sum_{i=1}^{n} v_i \inf\{x_{ij} | x_i \in D_i^-\} \le 0, \\ & j = 1, \ldots, k, \ j \ne p \\ & u_r, v_i \ge \varepsilon, \quad \forall r, i \end{aligned} \tag{12}$$

In this model, the D_i^- and D_r^+ sets represent the imprecise data. It should be noted that inf and sup can be replaced with min and max, respectively. An important drawback of model (12) is that feasibility is not considered in the calculation of the inf and sup. To illustrate this problem, consider the numerical example described in Park [17], where 8 telephone offices with three inputs and three outputs are analyzed. The third input is defined in ordinal format as follows:

$$D_3^- = \{x_3 \in R^8 | \ x_{34} \ge x_{35} \ge x_{33} \ge x_{37} \ge x_{31} \ge x_{36} \ge x_{32} \ge x_{38}\} \tag{13}$$

Table 5 Precise input–output data for Example 3 according to Wang et al. [21]

DMU	Input 1	Output 1
1	[0.1, 1]	[0.05, 1]
2	[0.1, 1]	[0.05, 1]

According to the method of Park [17], after normalization we have:

$$D_3'^{-} = \{x_3' \in R^8 \,|\, 1 \geq x_{34}' \geq x_{35}' \geq x_{33}' \geq x_{37}' \geq x_{31}' \geq x_{36}' \geq x_{32}' \geq x_{38}' \geq 0\} \quad (14)$$

When evaluating the lower bound efficiency of DMU_1, the corresponding data for DMU_1 and the other DMUs can be computed as follows:

$$\sup\{x_{31}' \,|\, x_3' \in D_3'^{-}\} = \max\{x_{31}' \,|\, x_3' \in D_3'^{-}\} = 1$$
$$\inf\{x_{3j}' \,|\, x_3' \in D_3'^{-}\} = \min\{x_{3j}' \,|\, x_3' \in D_3'^{-}\} = 0; \quad j = 2, 3, \ldots, 8 \quad (15)$$

Therefore, Park [17] used $x_3^* = (1, 0, 0, 0, 0, 0, 0, 0)$ to calculate the lower bound efficiency score of DMU_1, which is an infeasible solution. As mentioned above, the feasibility condition should be considered in the calculation of *inf* and *sup*. Applying the feasibility condition implies that Park should have used $x_3^* = (1, 0, 1, 1, 1, 0, 1, 0)$. Furthermore, similarly to the method of Kao [9], the method of Park [17] assigns only zeros and ones to all the ordinal data. As already stated, the probability of occurrence of these observations is equal to zero in practice.

The next example shows that in some cases the lower bound efficiencies cannot be calculated using the method of Park [17].

Example 4 Consider the data presented in Table 2. Given these ordinal data, the method of Park [17] assigns only zeros and ones to the corresponding observations, leading to the evaluation framework described in Table 6.

As can be seen from Table 6, some of the DMUs produce output without consuming any input. As a result, the efficiencies of several DMUs cannot be calculated using these data, which prevents us from obtaining a ranking of the alternatives.

2.4 Other methods

Within the set of recent alternative methods developed in the literature, we concentrate on those bearing some resemblance with the ones described throughout this section.

Kao [9] emphasized the fact that the efficiency scores should be imprecise in the presence of imprecise data. He proposed two mathematical programming models to calculate the lower and upper bound efficiencies in the presence of ordinal and bounded data. Similarly to the method of Despotis and Smirlis [7], his method uses different production frontiers for each DMU. In order to account for ordinal data, he set a lower and an upper bound for each ordinal observation after normalization.

Kao and Liu [10] argued that if the domain on the interval data is wide, then the interval efficiencies obtained using the previous methods are too wide to provide enough valuable information for a decision maker to make an accurate (good) decision. Therefore, they considered interval data as stochastic and estimated the distribution of the efficiency score for each DMU using a simulation method. They showed that this approach gives more reliable results than the interval data approaches available. Their method was used to rank Taiwan commercial banks by obtaining the distribution of their efficiencies.

Park [18] investigated the dual model of IDEA and its relationship with the primal problem in order to develop a computational method for it. Marbini et al. [15] studied performance evaluation in the presence of interval data without sign restrictions. They calculated the lower and upper bound efficiencies and categorized DMUs in three groups: strictly efficient, weakly efficient, and inefficient. Chen et al. [3] presented some models to account for discrete and bounded, and Likert scale data in DEA. They used the DEA models developed to evaluate regional energy efficiency in China.

In summary, after reviewing the IDEA literature and given the results of the numerical examples presented, the following drawbacks of the models described should be highlighted since they give place to potential improvements that can be implemented in this type of evaluation framework:

Table 6 Ordinal data used for lower bound efficiency in Park [17]

The values of variables	The DMU under evaluation					
	Without considering the feasibility			Considering the feasibility		
	DMU_1	DMU_2	DMU_3	DMU_1	DMU_2	DMU_3
$x_{11}'^*$	1	0	0	1	0	0
$x_{12}'^*$	0	1	0	1	1	0
$x_{13}'^*$	0	0	1	1	1	1
Lower bound efficiencies	Cannot be calculated	Cannot be calculated	Cannot be calculated	0.17	Cannot be calculated	Cannot be calculated

(a) Ranking the DMUs based only on the upper bound efficiencies, which can lead to unacceptable results [4, 5, 6, 22, 23, 16].
(b) Considering an upper bound for ordinal data such that the efficiencies obtained depend on the value of the upper bound chosen [22, 23].
(c) Converting ordinal data into interval data that yield unacceptable results [21, 22, 23].
(d) Replacing ordinal data with zero, one, and M values that lead to incorrect results [22, 23]. Also, in some cases, we may be unable to calculate the relative efficiencies [17].
(e) Replacing imprecise data with precise data that have a zero probability of occurrence in practice [9, 17, 22, 23].
(f) Using infeasible precise data instead of ordinal data [17].
(g) Violating the inclusion of observations axiom (we will focus on this problem in the next section).

The next section illustrates formally the main source of these drawbacks and proposes a new approach to eliminate these problems.

3 The proposed method for ranking DMUs with ordinal data

In this section, we start by analyzing the basic properties of ordinal data using a numerical example and a theorem. Then, the cause of the occurrence of the drawbacks described in the previous section will be presented. Finally, a new ranking method will be developed to overcome these drawbacks.

3.1 Properties of ordinal data

Direct solving of model (3) in the presence of ordinal data, as in Cooper et al. [4–6], Zhu [22, 23], and Park [16], allocates unacceptable (very large or small) numerical values to the ordinal data. To illustrate the problem, consider the following example.

Example 5 Consider the two DMUs described in Table 7, each using one precise input to produce one ordinal output.

Table 7 Input–output data for Example 5 with two DMUs

DMU	Input 1[a]	Output 1[a]
1	1	y_{11}
2	1000	y_{12}

[a] Where $y_{12} \geq y_{11}$

The efficiency score of DMU_2 is $\frac{y_{12}/1000}{\text{Max}\{y_{11}, y_{12}/1000\}}$. Thus, this DMU will be efficient if and only if $y_{12} \geq 1000 y_{11}$. As can be easily seen, maximizing the relative efficiency of DMU_2 implies that its ordinal output must take a very large value for this unit to be efficient. Indeed, this DMU will be efficient for any given input data of the DMUs. The following theorem is motivated by this example.

Theorem 1 *Assume that in model (3) the dth output of the DMUs is given in ordinal format and that DMU_p has the best rank. In other words, assume that $y_{dp} \geq y_{dj}$, $\forall j$. Then, DMU_p is always efficient, i.e., the upper bound efficiency score of DMU_p is equal to unity.*

Proof Intuitively, when calculating the relative efficiency score of DMU_p, model (3) could select y_{dp} to be a large enough positive number and y_{dj}, $\forall j \neq p$, relatively small numbers such that DMU_p dominates all the other DMU*s*. Note that a similar theorem can be defined for ordinal input data.

A formal proof of the theorem can be given as follows: The max–min model employed to calculate the relative efficiency of DMU_p is given by:

$$\text{Relative efficiency of DMU}_p = \text{Max}_{u,v \geq 0}\left\{\frac{\sum_r u_r y_{rp}}{\sum_i v_i x_{ip}} \middle/ \text{Max}_j\left\{\frac{\sum_r u_r y_{rj}}{\sum_i v_i x_{ij}}\right\}\right\} \tag{16}$$

Set $u_r = v_i = 1,\ \forall i, r$. Then, we must have:

$$\text{Relative efficiency of DMU}_p \geq \text{Max}\left\{\frac{\sum_r y_{rp}}{\sum_i x_{ip}} \middle/ \text{Max}_j\left\{\frac{\sum_r y_{rj}}{\sum_i x_{ij}}\right\}\right\} \tag{17}$$

It is now enough to set $y_{dj} = 1,\ \forall j \neq p$, and $y_{dp} = M_1$, where M_1 is a large positive number such that the following relation is satisfied:

$$\frac{M_1 + \sum_{\forall r \neq d} y_{rp}}{\sum_i x_{ip}} \geq \frac{\sum_r y_{rj}}{\sum_i x_{ij}},\ \forall j \neq p \tag{18}$$

The relation described in Eq. (18) implies that the efficiency score of DMU_p is greater than or equal to one. It should be noted that, according to the definition of efficiency in the DEA literature, DMU_p is efficient if and only if there exists at least a common set of weights, $u^* > 0, v^* > 0$, such that $\frac{\sum_r u_r^* y_{rp}}{\sum_i v_i^* x_{ip}} \geq \frac{\sum_r u_r^* y_{rj}}{\sum_i v_i^* x_{ij}},\ \forall j$. This completes the proof. □

Obviously, even though this result (a DMU with the best rank in an ordinal input or output will always be efficient) may not be reasonable in practice, it is formally correct.

3.2 Proposed approach

The standard CCR model and its corresponding PPS are both based on a set of basic axioms. Assume that we have k

DMUs (DMU_j, $j = 1, 2, 3, \ldots, k$) with n inputs ($x_{ij} \geq 0$, $i = 1, 2, 3, \ldots, n$; $j = 1, 2, 3, \ldots, k$) and m outputs (y_{rj}, $r = 1, 2, 3, \ldots, m$; $j = 1, 2, 3, \ldots, k$) such that at least one input and one output are different from zero for each DMU. The axioms on which the standard CCR framework is built are:

1. Inclusion of observations: each observed DMU_j ($j = 1,2,\ldots, k$) belongs to T, with T denoting the PPS.
2. Free disposability of inputs and outputs: if $(x, y) \in T$, $y' \leq y$, then $(x, y') \in T$; similarly, if $(x, y) \in T$, $x' \geq x$, then $(x', y) \in T$.
3. Convexity: if $(x, y) \,\&\, (x', y') \in T \Rightarrow \lambda(x, y) + (1 - \lambda)(x', y') \in T$, $\forall 0 \leq \lambda \leq 1$.
4. Constant returns to scale: if $(x, y) \in T \Rightarrow (\lambda x, \lambda y) \in T$, $\forall \lambda \geq 0$.
5. Minimum extrapolation: T is the intersection of all the sets satisfying 1–4.

Note that when the data are given in ordinal or bounded format, there are actually no data being observed, which implies that the first axiom (inclusion of observations) cannot be verified. In this situation, we cannot build the PPS and the verifiability problem extends through the remaining axioms. To clarify the topic, consider the data presented in Table 4. For these data, the PPS is $\left\{(x, y) \in R^2 : \; x, y \geq 0\right\}$, which is illustrated in Fig. 1.

It can be immediately seen that DMUs can produce output without consuming any input. In other words, $(0\,,\, y)\,, \forall y \geq 0$, belongs to the PPS. This contradiction highlights the fact that observing the data from the different DMUs is a necessary condition and that the PPS cannot be built correctly without the first axiom. Also, the production frontier for this PPS is the line between $(0\,,\, 0)$ and $(0\,,\, +\infty)$, a result that is not acceptable. This problem exists in all of the DEA methods described in the previous section, since all of them tried to construct the PPS without observed data. Therefore, using the unit-invariant property of DEA, the ordinal data were replaced with precise numbers (zero, one, or M) chosen ad hoc.

We introduce now a new method based on simulation (data generation) to rank DMUs in the presence of imprecise (ordinal and interval) data. The values of the ordinal data are unknown, and only a relation is established among them. In order to account for the inclusion of observations axiom, the imprecise data can be regarded as stochastic realizations. Since the probability distribution of these data is unknown, we will assume that they are drawn from a uniform density endowed with the highest information entropy. Consequently, the steps of the proposed algorithm are defined as follows:

1. Generating a uniformly distributed set of data (considering the relations defined among the ordinal data) through N iterations.
   Let X_i and Y_i, $i = 1, 2, \ldots, N$, be matrices of input and output data, respectively. The elements composing these matrices will be independently generated using uniform distributions so as to obtain a total of N matrix pairs, (X_1, Y_1), $(X_2, Y_2), \ldots, (X_N, Y_N)$. We apply the following process to generate the stochastic data that will be used in place of the ordinal data.
   Consider, for example, the set of ordinal input data defined in Eq. (1). First, a random real number is generated for the highest ranked input, x_{in}, using an uniform distribution defined within the [0, 1] domain. Then, a random real number is generated for $x_{i,n-1}$ within the [0, x_{in}] domain. The values of the remaining stochastic data are generated using the same process so as to preserve the ordinal relation $x_{i1} \leq x_{i2} \leq \cdots \leq x_{in}$. It should be noted that stochastic data are generated from interval data by defining uniform distributions within the domains limited by their corresponding lower and upper bounds.
2. Calculating efficiencies for all the DMUs. That is, given the N precise datasets generated in the previous step, N efficiency scores are obtained for each DMU by solving the standard DEA model.
3. Calculating the average efficiency score and the standard deviation of the efficiencies for each DMU. Note that the average efficiency is an estimation of the expected value of the efficiencies.
4. Ranking the DMUs based on the average efficiency scores and the standard deviation of the efficiencies as follows: DMU_1 dominates DMU_2 if and only if the average efficiency score of DMU_1 is greater than that of DMU_2. Moreover, if two DMUs have the same average efficiency score, the DMU with less standard deviation has better rank.

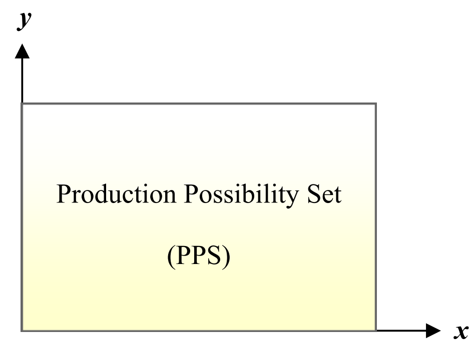

Fig. 1 PPS for Example 3 data

Obviously, increasing the number of realizations, N, yields a more exact average efficiency. If we were able to extract the efficiency distribution, we could calculate the expected value of the relative efficiency for each DMU.

However, given the complexities involved in the extraction of the efficiency distribution [10], and the fact that the average efficiency is an estimation of the expected relative efficiency—so these values will be approximately equal as the value of N increases—we will compute and compare different rankings obtained for increasing numbers of realizations.

In order to illustrate the improvements from implementing the suggested approach, consider the data presented in Example 3. The methods described in the previous section are unable to rank these DMUs correctly. In this regard, Table 8 presents the results obtained after applying the proposed algorithm for different values of N. As expected, DMU_2 is always efficient. Note also that DMU_1 is inefficient and its average efficiency score converges to 0.25.

In more detail, the steps of the algorithm applied to the example described in Table 4 and whose results are presented in Table 8 are the following ones:

Step 1 Let $X_i = \begin{bmatrix} x_{11}^i \\ x_{12}^i \end{bmatrix}$ and $Y_i = \begin{bmatrix} y_{11}^i \\ y_{12}^i \end{bmatrix}$, $i = 1, 2, \ldots, N$, be the matrices of input and output data, where x_{11}^i, x_{12}^i, y_{11}^i, and y_{12}^i are the precise numbers generated using uniform distributions as follows.

First, we generate x_{11}^i, $\forall i$, in the interval [0, 1]. Then, the value of x_{12}^i is generated in the interval $[0, x_{11}^i]$ in order to preserve the relation $x_{11} \geq x_{12}$. The values of $Y_i = \begin{bmatrix} y_{11}^i \\ y_{12}^i \end{bmatrix}$ are generated in the same way.

For example, assume that $N = 3$. The first step of the algorithm delivers the following data

$$X_1 = \begin{bmatrix} 0.6 \\ 0.2 \end{bmatrix}, \; X_2 = \begin{bmatrix} 0.8 \\ 0.5 \end{bmatrix}, \; X_3 = \begin{bmatrix} 0.4 \\ 0.1 \end{bmatrix};$$

$$Y_1 = \begin{bmatrix} 0.5 \\ 0.7 \end{bmatrix}, \; Y_2 = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix}, \; Y_3 = \begin{bmatrix} 0.6 \\ 0.9 \end{bmatrix}$$

Step 2 Three efficiency scores are computed for each DMU using the above data:

DMUs	Input	Output	Efficiency scores
1	0.6	0.5	0.238
2	0.2	0.7	1

DMUs	Input	Output	Efficiency scores
1	0.8	0.2	0.25
2	0.5	0.5	1

DMUs	Input	Output	Efficiency scores
1	0.4	0.6	0.167
2	0.1	0.9	1

Step 3 The average efficiency score and the standard deviation of the efficiencies are calculated for each DMU.

N	Average efficiency scores		Standard deviation	
	DMU_1	DMU_2	DMU_1	DMU_2
3	0.218	1	0.037	0

Step 4 The DMUs are ranked based on their average efficiency scores and the standard deviations of their efficiencies.

DMUs	Rank
1	2
2	1

Remark The average values of the X_i and Y_i, $i = 1, 2, 3$, data defined in Step 1 are

$$\bar{X} = \begin{bmatrix} 0.6 \\ 0.27 \end{bmatrix}, \; \bar{Y} = \begin{bmatrix} 0.43 \\ 0.7 \end{bmatrix}$$

DMUs	Input	Output	Efficiency scores
1	0.6	0.43	0.276
2	0.27	0.7	1

Using these averaged data, the resulting efficiency scores of the DMUs are given by

Note that, in this latter case, only two DEA models have been solved to estimate the efficiency scores of the DMUs, while our proposed algorithm requires solving six DEA

Table 8 Results of the proposed algorithm for the data presented in Table 4

N	Average efficiency score	
	DMU_1	DMU_2
100	0.2686	1
1000	0.2543	1
10,000	0.2506	1
100,000	0.2505	1
1,000,000	0.2500	1

models to estimate the expected efficiencies. Conventional methods tend to use the average of imprecise data to calculate the efficiency scores, which, as can be seen from the previous numerical example, differ from those of our algorithm. In other words, the efficiency scores obtained from the averaged data are different from the average of the efficiency scores, though both these scores converge for large values of N ($N \geq 5000$) in the numerical example presented in Table 8.

We must emphasize that the unit-invariant property of DEA guarantees that the normalization of the input and output data implemented does not affect the efficiency scores of the DMUs. For this reason, we have generated the random data for the ordinal variables in the interval [0, 1]. That is, we could have used the interval $[0, b]$, b given by an arbitrary positive real number, and the efficiency and ranking results obtained would have remain unchanged.

Final Remark In most of the methods described previously, such as those of Despotis and Smirlis [7] and Park [17], DMUs are categorized into three groups as follows:

(a) Strong efficient: the lower bound of the efficiency score is equal to one.
(b) Potentially efficient: the upper bound of the efficiency score is equal to one, but the lower bound efficiency is less than one.
(c) Inefficient: the upper bound of the efficiency score is less than one.

Based on this categorization, a DMU_1 with an interval efficiency of [0.1, 1] has a better rank than a DMU_2 with an interval efficiency of [0.95, 0.99]. Note that DMU_1 is efficient in its best potential situation. However, the lower bound of its efficiency is 0.1, while that of DMU_2 is 0.95. Thus, if we assume a uniform distribution of efficiencies, the expected efficiencies of DMU_1 and DMU_2 will be equal to 0.55 and 0.97, respectively. It therefore follows that DMU_2 is more reliable than DMU_1. This example has been introduced to emphasize the fact that considering the distribution of efficiencies and the corresponding expected efficiencies provides more reasonable results.

4 Numerical example

In this section, we apply the methods overviewed in Sect. 2 and the approach proposed in Sect. 3 to measure the efficiency of the five DMUs studied in Cooper et al. [4]. These DMUs have two inputs (one exact and one interval) and two outputs (one exact and one ordinal), whose values are presented in Table 9.

Table 10 summarizes the efficiency scores derived from the different methods described in Sect. 2 for $\varepsilon = 10^{-6}$, while the corresponding rankings are presented in Table 11. As shown in Tables 10 and 11, different rankings are obtained for these five DMUs. As emphasized previously, these methods differ in the values of the ordinal and interval data considered and follow different approaches to calculate the relative efficiencies.

For example, it should be remarked that the methods presented in Cooper et al. [4–6] and Zhu [22] calculate only the upper bound efficiencies. In this regard, by considering $M = 5$ in the method of Zhu [22, 23], we obtain efficiency scores for the five DMUs that are lower than or equal to their upper bound efficiencies. Note also the different relative rankings obtained for DMU_3 and DMU_4 when comparing the method of Wang et al. [21] with all the other ones.

In order to apply the method proposed in Sect. 3, we generated $N = 1$, 10, 100, 1000, 5000 and 10,000 random observations for ordinal and interval data and computed both the average efficiency score and the corresponding standard deviation for all five DMUs. The results obtained are summarized in Table 12. Note that the variations of the average efficiencies and the standard deviations decrease as the value of N increases. Indeed, the numerical results show that generating $N \geq 1000$ random observations yields a sufficiently reliable ranking. That is, the rankings obtained based on 5000 and 10,000 observations are the same as the one obtained for $N = 1000$.

5 Conclusion

DEA has been widely used for performance evaluation in many real-world problems. In most of these problems, the values of inputs and outputs are imprecise. As a result, researchers have developed different approaches to deal with imprecise data in DEA. We have briefly reviewed several approaches designed to deal with weak ordinal and interval data and described some of their main drawbacks, ranging from unacceptable results to infeasibility problems.

The importance of the inclusion of observations axiom has been emphasized, particularly the fact that this axiom

Table 9 Input–output values for the five DMU case [4]

DMUs	Inputs		Outputs	
	x_{1j} (exact)	x_{2j} (interval)	y_{1j} (exact)	y_{2j} (ordinal)[a]
1	100	[0.6, 0.7]	2000	4
2	150	[0.8, 0.9]	1000	2
3	150	1	1200	5
4	200	[0.7, 0.8]	900	1
5	200	1	600	3

[a] Ranking such that: $y_{23} \geq y_{21} \geq y_{25} \geq y_{22} \geq y_{24}$

Table 10 Efficiency scores for the five DMUs calculated using different methods

DMUs	Efficiency scores					
	Cooper et al. [4–6]	Zhu [22] first approach	Zhu [22] second approach[a]	Wang et al. [21][b]	Kao [9]	Park [17]
1	1	1	1	[0.99999, 1]	[1, 1]	[1, 1]
2	0.87499	0.87499	0.87397	[0.33333, 0.74899]	[0.66566, 0.87397]	[0.33333, 0.87499]
3	1	1	1	[0.39999, 0.66587]	[1, 1]	[0.4, 1]
4	0.99999	0.99999	0.99880	[0.33750, 0.85597]	[0.74885, 0.99880]	[0.33748, 0.99999]
5	0.69999	0.69999	0.69856	[0.17999, 0.59858]	[0.59858, 0.69856]	[0.17999, 0.69999]

[a] $M = 5$ according to Zhu [23]

[b] Assuming $\sigma_1 = 0.1$

Table 11 DMU rankings according to different approaches

DMUs	Ranking					
	Cooper et al. [4–6]	Zhu [22]	Wang et al. [21]	Kao [9]	Park [17]	Proposed algorithm
1	1	1	1	1	1	1
2	3	3	3	3	3	3
3	1	1	4	1	2	2
4	2	2	2	2	3	4
5	4	4	5	4	3	5

Table 12 Results of the proposed algorithm for the data presented in Table 9

N	Average efficiency scores and standard deviation									
	DMU_1		DMU_2		DMU_3		DMU_4		DMU_5	
	AES[a]	SD[a]	AES	SD	AES	SD	AES	SD	AES	SD
1	1	0	0.4179	0	1	0	0.3635	0	0.4519	0
10	1	0	0.4014	0.0787	0.8922	0.1409	0.3837	0.0245	0.3478	0.1325
100	1	0	0.3918	0.0338	0.9351	0.1023	0.3939	0.0357	0.2846	0.1234
1000	1	0	0.3952	0.0478	0.9414	0.1014	0.3939	0.0305	0.2947	0.1253
5000	1	0	0.3944	0.0448	0.9403	0.1020	0.3937	0.0328	0.2945	0.1253
10,000	1	0	0.3950	0.0495	0.9401	0.1034	0.3933	0.0417	0.2941	0.1236

[a] AES is average efficiency score, and SD is standard deviation

cannot be verified when data are imprecise. Such a restriction prevents the correct definition of the PPS, which implies that the production frontier does not exist in some cases. One of our main contributions has consisted of showing that if a DMU has the best rank in an ordinal input or output, then it will always be efficient.

In order to overcome the drawbacks described and account for the inclusion of observations axiom, we have proposed a novel algorithm based on data generation that considers the average and the standard deviation of the efficiency scores obtained to rank the DMUs. A limitation of the proposed algorithm can be observed in the numerical example presented in Table 12, namely the variability of the rankings in the initial evaluation stages. In this regard, given the large number of iterations required to obtain a stable ranking, the convergence requirements of the model may constitute a computational burden, particularly when considering multiple ordinal or interval variables.

We conclude by emphasizing the fact that the proposed algorithm can be used with other types of imprecise data, such as ratio bound.

Compliance with ethical standards

Conflict of interest None.

References

1. Asosheh A, Nalchigar S, Jamporazmey M (2010) Information technology project evaluation: an integrated data envelopment analysis and balanced scorecard approach. Expert Syst Appl 37:5931–5938
2. Charnes A, Cooper WW, Rhodes E (1978) Measuring the efficiency of decision making units. Eur J Oper Res 2:429–444
3. Chen Y, Cook WD, Du J, Hu H, Zhu J (2015) Bounded and discrete data and Likert scales in data envelopment analysis: application to regional energy efficiency in China. Ann Oper Res. doi:10.1007/s10479-015-1827-3
4. Cooper WW, Park KS, Yu G (1999) IDEA and AR-IDEA: models for dealing with imprecise data in DEA. Manag Sci 45:597–607
5. Cooper WW, Park KS, Yu G (2001) IDEA (imprecise data envelopment analysis) with CMDs (column maximum decision making units). J Oper Res Soc 52(2):176–181
6. Cooper WW, Park KS, Yu G (2001) An illustrative application of IDEA (imprecise data envelopment analysis) to a Korean mobile telecommunication company. Oper Res 49(6):807–820
7. Despotis DK, Smirlis YG (2002) Data envelopment analysis with imprecise data. Eur J Oper Res 140:24–36
8. Farzipoor Saen R (2007) Suppliers selection in the presence of both cardinal and ordinal data. Eur J Oper Res 183:741–747
9. Kao C (2006) Interval efficiency measures in data envelopment analysis with imprecise data. Eur J Oper Res 174:1087–1099
10. Kao C, Liu ST (2009) Stochastic data envelopment analysis in measuring the efficiency of Taiwan commercial banks. Eur J Oper Res 196:312–322
11. Karsak EE, Dursun M (2014) An integrated supplier selection methodology incorporating QFD and DEA with imprecise data. Expert Syst Appl 41:6995–7004
12. Khalili-Damghani K, Tavana M, Haji-Saami S (2015) Data envelopment analysis model with interval data and undesirable output for combined cycle power plant performance assessment. Expert Syst Appl 42:760–773
13. Kim SH, Park CK, Park KS (1999) An application of data envelopment analysis in telephone offices evaluation with partial data. Comput Oper Res 26:59–72
14. Lee YK, Park KS, Kim SH (2002) Identification of inefficiencies in an additive model based IDEA (imprecise data envelopment analysis). Comput Oper Res 29:1661–1676
15. Marbini AH, Emrouznejad A, Agrell PJ (2014) Interval data without sign restrictions in DEA. Appl Math Model 38:2028–2036
16. Park KS (2004) Simplification of the transformations and redundancy of assurance regions in IDEA (imprecise DEA). J Oper Res Soc 55:1363–1366
17. Park KS (2007) Efficiency bounds and efficiency classifications in DEA with imprecise data. J Oper Res Soc 58:533–540
18. Park KS (2010) Duality, efficiency computations and interpretations in imprecise DEA. Eur J Oper Res 200:289–296
19. Toloo M (2014) Selecting and full ranking suppliers with imprecise data: a new DEA method. Int J Adv Manuf Technol 74:1141–1148
20. Toloo M, Nalchigar S (2011) A new DEA method for supplier selection in presence of both cardinal and ordinal data. Expert Syst Appl 38:14726–14731
21. Wang YM, Greatbanks R, Yang JB (2005) Interval efficiency assessment using data envelopment analysis. Fuzzy Sets Syst 153:347–370
22. Zhu J (2003) Imprecise data envelopment analysis (IDEA): a review and improvement with an application. Eur J Oper Res 144:513–529
23. Zhu J (2004) Imprecise DEA via standard linear DEA models with a revisit to a Korean mobile telecommunication company. Oper Res 52:323–329