A data envelopment analysis model with discretionary and non-discretionary factors in fuzzy environments

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Abstract: Data envelopment analysis (DEA) is a methodology for measuring the relative efficiencies of a set of decision-making units (DMUs) that use multiple inputs to produce multiple outputs. The standard DEA models assume that all inputs and outputs are crisp and can be changed at the discretion of management. While crisp input and output data are fundamentally indispensable in the standard DEA evaluation process, input and output data in real-world problems are often imprecise or ambiguous. In addition, real-world problems may also include non-discretionary factors that are beyond the control of a DMU's management. Fuzzy logic and fuzzy sets are widely used to represent ambiguous, uncertain or imprecise data in DEA by formalising the inaccuracies inherent in human decision-making. In this paper, we show that considering bounded factors in DEA models results in a disregard to the concept of relative efficiency since the efficiency of the DMUs are calculated by comparing the DMUs with their lower and/or upper bounds. In addition, we present a fuzzy DEA model with discretionary and non-discretionary factors in both the input and output-oriented CCR models. A numerical example is used to demonstrate the applicability and the efficacy of the proposed models.

Keywords: DEA; data envelopment analysis; efficiency; non-discretionary, discretionary and bounded factors; fuzzy mathematical programming.

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1 Introduction

Data envelopment analysis (DEA) is a powerful mathematical method that utilises linear programming (LP) to determine the relative efficiencies of a set of functionally similar decision-making units (DMUs). A DMU is considered efficient when no other DMUs can produce more outputs using an equal or lesser amount of inputs. The DEA generalises the usual efficiency measurement from a single-input single-output ratio to a multiple-input multiple-output ratio by using a ratio of the weighted sum of outputs to the weighted sum of inputs. A score of one is assigned to the frontier (efficient) units. The frontier units in DEA are those with maximum output levels for given input levels or with minimum input levels for given output levels. Charnes et al. (1978) originally proposed the first DEA model known as the CCR model. The conventional DEA methods require accurate measurement of both the inputs and outputs. In recent years, DEA has been used to measure the efficiency and effectiveness in supply chain management (Wong et al., 2008), operating entities (Parkan, 2006), banking (Azadeh et al., 2010a,b; Cooper et al., 2008), stock market (Emrouznejad and Thanassoulis, 2010), farming (Mulwa et al., 2009), hotel industry (Cheng et al., 2010), financial statement analysis (Ho, 2007) and healthcare (Dharmapala, 2009).

The standard DEA models assume that all inputs and outputs are known precisely and can be modified at the discretion of management. While crisp input and output data are fundamentally indispensable in the standard DEA models, input and output data in real-world problems are often imprecise or ambiguous. In addition, real-world problems may also include non-discretionary factors that are beyond the control of a DMU's management. Fuzzy logic and fuzzy sets are widely used to represent ambiguous, uncertain or imprecise data in DEA by formalising the inaccuracies inherent in human decision-making. In this paper, we show that considering bounded factors in DEA models results in the desertion of the concept of relative efficiency since the efficiency of the DMUs are calculated by comparing the DMUs with their lower and/or upper bounds. In addition, we present a fuzzy DEA model with discretionary and non-discretionary factors in both the input and output-oriented CCR models.

This paper is organised into eight sections. In Section 2, we present a comprehensive review and critical analysis of previously published literature. We discuss an overview of the fuzzy sets in Section 3. In Section 4, we present the original CCR model and in Section 5 we discuss bounded factors in the CCR model. In Section 6, we present the details of the proposed model and in Section 7, we present a numerical example to demonstrate the applicability of the proposed model and exhibit the efficacy of the procedures and algorithms. In Section 8, we conclude with our conclusions and future research directions.

2 Literature review

The classical DEA model proposed by Charnes et al. (1978) assumes that all inputs and outputs are discretionary and can be varied at the discretion of managers. However, real-life problems often involve non-discretionary inputs or outputs that are beyond the control of a DMUs management but should be considered in the performance assessment to ensure fair comparisons (Ebadi and Shiri Shahraki, 2010; Muñiz et al., 2006;

Ruggiero, 1998; Syrjänen, 2004). Charnes et al. (1980) coined with the idea of non-discretionary inputs. Charnes and Cooper (1985) and Charnes et al. (1985) laid out the blueprint for possible formulations of the DEA problems with non-discretionary factors. The first approach to explain the differences in non-discretionary factors was introduced by Banker and Morey (1986). Banker and Morey (1986) showed the impact of non-discretionary inputs in an analysis of a network of fast food restaurants. They considered 60 restaurants in the fast food industry with six inputs and three outputs. All three output variables (the breakfast, lunch and dinner sales) and two of the six input variables (supplies and labour expenses) were discretionary. The other four inputs (the age of the restaurant, advertising expenses determined by the corporate headquarters, restaurant location and the drive-in capability) were beyond the control of the individual restaurants management in this chain.

Ray (1991) excluded the non-discretionary inputs from the DEA model in a first stage and controlled the non-discretionary inputs in a second stage regression. Golany and Roll (1993) simultaneously accounted for both non-discretionary inputs and non-discretionary outputs and generalised the approach introduced by Banker and Morey (1986). Lovell (1994) and Ruggiero (1996) further extended the approach proposed by Banker and Morey (1986). These methods considered continuous fully discretionary or fully non-discretionary factors in a one stage DEA model. Ruggiero (1998) developed a hybrid model with three stages to allow for multiple non-discretionary inputs and used simulation to show that the multiple stage models of Ray (1991) and Ruggiero (1996) were preferred to the Banker and Morey (1986) model. Ruggiero (2004) argued that the current DEA literature on non-discretionary inputs ignores the possibility that efficiency may be correlated with the non-discretionary factors. He showed that if the true technical efficiency is negatively correlated with the non-discretionary inputs, the existing DEA efficiency estimates will be biased upward.

Muñiz et al. (2006) summarised the state of the current knowledge and Syrjänen (2004) identified several problems with different returns to scale assumptions related to continuous non-discretionary variables. Hosseinzadeh Lotfi et al. (2007) discussed and reviewed the use of the super-efficiency approach in DEA sensitivity analyses when some inputs were exogenously fixed. They proposed a super-efficiency DEA model by means of modified Banker and Morey's (1986) model, when the DMU under evaluation was excluded from the reference set. They were able to determine what perturbations of discretionary data could be tolerated before frontier DMUs become non-frontier. Hua et al. (2007) proposed a non-radial output-oriented DEA model by analysing the impacts of the non-discretionary input on DMU desirable and undesirable outputs. They also analysed the impacts of the non-discretionary input on DMUs' returns. Esmaeili (2009) proposed a slacks-based measure of efficiency in the presence of non-discretionary factors in DEA. The new measure has a close connection with the measure proposed by Banker and Morey (1986). This scalar measure dealt directly with the discretionary input excess and the discretionary output shortfall of the DMUs concerned.

Camanho et al. (2009) distinguished non-discretionary factors into two groups: internal and external factors, as in Thanassoulis et al. (2008). They constructed the production possibilities set based only on discretionary factors and internal non-discretionary factors, and implemented a procedure for peer selection according to the values of the external non-discretionary factors. Estelle et al. (2010) discussed the three-stage models that control for exogenous, non-discretionary inputs in DEA and introduced new second-stage models and compared and contrasted them with simulated data. Wu and Lee (2010) proposed a new DEA model by simultaneously considering stochastic variables, non-discretionary variables and ordinal data in a condominium pricing problem. They developed a performance measurement tool to provide a basis for understanding the condominium pricing problem, to direct and monitor the implementation of pricing strategy and to provide information regarding the results of pricing efforts for units sold as well as insights for future building design. Tone and Tsutsui (2010) showed that DEA models for measuring efficiency changes over time (e.g. the window analysis and the Malmquist index) usually neglect carry-over activities between two consecutive terms. They classified these models into four categories (i.e. desirable, undesirable, free and fixed) according to the characteristics of carry-overs. Desirable carry-overs corresponded, e.g. to profit carried forward and net earned surplus carried to the next term, while undesirable carry-overs included, e.g. loss carried forward, bad debt and dead stock. Free and fixed carry-overs indicated, respectively, discretionary and non-discretionary ones.

Löber and Staat (2010) introduced a method to incorporate categorical non-discretionary variables in DEA and surmised that the existing solution concepts pose problems for applied researchers. They developed a simple and straightforward alternative based on indicator variables and provided a flexible tool for models with categorical variables that could be solved with standard DEA software irrespective of scale assumptions even when no option for non-discretionary variables was available. Lozano and Gutiérrez (2011) proposed a DEA approach with variable returns to scale and joint weak disposability of the desirable and undesirable outputs. They showed that their approach has more discriminatory power than the common directional distance function approach.

One of the main challenges associated with the application of DEA is the difficulty in quantifying some of the input and output data in real-world problems where the observed values are often imprecise or vague. Imprecise or vague data may be the result of unquantifiable, incomplete and non-obtainable information. In recent years, many researchers have formulated DEA models to deal with the uncertain input and output data (Hatami-Marbini et al., 2010; Hatami-Marbini and Saati, 2009; Hatami-Marbini and Tavana, 2011; Saati et al., 2011; Shokouhi et al., 2010). One way to manipulate uncertain data in DEA is via probability distributions. Nevertheless, probability distributions require either a priori predictable regularity or a posteriori frequency determinations which are difficult to construct. An alternative approach is to represent the imprecise or vague values by membership functions of the fuzzy sets theory.

3 Overview of fuzzy sets

In this section, we review some basic definitions of fuzzy sets (Dubois and Prade, 1978; Kaufmann and Gupta, 1991; Zimmermann, 2001).

Definition 1. *Let U be a universe set. A fuzzy set* $\tilde{A}$ *of U is defined by a membership function* $\mu_{\tilde{A}}(x) \rightarrow [0,1]$, *where* $\mu_{\tilde{A}}(x)$, $\forall x \in U$, *indicates the degree of membership of* $\tilde{A}$ *to U.*

Definition 2. *A fuzzy subset* $\tilde{A}$ *of real number R is convex if and only if*

$$\mu_{\tilde{A}}\left(\lambda x+(1-\lambda)y\right)\geq\left(\mu_{\tilde{A}}(x)\wedge\mu_{\tilde{A}}(y)\right),\quad \forall x,y\in R,\forall\lambda\in[0,1]$$

where ' $\wedge$ *' denotes the minimum operator.*

Definition 3. *The* γ *-level of fuzzy set* $\tilde{A}$ *,* $\tilde{A}_\gamma$ *, is the crisp set* $\tilde{A}_\gamma=\{x|\mu_{\tilde{A}}(x)\geq\gamma\}$. *The support of* $\tilde{A}$ *is the crisp set* $Sup(\tilde{A})=\{x|\mu_{\tilde{A}}(x)>0\}$. $\tilde{A}$ *is normal if and only if* $Sup_{x\in U}\mu_{\tilde{A}}(x)=1$, *where U is the universal set.*

Definition 4. $\tilde{A}$ *is a fuzzy number iff* $\tilde{A}$ *is a normal and convex fuzzy subset of R.*

Definition 5. *A real fuzzy number* $\tilde{A}$ *denoted by* $\tilde{A}=(a,b,c,d;w)$ *is described as any fuzzy subset of the real line R with membership function* $\mu_{\tilde{A}}$ *which satisfies the following properties:*

- $\mu_{\tilde{A}}$ *is a semi continuous mapping from R to the closed interval* $[0,w], 0\leq w\leq 1$
- $\mu_{\tilde{A}}(x)=0$, for all $x\in[-\infty,a]$
- $\mu_{\tilde{A}}$ *is increasing on* $[a,b]$
- $\mu_{\tilde{a}}(x)=w$ *for all* $x\in[b,c]$, *where w is a constant and* $0<w\leq 1$
- $\mu_{\tilde{A}}$ *is decreasing on* $[c,d]$
- $\mu_{\tilde{A}}(x)=0$, *for all* $x\in[d,\infty]$

where a, b, c and d are real numbers.

Unless elsewhere specified, it is assumed that $\tilde{A}$ *is convex and bounded; i.e.* $-\infty<a,\ d<\infty$. *If* $w=1$, $\tilde{A}$ *is a normal fuzzy number, and if* $0<w<1$, $\tilde{A}$ *is a non-normal fuzzy number.*

The membership function $\mu_{\tilde{A}}$ *of* $\tilde{A}$ *can be expressed as:*

$$\mu_{\tilde{A}}(x)=\begin{cases}f^L(x), & a\leq x\leq b\\ w, & b\leq x\leq c\\ f^R(x), & c\leq x\leq d\\ 0, & otherwise\end{cases}\qquad(1)$$

where $f^L:[a,b]\to[0,w]$ *and* $f^R:[c,d]\to[0,w]$.

In addition, a fuzzy number $\tilde{A}$ *in parametric form is denoted by* $(\underline{a},\overline{a})$ *of function* $\underline{a}(r)$, $\overline{a}(r)$, $0\leq r\leq 1$, *which satisfies the following requirements:*

1 $\underline{a}(r)$ *is a bounded increasing left continuous function*

2 $\overline{a}(r)$ *is a bounded decreasing right continuous function*

3 $\underline{a}(r)\leq\overline{a}(r)$, *where* $0\leq r\leq 1$.

Definition 6. *A trapezoidal fuzzy number is widely best used for solving practical problems. Trapezoidal fuzzy numbers are determined by quadruples* $\tilde{u} = (u^{(1)}, u^{(2)}, u^{\alpha}, u^{\beta})$ *of two defuzzifiers* $u^{(1)}$ *and* $u^{(2)}$*, and left fuzziness* $u^{\alpha} > 0$ *and right fuzziness* $u^{\beta} > 0$*. The membership function can be defined as follows:*

$$\mu_{\tilde{u}}(x) = \begin{cases} \dfrac{1}{u^{\alpha}}\left(x - u^{(1)} + u^{\alpha}\right), & u^{(1)} - u^{\alpha} \le x \le u^{(1)} \\ 1, & x \in \left[u^{(1)}, u^{(2)}\right] \\ \dfrac{1}{u^{\beta}}\left(u^{(2)} - x + u^{\beta}\right) & u^{(2)} \le x \le u^{(2)} + u^{\beta} \\ 0, & otherwise \end{cases} \tag{2}$$

Note that the trapezoidal fuzzy number is a triangular fuzzy number if $u^{(1)} = u^{(2)} = u$*, denoted by a triple* $(u, u^{\alpha}, u^{\beta})$.

Definition 7. *In fuzzy LP, the min T-norm is usually applied to assess a linear combination of fuzzy quantities. Therefore, for a given set of trapezoidal fuzzy numbers* $\tilde{u}_j = (u_j^{(1)}, u_j^{(2)}, u_j^{\alpha}, u_j^{\beta})$*, j=1,2,…,n and* $\lambda_j \ge 0$, $\sum_{j=1}^{n} \lambda_j \tilde{u}_j$ *is defined as follows:*

$$\sum_{j=1}^{n} \lambda_j \tilde{u}_j = \left(\sum_{j=1}^{n} \lambda_j u_j^{(1)}, \sum_{j=1}^{n} \lambda_j u_j^{(2)}, \sum_{j=1}^{n} \lambda_j u_j^{\alpha}, \sum_{j=1}^{n} \lambda_j u_j^{\beta} \right) \tag{3}$$

where $\sum_{j=1}^{n} \lambda_j \tilde{u}_j$ *denotes the combination* $\lambda_1 \tilde{u}_1 \oplus \lambda_2 \tilde{u}_2 \oplus \cdots \oplus \lambda_n \tilde{u}_n$.

4 The original CCR model

Charnes et al. (1978) proposed the CCR model to evaluate the technical efficiency of a given DMU_p. The original CCR model considers a set of n DMUs where each DMU_j $(j = 1, 2, ..., n)$ uses m inputs $x_{ij}\,(i = 1, 2, ..., m)$ to generate s outputs $y_{rj}\,(r = 1, 2, ..., s)$. The primal and the dual LP statements for the (input oriented) CCR model are:

Primal CCR model	*Dual CCR model*
$\min \theta$	$\max \sum_{r=1}^{s} u_r y_{rp}$
s.t.	s.t.
$\theta x_{ip} - \sum_{j=1}^{n} \lambda_j x_{ij} \ge 0, \quad \forall i$	$\sum_{i=1}^{m} v_i x_{ip} = 1$
$\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{rp}, \quad \forall r$	$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad \forall j$
$\lambda_j \ge 0, \quad \forall j$	$u_r, v_i \ge 0, \quad \forall r, i$

(4)

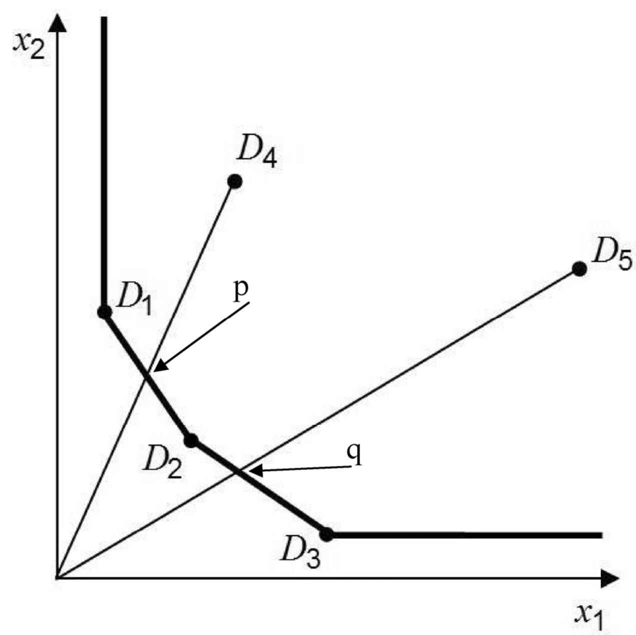
where u_r and v_i are the weights assigned to the rth output and the ith input, respectively. The primal model and its dual are referred to as the envelopment and the multiplier, respectively.

5 Discussion on bounded factors in the CCR model

In this section, we argue that considering bounded factors in DEA models results in the disregard to the concept of relative efficiency since the efficiency of the DMUs are calculated by comparing the DMUs with their bounds. Maital and Vaninsky (2001) and Cooper et al. (2003) developed the CCR model with bounded factors to measure the efficiency of the DMUs while considering a number of constraints in the model. These constraints results in some negative aspects in the CCR model as shown in the problem presented in Figure 1 with two inputs and five DMUs.

Figure 1 shows the efficient frontier for the five DMUs under consideration as $D_1(1,6)$, $D_2(3,3)$, $D_3(6,1)$, $D_4(4,9)$ and $D_5(12,7)$. As shown in this figure, D_1, D_2 and D_3 are clearly on the efficient frontier. Here we consider the following scenarios.

Figure 1 An example with two controllable inputs (Scenario I)



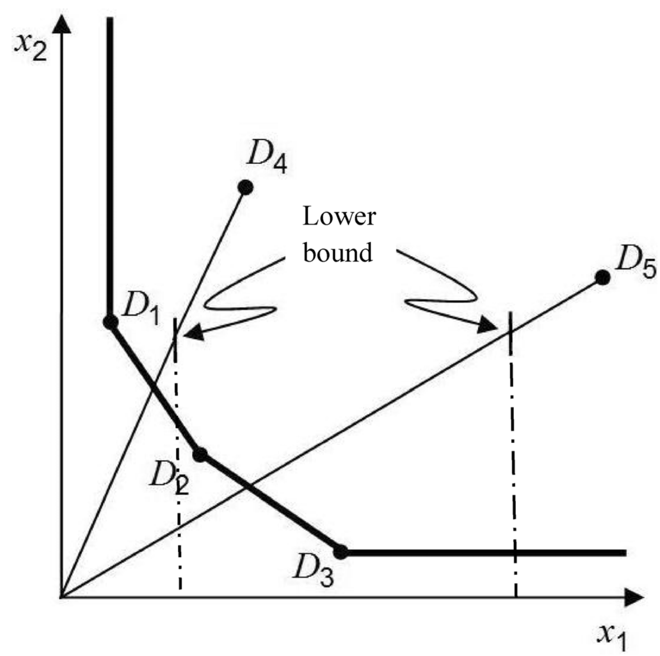
Scenario I: Both x_1 and x_2 are controllable

In this case, the efficiency of D_4 is equal to 0.5 and its projected (frontier) value is represented by the point *p*. As a result, D_1 and D_2 are suitable benchmarks for D_4. Similarly, the efficiency of D_5 is 0.33 and *q* is a projected point on the frontier. Hence, D_2 and D_3 are suitable benchmarks for D_5.

Scenario II: x_1 is bounded and x_2 is controllable

Consider Figure 2 and assume that the lower bound of x_1 for D_4 and D_5 are 2.5 and 10, respectively. In this case, the efficiency values of D_4 and D_5 are 0.625 and 0.83, respectively. In contrast to scenario I, we can see an increase in the efficiency values of both D_4 and D_5 since the lower bound has been imposed on the model. D_5, with a lower efficiency than D_4 in scenario I, has a bigger efficiency in this assessment. On the other hand, this efficiency is obtained in comparison with the lower bound of x_1 of D_5 and not in comparison with the other DMUs. This is the result of applying the bounds in the model for the various factors.

Figure 2 x_1 is bounded and x_2 is controllable (Scenario II)



Scenario III: x_1 is controllable and x_2 is non-controllable

Consider Figure 3 where x_2 is a non-controllable factor, and D_4 and D_5 are projected as parallel lines to the x_1 axis on the frontier. In this case, the DMUs can be projected on the frontier since x_1 is not bounded. Although the efficiency of the DMUs is not the same as the efficiency in scenario I, we can consider this case because

1 the DMUs are compared with the efficient frontier

2 a reference set is determined for each DMU.

Scenario IV: x_1 is bounded and x_2 is non-controllable

This case is a combination of scenarios II and III. Therefore, the problem discussed for scenario II also applies to this scenario (see Figure 4).

Figure 3 x_1 is controllable and x_2 is non-controllable (Scenario III)

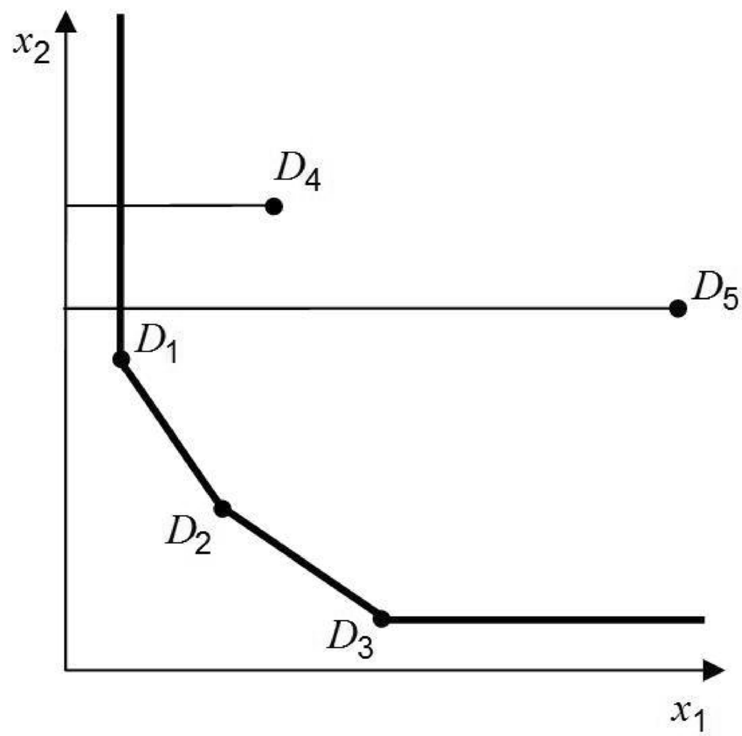
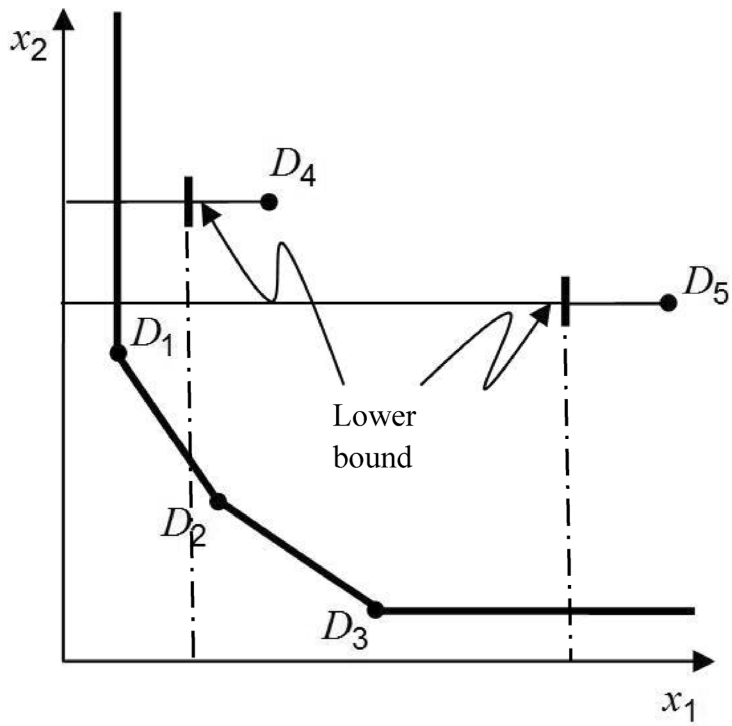


Figure 4 x_1 is bounded and x_2 is non-controllable (Scenario IV)



With regard to the aforementioned scenarios, we have shown that the DEA models with bounded data require a comparison between the DMUs and their bounds. In other words, the efficiency of the DMUs are calculated by comparing the DMUs with their lower and/or upper bounds and, as a result, the concept of 'relative efficiency' is disregarded. In the case with bounded factors, although in practice it is possible that a DMU is not capable of reducing the input values and/or increasing the output values, this should not be considered an advantage for the DMU. Therefore, it is better to consider the controllable and uncontrollable factors in the DEA models and apply the bounded factors just in the improvement step.

6 Proposed model

In this section, we discuss the discretionary and non-discretionary factors in the CCR and the fuzzy CCR model with discretionary and non-discretionary factors.

6.1 Discretionary and non-discretionary factors in CCR

Considering those factors affecting the production in DEA, some could lie outside the control of the DMU under evaluation. For example, in a study of schools, it is necessary to allow for variations in 'minority', 'economically disadvantaged' and 'low English proficiency' students who have to be dealt with in the different schools. These input variables are non-discretionary. That is, they cannot be varied at the discretion of the individual school managers but nevertheless need to be taken into consideration in arriving at relative efficiency evaluations. Assume that indexes I_0 and I_1 refer to sets of 'non-discretionary' and 'discretionary' inputs, respectively. The CCR envelopment model and the CCR multiplier model with non-discretionary and discretionary factors in the input-oriented case are:

Envelopment model (input oriented) *Multiplier model (input oriented)*

$$\begin{array}{ll}
\min \theta & \max \sum_{r=1}^{s} u_r y_{rp} - \sum_{i \in I_0} v_i x_{ip} \\
\text{s.t.} & \text{s.t.} \\
-\sum_{j=1}^{n} \lambda_j x_{ij} \geq -x_{ip}, \quad i \in I_0 & \sum_{i \in I_1} v_i x_{ip} = 1 \\
\theta x_{ip} - \sum_{j=1}^{n} \lambda_j x_{ij} \geq 0, \quad i \in I_1 & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad \forall j \\
\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rp}, \quad \forall r & u_r, v_i \geq 0, \quad \forall r, i \\
\lambda_j \geq 0 \quad \forall j &
\end{array} \tag{5}$$

Similarly, the output-oriented case of the CCR envelopment and multiplier models with non-discretionary and discretionary factors are:

Envelopment model (output oriented) *Multiplier model (output oriented)*

$$\begin{array}{ll}
\max \phi & \min \sum_{i=1}^{m} v_i x_{ip} - \sum_{r \in O_0} u_r y_{rp} \\
\text{s.t.} & \text{s.t.} \\
\sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{ip}, \quad \forall i & \sum_{r \in O_1} u_r y_{rp} = 1 \\
-\sum_{j=1}^{n} \lambda_j y_{rj} \leq -y_{rp}, \quad r \in O_0 & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad \forall j \\
\phi y_{rp} - \sum_{j=1}^{n} \lambda_j y_{rj} \leq 0, \quad r \in O_1 & u_r, v_i \geq 0, \quad \forall r, i \\
\lambda_j \geq 0, \quad \forall j &
\end{array} \tag{6}$$

where the indexes O_0 and O_1 are the sets of 'non-discretionary' and 'discretionary' outputs, respectively.

6.2 Fuzzy CCR model with discretionary and non-discretionary factors

All data including discretionary and non-discretionary factors in the conventional CCR model are known precisely or given as crisp values. However, under many conditions, crisp data are inadequate or insufficient to model real-life evaluation problems. The multiplier models (5) and (6) can be formulated by the following fuzzy LP model:

Fuzzy multiplier model (input oriented)

$$\begin{aligned}
&\max \sum_{r=1}^{s} u_r \tilde{y}_{rp} - \sum_{i\in I_0} v_i \tilde{x}_{ip} \\
&\text{s.t.} \\
&\sum_{i\in I_1} v_i \tilde{x}_{ip} = 1 \\
&\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \le 0, \quad \forall j \\
&u_r, v_i \ge 0, \quad \forall r, i
\end{aligned}$$

Fuzzy multiplier model (output oriented)

$$\begin{aligned}
&\min \sum_{i=1}^{m} v_i \tilde{x}_{ip} - \sum_{r\in O_0} u_r \tilde{y}_{rp} \\
&\text{s.t.} \\
&\sum_{r\in O_1} u_r \tilde{y}_{rp} = 1 \\
&\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \le 0, \quad \forall j \\
&u_r, v_i \ge 0, \quad \forall r, i
\end{aligned} \tag{7}$$

where $\tilde{x}_{ij} = (x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{\alpha}, x_{ij}^{\beta})$ and $\tilde{y}_{rj} = (y_{rj}^{(1)}, y_{rj}^{(2)}, y_{rj}^{\alpha}, y_{rj}^{\beta})$ are the fuzzy inputs and fuzzy outputs of each DMU, respectively, expressed by trapezoidal fuzzy numbers. Thus, models (7) can be rewritten as follows:

$$\begin{aligned}
&\max \sum_{r=1}^{s} u_r \left(y_{rp}^{(1)}, y_{rp}^{(2)}, y_{rp}^{\alpha}, y_{rp}^{\beta}\right) - \sum_{i\in I_0} v_i \left(x_{ip}^{(1)}, x_{ip}^{(2)}, x_{ip}^{\alpha}, x_{ip}^{\beta}\right) \\
&\text{s.t.} \\
&\sum_{i\in I_1} v_i \left(x_{ip}^{(1)}, x_{ip}^{(2)}, x_{ip}^{\alpha}, x_{ip}^{\beta}\right) = 1 \\
&\sum_{r=1}^{s} u_r \left(y_{rj}^{(1)}, y_{rj}^{(2)}, y_{rj}^{\alpha}, y_{rj}^{\beta}\right) - \sum_{i=1}^{m} v_i \left(x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{\alpha}, x_{ij}^{\beta}\right) \le 0, \quad \forall j \\
&u_r, v_i \ge 0, \quad \forall r, i
\end{aligned}$$

$$\begin{aligned}
&\min \sum_{i=1}^{m} v_i \left(x_{ip}^{(1)}, x_{ip}^{(2)}, x_{ip}^{\alpha}, x_{ip}^{\beta}\right) - \sum_{r\in O_0} u_r \left(y_{rp}^{(1)}, y_{rp}^{(2)}, y_{rp}^{\alpha}, y_{rp}^{\beta}\right) \\
&\text{s.t.} \\
&\sum_{r\in O_1} u_r \left(y_{rp}^{(1)}, y_{rp}^{(2)}, y_{rp}^{\alpha}, y_{rp}^{\beta}\right) = 1 \\
&\sum_{r=1}^{s} u_r \left(y_{rj}^{(1)}, y_{rj}^{(2)}, y_{rj}^{\alpha}, y_{rj}^{\beta}\right) - \sum_{i=1}^{m} v_i \left(x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{\alpha}, x_{ij}^{\beta}\right) \le 0, \quad \forall j \\
&u_r, v_i \ge 0, \quad \forall r, i
\end{aligned} \tag{8}$$

Four approaches are proposed in the fuzzy LP literature to solve models (8). These four methods include:

1 the tolerance approach
2 the γ-level-based approach
3 the fuzzy ranking approach
4 the possibility approach.

In this paper, we apply the γ-level approach proposed by Saati et al. (2002) to develop a new fuzzy model for evaluating the efficiency of DMUs with fuzzy discretionary and non-discretionary factors. The resulting model can be solved as a crisp LP model with some variable substitutions to produce a crisp efficiency score for each DMU and for a given γ. In addition, this model can be solved for various values of γ to monitor how the efficiency scores of the DMUs change when the possibility level γ varies. This information is useful when considering the sensitivity of the results to small variations in various linguistic data. Therefore, the following models would follow:

$$\begin{aligned}
&\max \sum_{r=1}^{s} \bar{y}_{rp} - \sum_{i \in I_0} \bar{x}_{ip} \\
&\text{s.t.} \\
&\sum_{i \in I_1} \bar{x}_{ip} = 1 \\
&\sum_{r=1}^{s} \bar{y}_{rj} - \sum_{i=1}^{m} \bar{x}_{ij} \le 0, \quad \forall j \\
&\left(x_{ij}^{(1)} - (1-\gamma)x_{ij}^{\alpha}\right)v_i \le \bar{x}_{ij} \\
&\quad \le \left(x_{ij}^{(2)} + (1-\gamma)x_{ij}^{\beta}\right)v_i, \quad \forall i, j \\
&\left(y_{rj}^{(1)} - (1-\gamma)y_{rj}^{\alpha}\right)u_r \\
&\quad \le \bar{y}_{rj} \le \left(y_{rj}^{(2)} + (1-\gamma)y_{rj}^{\beta}\right)u_r, \quad \forall r, j \\
&u_r, v_i \ge 0, \quad \forall i, r
\end{aligned}
\qquad
\begin{aligned}
&\min \sum_{i=1}^{m} \bar{x}_{ip} - \sum_{r \in O_0} \bar{y}_{rp} \\
&\text{s.t.} \\
&\sum_{r \in O_1} \bar{y}_{rp} = 1 \\
&\sum_{r=1}^{s} \bar{y}_{rj} - \sum_{i=1}^{m} \bar{x}_{ij} \le 0, \quad \forall j \\
&\left(x_{ij}^{(1)} - (1-\gamma)x_{ij}^{\alpha}\right)v_i \le \bar{x}_{ij} \\
&\quad \le \left(x_{ij}^{(2)} + (1-\gamma)x_{ij}^{\beta}\right)v_i, \quad \forall i, j \\
&\left(y_{rj}^{(1)} - (1-\gamma)y_{rj}^{\alpha}\right)u_r \le \bar{y}_{rj} \\
&\quad \le \left(y_{rj}^{(2)} + (1-\gamma)y_{rj}^{\beta}\right)u_r, \quad \forall r, j \\
&u_r, v_i \ge 0, \quad \forall i, r
\end{aligned}
\tag{9}$$

where $\bar{x}_{ij}$ and $\bar{y}_{rj}$ are the decision variables obtained from variable substitutions used to transform the fuzzy models into parametric LP models. Hence, models (9) are parametric LP problems, while $\gamma \in [0,1]$ is a parameter.

7 Numerical example

In this section, we use a numerical example with non-controllable factors in the fuzzy environment to demonstrate the applicability and efficacy of the proposed method. In this example, two trapezoidal fuzzy inputs are used to obtain two trapezoidal fuzzy outputs in five DMUs as shown in Table 1.

The efficiencies of the DMUs for different γ values using the method proposed by Saati et al. (2002) are presented in Table 2.

We assume that the first and the second input of each DMU are uncontrollable and controllable, respectively. The efficiencies of the DMUs presented in Table 3 are calculated using the input-oriented model (9) for different γ.

As it is shown in Table 3, the efficiencies of the DMUs are changed. Note that these values are always lower than or equal to the values in Table 2. In other words, this reduction of the objective function is for the penalty which should be considered for the first uncontrollable input of the DMUs. Now assume that the first and the second input of each DMU are controllable and uncontrollable, respectively. Using the input-oriented model (9) for different γ, the efficiencies are obtained and shown in Table 4.

Table 1 Fuzzy input and output data

	Fuzzy inputs		*Fuzzy outputs*	
DMU	*1*	*2*	*1*	*2*
1	(10, 11, 1, 2)	(17, 18, 2, 5)	(73, 78, 5, 3)	(5, 6, 1, 0.1)
2	(12, 13, 1, 3)	(25, 27, 3, 4)	(75, 79, 3, 1)	(4, 5, 1, 0.2)
3	(13, 14, 2, 1)	(31, 32, 5, 3)	(68, 71, 2, 6)	(3, 4, 0.75, 0.2)
4	(20, 21, 2, 1)	(41, 43, 6, 7)	(25, 29, 3, 2)	(2, 3, 0.5, 0.3)
5	(20, 26, 1, 2)	(40, 41, 8, 6)	(20, 21, 1, 1)	(1, 2, 0.25, 0.5)

Table 2 The efficiencies of five DMUs using Saati et al. (2002)'s method

	γ				
DMU	*0.00*	*0.25*	*0.50*	*0.75*	*1.00*
1	1.00	1.00	1.00	1.00	1.00
2	1.00	1.00	1.00	1.00	0.99
3	1.00	1.00	1.00	0.93	0.82
4	0.60	0.51	0.44	0.38	0.33
5	0.45	0.36	0.31	0.26	0.22

Table 3 The efficiencies with controllable and uncontrollable data

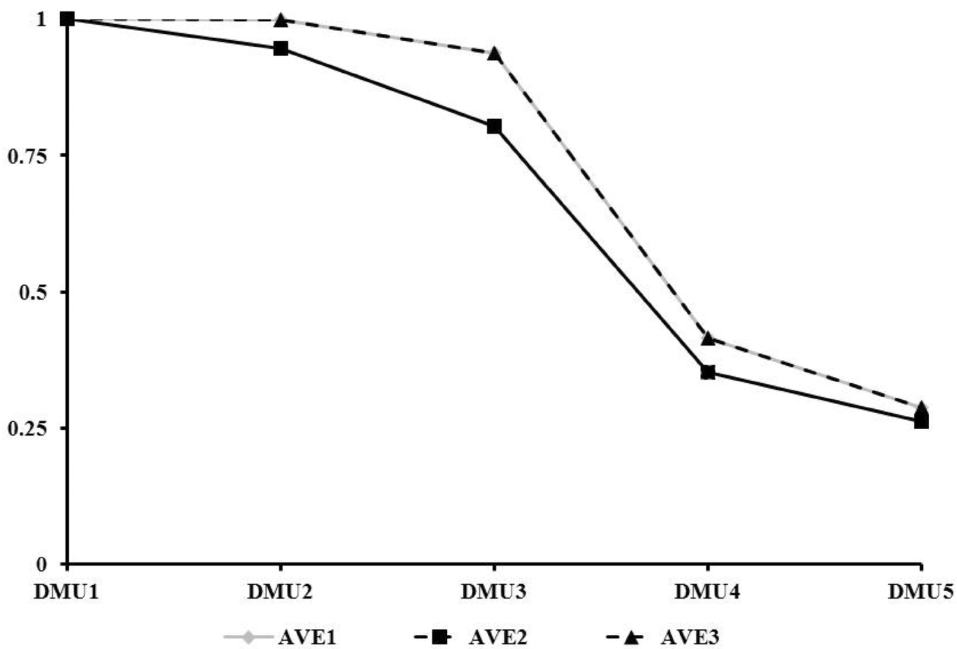
	γ				
DMU	*0.00*	*0.25*	*0.50*	*0.75*	*1.00*
1	1.00	1.00	1.00	1.00	1.00
2	1.00	1.00	1.00	1.00	0.78
3	1.00	1.00	1.00	0.65	0.56
4	0.54	0.45	0.38	0.32	0.26
5	0.45	0.36	0.28	0.23	0.18

Table 4 The efficiencies with controllable and uncontrollable data

	γ				
DMU	*0.00*	*0.25*	*0.50*	*0.75*	*1.00*
1	1.00	1.00	1.00	1.00	1.00
2	1.00	1.00	1.00	1.00	0.99
3	1.00	1.00	1.00	0.93	0.82
4	0.60	0.51	0.44	0.38	0.33
5	0.43	0.36	0.31	0.26	0.22

A comparison of the results in Tables 2–4 reveals that these DMUs are more sensitive to their first input in comparison to the second one. As is shown in Figure 5 the first and third cases overlap but the second one move below. Note that Figure 5 is depicted using the average of five γ-cuts for each DMU.

Figure 5 A comparison of the average efficiencies for five DMUs



8 Conclusions and future research directions

The field of DEA has grown exponentially since the pioneering paper of Charnes et al. (1978). DEA measures the relative efficiency of a DMU by comparing it against a peer group. Discretionary DEA models assume that all inputs and outputs can be varied at the discretion of management. However, many real-life situations may involve 'exogenously fixed' or non-discretionary factors that are beyond the control of a DMU's management. On the other hand, input and output data in the real-world problems are often imprecise or ambiguous. Fuzzy set theory is widely used to represent the impreciseness and the ambiguity in DEA by formalising the inaccuracies inherent in human decision-making. In this paper, we showed that considering bounded factors in DEA models results in a disregard to the concept of relative efficiency since the efficiency of the DMUs are calculated by comparing the DMUs with their lower and/or upper bounds. In addition, we developed a fuzzy DEA model with non-discretionary factors in order to measure the relative efficiency of the DMUs using both the input- and output-oriented CCR models. One of the tangible advantages of our approach is the ability to provide the decision maker with the necessary information to consider the sensitivity of the results to small variations in linguistic data which can be controllable and non-controllable. Finally, we presented a numerical example to illustrate the applicability and efficacy of the proposed model.

This study suggests future work in several directions including the development of:

- similar models for other non-radial DEA models such as slack-based measure or directional measure of DEA
- similar models for variable returns to scale model that allow the measurement of the scale efficiency change with fuzzy and interval data
- similar models for calculating the productivity change of a DMU over time with the Malmquist productivity index.

We hope that the concepts introduced here will provide inspiration for future research.

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