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A new non-radial directional distance model for data envelopment analysis problems with negative and flexible measures☆

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ABSTRACT

Data envelopment analysis (DEA) is a mathematical approach for evaluating the efficiency of decision-making units that convert multiple inputs into multiple outputs. Traditional DEA models measure technical (radial) efficiencies by assuming the input and output status of each performance measure is known, and the data associated with the performance measures are non-negative. These assumptions are restrictive and limit the applications of DEA to real-world problems. We propose a new extended non-radial directional distance model, which is a variant of the weighted additive model, to cope with negative data. We then extend our model and use flexible measures, which play the role of both inputs and outputs, to cope with the unknown status of the performance measures. We also present a case study in the automotive industry to exhibit the efficacy of the models proposed in this study.

1. Introduction

Data envelopment analysis (DEA) is a mathematical technique based on linear programming for measuring the performance efficiency of organizational units known as decision-making units (DMUs). The first DEA model was proposed by Charnes et al. [20] to evaluate the efficiency of an educational center and then further developed by Banker et al. [12]. The traditional DEA models assume that the input and output status of each performance measure is known, and the data associated with each performance measure is non-negative. However, these assumptions are restrictive and do not always hold in real-world problems. For example, net profit as an output can be negative. Scheel [71] studied various methods for handling undesirable outputs in DEA and proposed new radial measures. Portela et al. [68] developed the generic directional distance model of Chambers et al. [17,18] for problems with positive and negative inputs and/or outputs. Their range directional measure (RDM) was applicable to negative data without the need for any transformation, and it yielded a measure of efficiency similar to the radial measures in traditional DEA models. However, the RDM model fails to (i) evaluate weak-efficient (inefficient) DMUs, and (ii) project inefficient DMUs on the strong-efficient frontier. Additionally, Aparicio et al. [8] formulated the weighted additive model for measuring slacks-based technical inefficiency. We extend a non-radial directional distance function (DDF) model, which is a variant of the weighted additive model, to allow for simultaneous input contraction and output expansion with different scales. Furthermore, our formulated model not only handles negative data but also deals with the above failures.

In addition, some factors, called flexible measures, can play both input and output roles. For example, in the supplier selection problem, research and development (R&D) expenditures can be considered as both input and output factors. R&D, on the input side, imposes extraordinary expenses to the supplier and, on the output side, plays the role of proxy for innovation, which results in better products, services, and processes. Beasley [14,15] considered the research funding factor as a flexible measure in evaluating US universities. The research funding factor, on the one hand, is a proxy for the quality of the research program,

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which leads to acquiring the income for universities and, on the other hand, supports different outputs, such as graduate students [25].

The goal of this study is to propose a non-radial directional distance model and relax the restrictive assumptions of known status and non-negative data for the input and output measures in the traditional DEA models. We accomplish this goal by:

(i) including flexible measures in the envelopment form of the slacks-based measure (SBM) of inefficiency model and obtaining the minimum possible efficiency score (pessimistic approach),
(ii) imposing the flexible measures to the dual of the SBM of inefficiency model and obtaining the maximum possible efficiency score (optimistic approach),
(iii) establishing several theorems and validating our models, and
(iv) presenting a case study in the automotive industry and demonstrating the efficacy of the approach proposed in this study.

The remainder of this paper is organized as follows. In Section 2, we review the key developments in dealing with negative data, the DDF, and flexible measures in DEA. Section 3 briefly reviews the SBM of inefficiency and the conventional DEA models for handling flexible measures. Section 4 extends a pair of pessimistic and optimistic classifier DDF models. Section 5 presents a case study to demonstrate the efficacy of the model proposed in this study. Finally, we conclude with our conclusions and further research directions in Section 6.

2. Literature review

Traditional DEA models assume that (i) the data associated with the performance measures are non-negative, (ii) the technical (radial) efficiency is required, and (iii) the input or output status of each performance measure is known. This section reviews the approaches proposed in the literature for relaxing these basic assumptions.

2.1. Negative data in DEA

The traditional DEA models only allow for non-negative data. However, real-world problems often comprise negative data. To the best of our knowledge, Scheel [71] was the first study that suggested a DEA model with negative data by treating the absolute values of negative outputs as inputs and the absolute values of negative inputs as outputs. This method cannot be used to solve DEA problems with both negative and positive inputs and/or outputs. Sharp et al. [73] introduced a modified slack-based measure in which both negative outputs and negative inputs could be handled. The authors presented a modified SBM model and proved that their model overcomes the lack of translation invariance of the RDM model. Emrouznejad et al. [29] proposed a semi-oriented radial measure (SORM) that permits the presence of variables that can take both negative and positive values. They claimed that their model resolves some of the difficulties in the models proposed by Portela et al. [68] and Sharp et al. [73].

Kazemi Matin and Azizi [47] presented a two-phase approach based on a modified version of the classical additive DEA model in the presence of negative data and applied their model to evaluate efficiency in the banking industry. The authors showed the deficiency of the SORM when positive output levels in the benchmarks are in interest, and the models could not guarantee to provide boundary units. Cheng et al. [23] developed a variant of the traditional radial model whereby original values were replaced with absolute values. Their model was unit-invariant and could deal with negative data. Kazemi Matin et al. [46] pointed out some concerns with target-setting in the SORM models and introduced a modified approach. Kerstens and Van de Woestyne [48] showed that the Chang et al. [19] method contains some imprecisions and presented some further results.

Pastor and Aparicio [62] presented an overview of the translation and non-translation invariant DEA models for handling negative data and introduced the linear loss distance function as a powerful tool for revising translation invariance in DEA models. To the best of our knowledge, Allahyar and Rostamy-Malkhalifeh [2] were the first to suggest a method for determining the type of returns to scale in the presence of negative input and output values. Sahoo et al. [69] continued their effort and developed a non-radial DEA based methodology to determine both the most productive scale size and the returns to scale characterizations of production units in the presence of negative data. To develop other kinds of DEA models in the presence of negative data, Khoveyni et al. [50] presented a DEA model to determine DMUs with congestion in the presence of negative data. Lin & Chen [52] developed a new radial super-efficiency DEA model, which allowed input-output variables to take both negative and positive values. They demonstrated their model is always feasible (no matter whether the input-output data are non-negative or not), and their model can fully rank all DMUs.

The following DDF models are most commonly used to handle negative data in DEA. Portela et al. [68] developed an RDM based model for handling negative data based on a DDF approach and showed that the RDM model is units and translation invariant, which makes it suitable to deal with negative data. Portela and Thanassoulis [67] introduced a meta-Malmquist index for handling negative data and developed an index and an indicator of productivity change to be used with negative data based on the RDM model to measure the performance in the banking industry. Tavana et al. [75] developed a new dynamic RDM for the two-stage DEA models with negative data for both desirable and undesirable carryovers. Kaffash et al. [45] proposed a new version of the modified SORM model by using DDF and choosing a relevant direction to efficiently deal with variables with both positive and negative values.

2.2. DDF approach

Chambers et al. [17,18] studied the relationship between the directional technology, distance function, and the profit function, and used the directional technology distance function in various economic settings. Färe et al. [33] used the directional output distance function to model the joint production of good and bad outputs and the reduced disposability of bad output imposed by regulation in the utility industry. Portela et al. [68] proposed the RDM for handling positive and negative data in DEA. Following the introduction of RDM, several methods have been proposed to extend RDM and apply this method to real-world problems. Shetty et al. [74] proposed a new modification of the DDF of DEA to assess bankruptcy in information technology companies. Chen et al. [22] proposed a modification of the DDF by selecting the feasible reference bundles that ensured the resulting Nerlove–Luenberger measure of super-efficiency was always feasible. Sahoo et al. [70] extended the value-based models in a directional DEA set up to develop new directional cost- and revenue-based measures of efficiency in the banking industry. Falavigna et al. [31] compared DEA and DDF by investigating the technical efficiency in a judicial system. Both methods provide the efficiency of each observation as the radial distance from the efficient frontier. Cherchye et al. [24] extended the DEA methodology based on DDF for cost-efficiency analysis towards profit efficiency settings. Aparicio et al. [9] showed that different directional vectors

might lead to DDF models without translation invariance property. They established the necessary and sufficient conditions that the directional vector must fulfill to reach a translation-invariant DDF model. Izadikhah and Farzipoor Saen [41] developed a new two-stage DEA model based on RDM with negative input-intermediate-output data and evaluated the sustainability in medical device supply chains.

Lee and Choi [51] used non-radial DDF to evaluate greenhouse gas performance by decomposing the technical efficiency into pure technical efficiency and scale efficiency. Yang et al. [86] developed a DEA-based DDF model to investigate the appropriate (or the best) direction along which to measure efficiency. Zhang and Chen [88] applied the DEA window analysis method and the DEA-based DDF to comprehensively evaluate the dynamic performance of energy portfolios based on daily fossil-fuel future prices. Toloo et al. [83] developed a new classifier non-radial directional distance method by taking into consideration the input contraction and output expansion, simultaneously. Hatami-Marbini et al. [39] modified the super-efficiency RDM model to overcome the infeasibility problem occurring in conventional RDM models. Pastor et al. [63] extended a family of DDF models in DEA under constant returns to scale assumption and started a new subfamily of bounded or partially-bounded CRS-DDF models. Tavana et al. [75] developed a new dynamic RDM for two-stage DEA models that allow for negative data for both desirable and undesirable carryovers in the banking industry. Deng et al. [27] developed a new factor-analysis-based approach for choosing the optimal direction in a DDF analysis to measure the performance of public hospitals. Meng [56] proposed a DDF model to assess the carbon emission efficiency under natural and managerial disposability. Liu et al. [54] used DEA and DDF to investigate energy and environmental performance. Falavigna and Ippoliti [30] proposed a DDF model to investigate the sustainability of a healthcare project in municipalities promoting active and healthy aging. Lin and Yue [53] developed a DDF-based super-efficiency model with capabilities to deal with negative data and generated bounded super-efficiency scores. Pastor et al. [64] introduced a new Malmquist productivity index by modifying conventional DDF.

2.3. Flexible measures

Traditional DEA models assume the status of each factor is predetermined whether the factor is an input or an output, while in some real-world problems, the type of data is unclear. These factors are known as flexible measures and can play the role of both inputs and outputs. In the supplier selection context, R&D expenditure can be both an input and an output factor. Given that efficiency is the ratio of outputs over inputs, the output is a variable that increasing it causes an increase in efficiency. In contrast, an input is a variable that decreasing it causes an increase in efficiency. On the one hand, reducing the R&D expenditure increases the reaming financial resources, which shows it is an input factor. On the other hand, the R&D expenditure can be treated as an output because R&D activities are essential in achieving future growth, better production, and better technology capacities. Beasley [14] employed research income as a factor for evaluating university chemistry and physics departments in the United Kingdom. The author argued that research income is a flexible measure because it is used to support faculty and postgraduate students' research while it is also a measure of reputation.

Cook and Zhu [25] modified the multiplier form of the Charnes, Cooper, and Rhodes (CCR) model to develop a classifier DEA model to accommodate flexible measures. Toloo [78] dealt with some computational problems in Cook and Zhu [25] by showing their method may produce incorrect efficiency scores because of a large positive number used in their model. Amirteimoori and Emrouznejad [6] further revealed that the models of Cook and Zhu [25] and Toloo [78] suffer from the problem of overestimating the efficiency; however, Toloo [[80]a] mathematically proved that the Amirteimoori and Emrouznejad [6]'s claim was false. Amirteimoori and Emrouznejad [5] and Toloo [79] extended the envelopment form of the classifier DEA model. Farzipoor Saen [34] further extended the subject and provided an improvement for the conventional DEA models in the presence of both flexible factors and imprecise data.

Amirteimoori et al. [7] developed a flexible SBM of efficiency to evaluate higher education institutions by considering flexible measures. Tohidi and Matroud [77] developed a non-oriented DEA method where the model was able to select the status of each flexible measure as an input or output and determine the returns to scale status. Their model was also capable of handling negative data. Toloo et al. [83] proposed pessimistic and optimistic aspects based on a non-radial directional distance method for dealing with flexible measures and validated their approach through a case study in the banking industry. Abolghasem et al. [1] developed an integrated method for evaluating healthcare systems by combining flexible measures and cross efficiency. Sedighi Hassan Kiyadeh et al. [72] developed two DEA models based on Russell's measurement to deal with flexible measures. Navas et al. [58] evaluated the performance of education institutions in the presence of flexible measures. Bod'a [16] proposed a flexible SBM where each flexible measure played and input role for some DMUs, and output role for other units.

3. Preliminaries

In this section, we focus our attention on the mathematical definitions of the basic SBM of inefficiency and classifier models that form the basis for the approach proposed in this study.

3.1. SBM of inefficiency

Assume that there are n DMUs (DMU_j) with m inputs $\boldsymbol{x}_j = (x_{ij}, \ldots, x_{mj})$ and s outputs $\boldsymbol{y}_j = (y_{1j}, \ldots, y_{sj})$. The production possibility set (PPS) is defined as $T = \{(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{x} \text{ can produce } \boldsymbol{y}\}$. Considering $(\boldsymbol{g}^-, \boldsymbol{g}^+)$ as a directional vector, the following DDF can be defined for DMU_p [32]:

$$\vec{D}\left(\boldsymbol{x}_p, \boldsymbol{y}_p, \boldsymbol{g}^-, \boldsymbol{g}^+\right) = \sup\{\beta | \left(\boldsymbol{x}_p - \beta \boldsymbol{g}^-, \boldsymbol{y}_p + \beta \boldsymbol{g}^+\right) \in T\}$$

There are two types of DDF approaches in the literature: radial and non-radial. The radial DDF approach assumes that inputs are contracted, and outputs are expanded at the same rate (see [17,18] and [68]). In contrast, the non-radial DDF approaches abandon the proportionate contraction assumption in the inputs and aim at obtaining the maximum rate of reduction in the inputs that may forgo varying proportions of original input resources (see [83,89], and [65]). Zhou et al. [89] compares and contrasts the radial and no-radial DDFs. For an in-depth discussion of radial and no-radial DDF, please see Arabi et al. [10] and Portela et al. [66].

The DDF approach can be used with any directional vector. The DMU's observed inputs and outputs are often used as the directional vector [32]. However, using such directional vectors is problematic because the directional vector's negative components lead to worse rather than better values for the input or output when a positive step length is taken in the vector's direction.

One way to overcome this problem is to use a fixed directional vector-like unit directional vector [35], or average input and output vector [61]. However, applying directions that are based on

slacks can be regarded as useful tools for considering negative data. Aparicio et al. [8] developed the following weighted additive model that measures the technical inefficiency scores using the input and output slacks.

$$\begin{aligned}
&\max \sum_{i=1}^{m} w_i^- s_{ip}^- + \sum_{r=1}^{s} w_r^+ s_{rp}^+ \\
&\text{s.t.} \\
&\sum_{j=1}^{n} \lambda_j x_{ij} = x_{ip} - s_{ip}^-, && \forall i, \\
&\sum_{j=1}^{n} \lambda_j y_{rj} = y_{rp} + s_{rp}^+, && \forall r, \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\lambda_j, s_{ip}^-, s_{rp}^+ \geq 0, && \forall j,\ \forall i,\ \forall r,
\end{aligned} \tag{1}$$

where $\boldsymbol{w}^- = (w_1^-, \ldots, w_m^-)$ and $\boldsymbol{w}^+ = (w_1^+, \ldots, w_s^+)$ are the weights representing the relative importance of the unit inputs and the unit outputs. Fukuyama and Weber [36] developed a directional slacks-based measure of technical inefficiency, which can be obtained from Model (1) if one sets $w_i^- = \frac{1}{2mg_i^-} \forall i$ and $w_r^+ = \frac{1}{2sg_r^+}\ \forall r$.

The following model of Portela et al. [68] can be obtained from Model (1) by setting $s_{ip}^- = \beta R_{ip}^-$, $w_i^- = \frac{1}{2mR_{ip}^-} \forall i$ and $s_{rp}^+ = \beta R_{rp}^+$, $w_r^+ = \frac{1}{2sR_{rp}^+}\ \forall r$:

$$\begin{aligned}
&\max \beta \\
&\text{s.t.} \\
&\sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{ip} - \beta R_{ip}^-, && \forall i, \\
&\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rp} + \beta R_{rp}^+ && \forall r, \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\lambda_j, \beta \geq 0, && \forall j.
\end{aligned} \tag{2}$$

The RDM Model (2) can take both positive or negative input or output data by improving the performance of the DMUs based on a DDF model. However, this model has the following drawbacks:

(I) Model (2) cannot guarantee projections on the Pareto efficient frontier, as happens with the classical radial DEA model [11], and
(II) Model (2) fails to spot the difference between the strong- and weak-efficient (inefficient) DMUs [29].

We capture these drawbacks by employing the following non-radial DDF model, which has an intuitive appeal direction rather than a fixed direction in the RDM Model (2):

$$\begin{aligned}
&\rho_p^* = \min 1 - \tfrac{1}{m+s}\left(\sum_{i=1}^{m} \tfrac{\theta_i}{\bar{\theta}_{ip}} + \sum_{r=1}^{s} \tfrac{\varphi_r}{\bar{\varphi}_{rp}} \right) \\
&\text{s.t.} \\
&\sum_{j=1}^{n} \lambda_j x_{ij} = x_{ip} - \theta_i g_{ip}^-, && \forall i, \\
&\sum_{j=1}^{n} \lambda_j y_{rj} = y_{rp} + \varphi_r g_{rp}^+, && \forall r, \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\lambda_j, \theta_i, \varphi_r \geq 0, && \forall j,\ \forall i,\ \forall r.
\end{aligned} \tag{3}$$

Note that the above model can be obtained from Model (1) if we set $s_{ip}^- = \theta_i g_{ip}^-$, $w_i^- = \frac{1}{g_{ip}^- \bar{\theta}_{ip}}$ $(\forall i)$ and $s_{rp}^+ = \varphi_r g_{rp}^+$, $w_r^+ = \frac{1}{g_{rp}^+ \bar{\varphi}_{rp}}$ $(\forall r)$ where:

$$\bar{\theta}_{ip} = \frac{x_{ip} - \min\{x_{ij} :\ \forall j\}}{g_{ip}^-},\ (\forall i) \text{ and}$$
$$\bar{\varphi}_{rp} = \frac{\max\{y_{rj} : \forall j\} - y_{rp}}{g_{rp}^+},\ (\forall r). \tag{4}$$

Parameters (4) are defined as $\theta_i \leq \bar{\theta}_{ip}$, $(\forall i)$ and $\varphi_r \leq \bar{\varphi}_{rp}$, $(\forall r)$ to guarantee the objective function lies between zero and one.

We should note that Model (3) turns to an input (output) orientation model if $g_{rp}^+ = 0$, $\forall r$ $(g_{ip}^- = 0,\ \forall i)$. Moreover, it is easy to verify that Model (3) is always feasible because $(\boldsymbol{\lambda}, \boldsymbol{\theta}, \phi) = (\boldsymbol{e}_p, 0_m, 0_s)$ is a feasible solution where $\boldsymbol{e}_p$ is the pth unit vector[3] and 0_m is the zero vector (origin) in $\mathbb{R}^m$. According to (4), if $\bar{\theta}_{ip} = 0$, then, $x_{ip} = \min_j\{x_{ij}\}$ and thus the ith input has minimum input in PPS and therefore we cannot improve it. Consequently, we can remove its related constraint. Same is true for $\bar{\varphi}_{rp}$. Additionally, considering the optimality of Model (3), we have $0 \leq \rho_p^* \leq 1$.

Next, we consider the case of negative data. The use of observed input and output levels will violate the first constraint in Model (3) in the presence of negative data, which is intended to ensure inputs and outputs do not deteriorate from their observed levels in the solution yielded by the model [68]. We modify Model (3) by using the ideal point concept and ensuring that it improves the solutions even when some of the data are negative. Thus, by setting $g_{ip}^- = R_{ip}^-$ $(\forall i)$ and $g_{rp}^+ = R_{rp}^+$ $(\forall r)$, we propose the following non-radial model:

$$\begin{aligned}
&\xi_p^* = \min 1 - \tfrac{1}{m+s}\left(\sum_{\substack{i=1 \\ R_{ip}^- > 0}}^{m} \theta_i + \sum_{\substack{r=1 \\ R_{rp}^+ > 0}}^{s} \varphi_r \right) \\
&\text{s.t.} \\
&\sum_{j=1}^{n} \lambda_j x_{ij} = x_{ip} - \theta_i R_{ip}^-, && \forall i, \\
&\sum_{j=1}^{n} \lambda_j y_{rj} = y_{rp} + \varphi_r R_{rp}^+, && \forall r, \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\lambda_j, \theta_i, \varphi_r \geq 0, && \forall j,\ \forall i,\ \forall r.
\end{aligned} \tag{5}$$

where $R_{ip}^- = x_{ip} - \min_j\{x_{ij}\}$, $(\forall i)$ and $R_{rp}^+ = \max_j\{y_{rj}\} - y_{rp}$, $(\forall r)$ are the lower-sided range for inputs and the upper-sided range for output, respectively. From Relation (4), it is concluded that $\bar{\theta}_{ip} = \frac{x_{ip} - \min_j\{x_{ij}\}}{R_{ip}^-} = 1$ $(\forall i)$ and $\bar{\varphi}_{rp} = \frac{\max_j\{y_{rj}\} - y_{rp}}{R_{rp}^+} = 1$ $(\forall r)$. Therefore, we have $\theta_i, \varphi_r \leq 1$, $(\forall i, \forall r)$. From the above discussion, it is obvious that $0 \leq \xi_p^* \leq 1$.

The following properties point shows the proposed model can overcome some of the drawbacks in radial models.

Property 1. If DMU_A dominates DMU_B, then $\xi_B^* \leq \xi_A^*$.

Property 2. Model (5) projects each inefficient DMU on the strong-efficient frontier.

In other words, Property 1 shows that Model (5) can correctly recognize all kinds of inefficiencies, and Property 2 can help managers to find a convenient target for inefficient DMUs.

3.2. Classifier models

Assuming the data involves L flexible measures $\boldsymbol{z}_j = (z_{1j}, \ldots, z_{lj})$ with unknown input/output status to be chosen,

[3] A vector having zero components except for a 1 in the pth position.

the following mixed binary linear programming (MBLP) model classifies flexible measures in the multiplier form 1,25:

$$
\begin{aligned}
& e_p^* = \max \sum_{r=1}^{s} \mu_r y_{rp} + \sum_{l=1}^{L} \delta_l z_{lp} \\
& \text{s.t.} \\
& \sum_{i=1}^{m} v_i x_{ip} + \sum_{l=1}^{L} \gamma_l z_{lp} = 1, \\
& \sum_{r=1}^{s} \mu_r y_{rj} + \sum_{l=1}^{L} \delta_l z_{lj} - \sum_{i=1}^{m} v_i x_{ij} - \sum_{l=1}^{L} \gamma_l z_{lj} \le 0, && \forall j, \\
& 0 \le \delta_l \le M d_l, && \forall l, \\
& 0 \le \gamma_l \le M(1-d_l), && \forall l, \\
& d_l \in \{0,1\}, && \forall l, \\
& \mu_r, v_i, \gamma_l, && \forall i, \forall r, \forall l,
\end{aligned} \tag{6}
$$

where M is a large positive number, and γ_l and δ_l are the weights of the lth flexible measure when it is treated as input or output, respectively. The indicator binary variables $\boldsymbol{d} = (d_1, \ldots, d_L)$ accommodates flexible measures: if $d_l = 0$, then from the constraint $0 \le \delta_l \le M d_l$ we obtain δ_l which follows z_l plays an input role; otherwise (if $d_l = 1$), we get $\gamma_l = 0$ from $0 \le \gamma_l \le M(1-d_l)$ which means z_l plays an output role.

Toloo [78] has shown Model (6) might produce incorrect results because of a computational issue due to introducing a large positive number in the model. In response, Toloo [79] proposed the following alternative envelopment form of classifier DEA model:

$$
\begin{aligned}
& \min \theta_p \\
& \text{s.t.} \\
& \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta_p x_{ip}, && \forall i, \\
& \sum_{j=1}^{n} \lambda_j z_{lj} \le \theta_p z_{lp} + M d_l, && \forall l, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{rp}, && \forall r, \\
& \sum_{j=1}^{n} \lambda_j z_{lj} \ge z_{lp} - M(1-d_l), && \forall l, \\
& d_l \in \{0,1\}, && \forall l, \\
& \lambda_j \ge 0, && \forall j.
\end{aligned} \tag{7}
$$

Here, if $d_l = 0$, then z_l plays an input role because the constraint $\sum_{j=1}^{n} \lambda_j z_{lj} \le \theta_p z_{lp} + M d_l$ ($\sum_{j=1}^{n} \lambda_j z_{lj} \le \theta_p z_{lp}$) is active; otherwise, the active constraint $\sum_{j=1}^{n} \lambda_j z_{lj} \ge z_{lp}$ follows that z_l plays an output role. In contrast to conventional DEA models where the multiplier and envelopment forms are mutually dual, the envelopment classifier model is not dual of the multiplier classifier model. The optimal value of Model (7) is a lower bound for those of Model (6), i.e., $\theta_p^* \le e_p^*$ [80].

4. Proposed method

This section suggests a pair of pessimistic and optimistic classifier models for extending and enabling the DDF Model (5) to accommodate flexible measures. In this respect, we propose DEA models that can not only handle negative data but also deal with flexible measures. The technological constraints in Model (5) are in equation form, which may raise some difficulties in imposing binary variables to the model for including flexible measures. The following theorem proves the technological constraints in Model (5) can be considered as inequality:

Theorem 1. *Model (5) is equivalent to the following model.*

$$
\begin{aligned}
& \eta_p^* = \min 1 - \tfrac{1}{m+s} \left(\sum_{\substack{i=1 \\ R_{ip}^- > 0}}^{m} \theta_i + \sum_{\substack{r=1 \\ R_{rp}^+ > 0}}^{s} \varphi_r \right) \\
& \text{s.t.} \\
& \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{ip} - \theta_i R_{ip}^-, && \forall i, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{rp} + \varphi_r R_{rp}^+, && \forall r, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j, \theta_i, \varphi_r \ge 0, && \forall j, \ \forall i, \ \forall r.
\end{aligned} \tag{8}
$$

Proof.

It is sufficient to prove that all $m+s$ technological constraints of Model (8) are binding at optimality. Assume that the vector $(\boldsymbol{\lambda}^*, \boldsymbol{\theta}^*, \boldsymbol{\varphi}^*) \in \mathbb{R}^{n+m+s}$ is an optimal solution for Model (8). In contrast to our claim, suppose that the technological constraint for the tth input is in inequality form (and without loss of generality all the other constraints are in the equation form):

$$
\begin{aligned}
& \sum_{j=1}^{n} \lambda_j^* x_{tj} < x_{tp} - \theta_t^* R_{tp}^-, \\
& \sum_{j=1}^{n} \lambda_j^* x_{ij} = x_{ip} - \theta_i^* R_{ip}^-, && \forall i \ne t, \\
& \sum_{j=1}^{n} \lambda_j^* y_{rj} = y_{rp} + \varphi_r^* R_{rp}^+, && \forall r, \\
& \sum_{j=1}^{n} \lambda_j^* = 1.
\end{aligned}
$$

Obviously, there exist an $\alpha > 1$ such that $\sum_{j=1}^{n} \lambda_j^* x_{tj} = x_{tp} - \alpha \theta_t^* R_{tp}^-$. Let $\bar{\boldsymbol{\theta}} = (\theta_1^*, \ldots, \theta_{t-1}^*, \alpha\theta_t^*, \theta_{t+1}^*, \ldots, \theta_m^*)$. An inspection makes it clear that $(\boldsymbol{\lambda}^*, \bar{\boldsymbol{\theta}}, \boldsymbol{\varphi}^*)$ is a feasible solution for Model (8). The following expression points out the objective function of Model (8) for $(\boldsymbol{\lambda}^*, \bar{\boldsymbol{\theta}}, \boldsymbol{\varphi}^*)$ is less than those for the optimal solution which is impossible:

$$
1 - \frac{1}{m+s} \left(\sum_{\substack{i=1 \\ R_{ip}^- > 0}}^{m} \bar{\theta}_i + \sum_{\substack{r=1 \\ R_{rp}^+ > 0}}^{s} \varphi_r^* \right) < 1
$$

$$
- \frac{1}{m+s} \left(\sum_{\substack{i=1 \\ R_{ip}^- > 0}}^{m} \theta_i^* + \sum_{\substack{r=1 \\ R_{rp}^+ > 0}}^{s} \varphi_r^* \right) = \eta_p^*
$$

Analogously, we can verify that the technological constraint for all outputs must be in equation form. As a result, all constraints in Model (8) are in equation form at optimality, and Models (5) and (8) are equivalent. □

4.1. Flexible measures: pessimistic approach

If the lth flexible measure is treated as an input, then the input-related constraint $\sum_{j=1}^{n} \lambda_j z_{lj} \le \theta_f z_{lp}$ should be imposed on Model (8); otherwise, the output-related constraint $\sum_{j=1}^{n} \lambda_j z_{lj} \ge \varphi_f z_{lp}$ should be added to the model. As a result, a choice should be made between the input-related constraint $\sum_{j=1}^{n} \lambda_j z_{lj} \le \theta_f z_{lp}$ and the output-related constraint $\sum_{j=1}^{n} \lambda_j z_{lj} \ge \varphi_f z_{lp}$, so that *only*

one(either one) holds. The condition that only one of the constraints must hold cannot be formulated in an LP model because, in a linear program, all constraints must be satisfied. To tackle this issue, an indicator binary variable d_l can be used to express the problem. The constraints can be rewritten as below:

$$\begin{aligned} &\sum_{j=1}^{n} \lambda_j z_{lj} \le \theta_f z_{lp} + M d_l, && \forall l, \\ &\sum_{j=1}^{n} \lambda_j z_{lj} \ge \varphi_f z_{lp} - M(1-d_l), && \forall l, \\ &d_l \in \{0,1\}, && \forall l, \end{aligned} \tag{9}$$

where M is a large enough number. When $d_l = 0$, constraint $\sum_{j=1}^{n} \lambda_j z_{lj} \le \theta_f z_{lp} + M d_l$ is imposed which is, in fact, the input-related constraint $\sum_{j=1}^{n} \lambda_j z_{lj} \le \theta_f z_{lp}$, and constraint $\sum_{j=1}^{n} \lambda_j z_{lj} \ge \varphi_f z_{lp} - M(1-d_l)$ is weakened to $\sum_{j=1}^{n} \lambda_j z_{lj} \ge \varphi_f z_{lp} - M$ which will always be non-binding and in conclusion the output-related constraint $\sum_{j=1}^{n} \lambda_j z_{lj} \ge \varphi_f z_{lp}$ is redundant. Analogously, it is easy to verify that when $d_l = 1$, the input-related constraint $\sum_{j=1}^{n} \lambda_j z_{lj} \le \theta_f z_{lp}$ is redundant, while, the output-related constraint $\sum_{j=1}^{n} \lambda_j z_{lj} \le \theta_f z_{lp}$ is binding. In summary:

$$d_l = \begin{cases} 0, & z_l \text{ acts as an input} \\ 1, & z_l \text{ acts as an output} \end{cases}$$

Therefore, we formulate the following mixed binary non-linear programming (MBNLP) model in the presence of flexible measures:

$$\underline{\xi}_p^* = \min 1 - \frac{1}{m+s+L}\left(\left(\begin{array}{c} \sum_{i=1}^{m} \theta_i + \sum_{l=1}^{L} (1-d_l)\bar{\theta}_l \\ R_{ip}^- > 0 \qquad \bar{R}_{lp}^- > 0 \end{array}\right) + \left(\begin{array}{c} \sum_{r=1}^{s} \varphi_r + \sum_{l=1}^{L} d_l \bar{\varphi}_l \\ R_{rp}^+ > 0 \qquad \bar{R}_{lp}^+ > 0 \end{array}\right)\right)$$

s.t.

$$\begin{aligned} &\sum_{j=1}^{n} \lambda_j x_{ij} \le x_{ip} - \theta_i R_{ip}^-, && \forall i, \\ &\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{rp} + \varphi_r R_{rp}^+, && \forall r, \\ &\sum_{j=1}^{n} \lambda_j z_{lj} \le z_{lp} - \bar{\theta}_l \bar{R}_{lp}^- + M d_l, && \forall l, \\ &\sum_{j=1}^{n} \lambda_j z_{lj} \ge z_{lp} + \bar{\varphi}_l \bar{R}_{lp}^+ - M(1-d_l), && \forall l, \\ &\sum_{j=1}^{n} \lambda_j = 1, \\ &d_l \in \{0,1\}, && \forall l, \\ &\lambda_j \ge 0, && \forall l, \end{aligned} \tag{10}$$

where $\bar{R}_{lp}^- = z_{lp} - \min_j\{z_{lj}\}$, $(\forall l)$, $\bar{R}_{lp}^+ = \max_j\{z_{lj}\} - z_{lp}$, $(\forall l)$.

The MBNLP Model (10) involves the parameter M which its value must be set before solving the model. On the one hand, if the value of M is too small, then, suboptimal solutions or infeasibility issues may occur. On the other hand, unrealistically large values of M may result in some computational problems, such as round-off error. According to the third and fourth set of constraints, we have $z_{lp} + \bar{\varphi}_l \bar{R}_{lp}^+ - M(1-d_l) \le \sum_{j=1}^{n} \lambda_j z_{lj} \le z_{lp} - \bar{\theta}_l \bar{R}_{lp}^- + M d_l$ which leads to $z_{lp} - \bar{\theta}_l \bar{R}_{lp}^- + M d_l \ge z_{lp} + \bar{\varphi}_l \bar{R}_{lp}^+ - M(1-d_l)$ or equvalenetly $M \ge \bar{\theta}_l \bar{R}_{lp}^- + \bar{\varphi}_l \bar{R}_{lp}^+$. Since $0 \le \bar{\theta}_l, \bar{\varphi}_l \le 1$ $\forall l$, we can select $M \ge \max_{l,j}\{\bar{R}_{lj}^- + \bar{R}_{lj}^+\}$.

We need to eliminate the nonlinear terms $d_l \bar{\theta}_l$ and $d_l \bar{\varphi}_l$ in the objective function. Toward this end, we let $\Theta_l = (1-d_l)\bar{\theta}_l$ and $\Phi_l = d_l \bar{\varphi}_l$ and impose the following constraints to the model:

$$\begin{aligned} &0 \le \Theta_l \le M(1-d_l) && 0 \le \Phi_l \le M d_l \\ &\Theta_l \le \bar{\theta}_l \le \Theta_l + M d_l && \Phi_l \le \bar{\varphi}_l \le \Phi_l + M(1-d_l) \end{aligned}$$

An easy verification clarifies that if $d_l = 0$, then $\Theta_l = \Phi_l = 0$; otherwise $\Theta_l = \bar{\theta}_l$ and $\Phi_l = \bar{\varphi}_l$.

$$\underline{\xi}_p^* = \min 1 - \frac{1}{m+s+L}\left(\begin{array}{c} \sum_{i=1}^{m} \theta_i + \sum_{r=1}^{s} \varphi_r + \sum_{l=1}^{L} (\Theta_l + \Phi_l) \\ R_{ip}^- > 0 \qquad R_{rp}^+ > 0 \qquad \bar{R}_{lp}^-, \bar{R}_{lp}^+ > 0 \end{array}\right)$$

s.t.

$$\begin{aligned} &\sum_{j=1}^{n} \lambda_j x_{ij} \le x_{ip} - \theta_i R_{ip}^-, && \forall i, \\ &\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{rp} + \varphi_r R_{rp}^+, && \forall r, \\ &\sum_{j=1}^{n} \lambda_j z_{lj} \le z_{lp} - \bar{\theta}_l \bar{R}_{lp}^- + M d_l, && \forall l, \\ &\sum_{j=1}^{n} \lambda_j z_{lj} \ge z_{lp} + \bar{\varphi}_l \bar{R}_{lp}^+ - M(1-d_l), && \forall l, \\ &0 \le \Theta_l \le M(1-d_l), && \forall l, \\ &0 \le \Theta_l \le M(1-d_l), && \forall l, \\ &\bar{\theta}_l - M d_l \le \Theta_l \le \bar{\theta}_l, && \forall l, \\ &0 \le \Phi_l \le M d_l, && \forall l, \\ &\bar{\varphi}_l - M(1-d_l) \le \Phi_l \le \bar{\varphi}_l, && \forall l, \\ &\sum_{j=1}^{n} \lambda_j = 1, \\ &d_l \in \{0,1\}, && \forall l, \\ &\lambda_j \ge 0, && \forall l. \end{aligned} \tag{11}$$

Note that from the constraints of Model (11), we have (i) $0 \le \theta_i \le 1$ and hence $0 \le \sum_{i=1}^{m} \theta_i \le m$; (ii) $0 \le \phi_r \le 1$ and hence $0 \le \sum_{r=1}^{s} \varphi_r \le m$; (iii) from $\Theta_l = (1-d_l)\bar{\theta}_l$ we obtain $0 \le \bar{\theta}_i \le 1$; and (iv) from $\Phi_l = d_l \bar{\varphi}_l$ we have $0 \le \bar{\phi}_r \le 1$. As a result, $0 \le \sum_{l=1}^{L} \Theta_l + \sum_{l=1}^{L} \Phi_l \le L$ which results in $0 \le (\sum_{i=1}^{m} \theta_i + \sum_{l=1}^{L} \Theta_l + \sum_{l=1}^{L} \Phi_l) + (\sum_{r=1}^{s} \varphi_r + \sum_{l=1}^{L} \Phi_l) \le 1$. In other words, the objective function value of Model (11) is positive and at most 1, i.e., $\underline{\xi}_p^* \in [0,1]$.

In the presence of L flexible measures, there are 2^L various combinations (cases) for flexible measures [80]. Let e_k^* $(k = 1, .., 2^L)$ be the optimal objective value for Model (5) for each case. The following theorem compares the MBLP Model (10), or equivalently Model (11), with the LP Model (5):

Theorem 2. $\underline{\xi}_p^* = \min\{\xi_k^* k = 1, \ldots, 2^L\}$.

Proof.

Let $(\boldsymbol{\lambda}^*, \boldsymbol{\theta}^*, \bar{\boldsymbol{\theta}}^*, \phi^*, \bar{\phi}^*, \boldsymbol{d}^*) \in R^{n+m+s+3L}$ be the optimal solution for Model (10). Let $IN = \{l : d_l^* = 0\}$ and $OUT = \{l : d_l^* = 1\}$. It is

easy to verify that the optimal solution of Model (10) is also an optimal solution for the following model:

$$\xi_p^* = \min 1 - \frac{1}{m+s+L}\left(\begin{array}{cccc} \sum_{i=1}^{m} \theta_i + & \sum_{r=1}^{s} \varphi_r + & \sum_{l=1}^{L} \bar{\theta}_l + & \sum_{l=1}^{L} \bar{\varphi}_l \\ R_{ip}^- > 0 & R_{rp}^+ > 0 & \bar{R}_{lp}^- > 0 & \bar{R}_{lp}^+ > 0 \end{array}\right)$$

s.t.

$$\begin{aligned} &\sum_{j=1}^{n} \lambda_j x_{ij} \le x_{ip} - \theta_i R_{ip}^-, && \forall i, \\ &\sum_{j=1}^{n} \lambda_j z_{ij} \le z_{lp} - \bar{\theta}_l \bar{R}_{lp}^-, && \forall l \in IN, \\ &\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{rp} + \varphi_r R_{rp}^+, && \forall r, \\ &\sum_{j=1}^{n} \lambda_j z_{ij} \ge z_{lp} - \bar{\phi}_l \bar{R}_{lp}^+, && \forall l \in OUT, \\ &\sum_{j=1}^{n} \lambda_j = 1, \\ &\lambda_j \ge 0, && \forall j. \end{aligned} \tag{12}$$

Hence, we have $\underline{\xi_p^*} \in \{\xi_1^*, \ldots, \xi_{2L}^*\}$ and what is left is to prove $\underline{\xi_p^*} = \min\{\xi_1^*, \ldots, \xi_{2L}^*\}$. Let IN' and OUT' be the set of flexible measures that has been treated as input and output, respectively, when $\min\{\xi_1^*, \ldots, \xi_{2L}^*\}$ occurs. In contrast, assume that $\underline{\xi_p^*} > \xi_p^{'*} = \min\{\xi_1^*, \ldots, \xi_{2L}^*\}$ where $\xi_p^{'*}$ is the optimal objective value of the following model:

$$\xi_p^{'*} = \min 1 - \frac{1}{m+s+L}\left(\begin{array}{cccc} \sum_{i=1}^{m} \theta_i + & \sum_{r=1}^{s} \varphi_r + & \sum_{l \in IN'} \bar{\theta}_l + & \sum_{l \in OUT'} \bar{\phi}_l \\ R_{ip}^- > 0 & R_{rp}^+ > 0 & \bar{R}_{lp}^- > 0 & \bar{R}_{lp}^+ > 0 \end{array}\right)$$

s.t.

$$\begin{aligned} &\sum_{j=1}^{n} \lambda_j x_{ij} \le x_{ip} - \theta_i R_{ip}^-, && \forall i, \\ &\sum_{j=1}^{n} \lambda_j z_{ij} \le z_{lp} - \bar{\theta}_l \bar{R}_{lp}^-, && \forall l \in IN', \\ &\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{rp} + \varphi_r R_{rp}^+, && \forall r, \\ &\sum_{j=1}^{n} \lambda_j z_{ij} \ge z_{lp} - \bar{\phi}_l \bar{R}_{lp}^+, && \forall l \in OUT', \\ &\sum_{j=1}^{n} \lambda_j = 1, \\ &\lambda_j >= 0 && \forall j. \end{aligned} \tag{13}$$

Let the optimal solution of Model (13) be $(\boldsymbol{\lambda}^{*'}, \boldsymbol{\theta}^{*'}, \bar{\boldsymbol{\theta}}^{*'}, \phi^{*'}, \bar{\phi}^{*'})(\boldsymbol{\lambda}^{*'}, \boldsymbol{\theta}^{*'}, \bar{\boldsymbol{\theta}}^{*'}, \phi^{*'}, \bar{\phi}^{*'})$. We define

$$\hat{\phi}_l = \begin{cases} \bar{\phi}_l'^{*}, & l \in OUT' \\ 0, & \text{Otherwise} \end{cases}; \hat{\theta}_l = \begin{cases} \bar{\theta}_l'^{*}, & l \in IN' \\ 0, & \text{Otherwise} \end{cases};$$

$$\hat{d}_l = \begin{cases} 1, & l \in OUT' \\ 0, & l \in IN' \end{cases}.$$

A straightforward computation clarifies that $(\boldsymbol{\lambda}^{*'}, \boldsymbol{\theta}^{*'}, \hat{\boldsymbol{\theta}}^{*'}, \phi^{*'}, \hat{\phi}^{*'}, \hat{\boldsymbol{d}}) \in \mathbb{R}^{n+m+s+3L}$ is a feasible solution for Model (11) with an objective function, which is less than the optimal solution of Model (11). This contradiction completes the proof. □

Theorem 2 shows that the proposed Model (11) obtains the minimum value of efficiency score for DMU_p among all the possible efficiency scores, which can be achieved by various combinations for flexible measures. Next, we provided another approach for accommodating the flexible measures in measuring the maximum value of efficiency score for DMU_p.

4.2. Flexible measures: optimistic approach

To find the efficiency scores of DMUs in the presence of negative data based on the optimistic viewpoint, we formulate the following dual model of Model (5) which is in the multiplier form:

$$\tau_p^* = \max 1 + \sum_{r=1}^{s} u_r y_{rp} - \sum_{i=1}^{m} v_i x_{ip} + u_0 - M\left(\sum_{r=1}^{s} h_r R_{rp}^+ + \sum_{r=1}^{s} w_i R_{ip}^-\right)$$

s.t.

$$\begin{aligned} &\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_0 \le 0, && \forall j, \\ &u_r R_{rp}^+ + h_r \ge \frac{1}{m+s}, && \forall r, \\ &v_i R_{ip}^- + w_i \ge \frac{1}{m+s}, && \forall i, \\ &u_r,\ v_i,\ h_r,\ w_i \ge 0, && \forall j,\ \forall i,\ \forall r, \end{aligned} \tag{14}$$

where v_i is the weights for the ith input along with R_{ip}^- and u_r are the weights for r^{th} output. Moreover, the second and third set of constraints show that each pair of variables (u_r, h_r) and (v_i, w_i) cannot be zero. These constraints show how these weights can vary based on the values of the lower-sided range for the inputs and the upper-sided range for the outputs. The term $(\sum_{r=1}^{s} h_r R_{rp}^+ + \sum_{r=1}^{s} w_i R_{ip}^-)$ in the objective function can be viewed as a penalty function. We can intuitively see that an optimal solution to the above problem must have $(\sum_{r=1}^{s} h_r R_{rp}^+ + \sum_{r=1}^{s} w_i R_{ip}^-)$ as close as possible to zero, because otherwise, a large penalty $M(\sum_{r=1}^{s} h_r R_{rp}^+ + \sum_{r=1}^{s} w_i R_{ip}^-)$ will be incurred [13]. The following theorem guarantees that the feasible region of Model (14) is a nonempty set.

Theorem 3. *Model (14) is feasible.*

Proof. The proof is trivial.

To extend the proposed Model (14) in the presence of flexible measures, let us again assume that for each DMU_j there are L flexible measures $\boldsymbol{z_j} = (z_{1j}, \ldots, z_{Lj})$. To consider the flexible factors, we define a binary variable d_l for each $l \in \{1, \ldots, L\}$ where $d_l = 1$ indicates that z_{lj} is an output, and $d_l = 0$ indicates that z_{lj} is an input factor. Therefore, we formulate the following mixed-binary non-linear programming model in the presence of flexible measures:

$$\bar{\xi}_p^* = \max\ 1 + \sum_{r=1}^{s} u_r y_{rp} + \sum_{l=1}^{L} d_l w_l^{out} z_{lp} - \sum_{i=1}^{m} v_i x_{ip} - \sum_{l=1}^{L} (1-d_l) w_l^{in} z_{lp} + u_0$$
$$-M\left(\sum_{r=1}^{s} h_r R_{rp}^+ + \sum_{l=1}^{L} d_l q_l^{out} \bar{R}_{lp}^+ + \sum_{r=1}^{s} w_i R_{ip}^- + \sum_{l=1}^{L} (1-d_l) q_l^{in} \bar{R}_{lp}^-\right)$$

s.t.

$$\begin{aligned} &\sum_{r=1}^{s} u_r y_{rj} + \sum_{l=1}^{L} d_l w_l^{out} z_{lj} - \sum_{i=1}^{m} v_i x_{ij} - \sum_{l=1}^{L} (1-d_l) w_l^{in} z_{lj} + u_0 \le 0, && \forall j, \\ &u_r R_{rp}^+ + h_r \ge \frac{1}{m+s+L}, && \forall r, \\ &w_l^{out} \bar{R}_{lp}^+ + q_l^{out} \ge \frac{d_l}{m+s+L}, && \forall l, \\ &v_i R_{ip}^- + w_i \ge \frac{1}{m+s+L}, && \forall i, \\ &w_l^{in} \bar{R}_{lp}^- + q_l^{in} \ge \frac{(1-d_l)}{m+s+L}, && \forall l, \\ &d_l \in \{0, 1\}, && \forall l, \\ &u_r,\ v_i,\ h_r,\ w_i \ge 0, && \forall i,\ \forall r, \\ &a_l,\ t_l,\ e_l,\ q_l \ge 0, && \forall l, \end{aligned} \tag{15}$$

where w_l^{in} and w_l^{out} are the weights for l^{th} flexible measure. We can see that Model (15) is nonlinear, and we need to eliminate the nonlinear terms $d_l w_l^{out}$, $(1-d_l)w_l^{in}$, $d_l q_l^{out}$, and $(1-d_l)q_l^{in}$ in the objective function and constraints. To do so, we let $\varphi_l^{out} = d_l w_l^{out}$, $\varphi_l^{in} = (1-d_l)w_l^{in}$, $\chi_l^{out} = d_l q_l^{out}$, and $\chi_l^{in} = (1-d_l)q_l^{in}$ and impose the following constraints into the model:

$$\begin{array}{lll} 0 \le \varphi_l^{out} \le M d_l, & ; & 0 \le \varphi_l^{in} \le M(1-d_l) \\ \varphi_l^{out} \le w_l^{out} \le \varphi_l^{out} + M(1-d_l), & ; & \varphi_l^{in} \le w_l^{out} \le \varphi_l^{in} + M d_l \end{array}$$

$$\begin{array}{lll} 0 \le \chi_l^{out} \le M d_l, & ; & 0 \le \chi_l^{in} \le M(1-d_l) \\ \chi_l^{out} \le q_l^{out} \le \chi_l^{out} + M(1-d_l), & ; & \chi_l^{in} \le q_l^{in} \le \chi_l^{in} + M d_l \end{array}$$

An easy verification clarifies that if $d_l = 0$, then $\varphi_l^{out} = \chi_l^{out} = 0$ and $\varphi_l^{in} = w_l^{out}$, $\chi_l^{in} = q_l^{in}$; otherwise $\varphi_l^{in} = \chi_l^{in} = 0$ and $\varphi_l^{out} = w_l^{out}$, $\chi_l^{out} = q_l^{out}$. Therefore, we linearize Model (15) as below:

$$\begin{aligned}
&\bar{\xi}_p^* = \max 1 + \sum_{r=1}^{s} u_r y_{rp} + \sum_{l=1}^{L} \varphi_l^{out} z_{lp} - \sum_{i=1}^{m} v_i x_{ip} - \sum_{l=1}^{L} \varphi_l^{in} z_{lp} + u_0 \\
&\quad -M\left(\sum_{r=1}^{s} h_r R_{rp}^{+} + \sum_{l=1}^{L} \chi_l^{out} \bar{R}_{lp}^{+} + \sum_{r=1}^{s} w_i R_{ip}^{-} + \sum_{l=1}^{L} \chi_l^{in} \bar{R}_{lp}^{-}\right) \\
&\text{s.t.} \\
&\sum_{r=1}^{s} u_r y_{rj} + \sum_{l=1}^{L} \varphi_l^{out} z_{lj} - \sum_{i=1}^{m} v_i x_{ij} - \sum_{l=1}^{L} \varphi_l^{in} z_{lj} + u_0 \le 0, && \forall j, \\
&u_r R_{rp}^{+} + h_r \ge \tfrac{1}{m+s+L}, && \forall r, \\
&w_l^{out} \bar{R}_{lp}^{+} + q_l^{out} \ge \tfrac{d_l}{m+s+L}, && \forall l, \\
&v_i R_{ip}^{-} + w_i \ge \tfrac{1}{m+s+L}, && \forall i, \\
&w_l^{in} \bar{R}_{lp}^{-} + q_l^{in} \ge \tfrac{(1-d_l)}{m+s+L}, && \forall l, \\
&0 \le \varphi_l^{out} \le M d_l, && \forall l, \\
&\varphi_l^{out} \le w_l^{out} \le \varphi_l^{out} + M(1-d_l), && \forall l, \\
&0 \le \varphi_l^{in} \le M(1-d_l), && \forall l, \\
&\varphi_l^{in} \le w_l^{out} \le \varphi_l^{in} + M d_l, && \forall l, \\
&0 \le \chi_l^{out} \le M d_l, && \forall l, \\
&\chi_l^{out} \le q_l^{out} \le \chi_l^{out} + M(1-d_l), && \forall l, \\
&0 \le \chi_l^{in} \le M(1-d_l), && \forall l, \\
&\chi_l^{in} \le q_l^{in} \le \chi_l^{in} + M d_l, && \forall l, \\
&d_l \in \{0,1\}, && \forall l, \\
&u_r,\ v_i,\ h_r,\ w_i \ge 0, && \forall i,\ \forall r, \\
&w_l^{out},\ w_l^{in},\ q_l^{out}, q_l^{in} \ge 0, && \forall l.
\end{aligned} \tag{16}$$

The following theorem proves an important property about the efficiency concept in the proposed Model (16).

Theorem 4. $\bar{\xi}_p^* \in [0,1]$.

Proof.

First, we prove $\bar{\xi}_p^* \le 1$.

From the objective function of Model (16), we can see that the value of the statement $-M(\sum_{r=1}^{s} h_r R_{rp}^{+} + \sum_{l=1}^{L} \chi_l^{out} \bar{R}_{lp}^{+} + \sum_{r=1}^{s} w_i R_{ip}^{-} + \sum_{l=1}^{L} \chi_l^{in} \bar{R}_{lp}^{-}) \le 0$ and also we have $\sum_{r=1}^{s} u_r y_{rj} + \sum_{l=1}^{L} \varphi_l^{out} z_{lj} - \sum_{i=1}^{m} v_i x_{ij} - \sum_{l=1}^{L} \varphi_l^{in} z_{lj} + u_0 \le 0$. Hence, the objective function value of Model (16) is less than or equal to 1.

To prove $\bar{\xi}_p^* \ge 0$, it is sufficient to find a feasible solution with zero objective function value since the model is a max-type optimization problem. For this purpose, we consider two cases as follows:

Case (*i*): $R_{rp}^{+} \ne 0,\ \forall r;\ R_{ip}^{-} \ne 0,\ \forall i;\ \bar{R}_{lp}^{+}, \bar{R}_{lp}^{-} \ne 0,\ \forall l;$

In this case, we define $(\hat{\boldsymbol{h}}, \hat{\boldsymbol{q}}^{out}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{q}}^{in}) = (0_s, 0_L, 0_s, 0_L)$, and $(\hat{\boldsymbol{\chi}}^{out}, \hat{\boldsymbol{\chi}}^{in}) = (0_L, 0_L)$. According to these values, we define $\hat{u}_r$, $\hat{w}_l^{out}$, $\hat{v}_i$ and $\hat{w}_l^{in}$ as follows:

- $\hat{u}_r R_{rp}^{+} = \frac{1}{m+s+L}$, therefore we have $\hat{u}_r\{\max_j y_{rj} - y_{rp}\} = \frac{1}{m+s+L}$ and thus $\hat{u}_r\{\max_j y_{rj}\} - \hat{u}_r y_{rp} = \frac{1}{m+s+L}$. Now, by summation over r, we obtain $\sum_{r=1}^{s} \hat{u}_r\{\max_j y_{rj}\} - \sum_{r=1}^{s} \hat{u}_r y_{rp} = \frac{s}{m+s+L}$.
- $\hat{w}_l^{out} \bar{R}_{lp}^{+} = \frac{d_l}{m+s+L}$, in a similar manner, we obtain $\sum_{l=1}^{L} \hat{w}_l^{out}\{\max_j z_{lj}\} - \sum_{l=1}^{L} \hat{w}_l^{out} z_{lp} = \frac{\sum_{l=1}^{L} d_l}{m+s+L}$.
- $\hat{v}_i R_{ip}^{-} = \frac{1}{m+s+L}$, analogously we have $\sum_{i=1}^{m} \hat{v}_i x_{ip} - \sum_{i=1}^{m} \hat{v}_i\{\min_j x_{ij}\} = \frac{m}{m+s+L}$.
- $\hat{w}_l^{in} \bar{R}_{lp}^{-} = \frac{(1-d_l)}{m+s+L}$, similarly, we obtain $\sum_{l=1}^{L} \hat{w}_l^{in} z_{lp} - \sum_{l=1}^{L} \hat{w}_l^{in}\{\min_j z_{lj}\} = \frac{\sum_{l=1}^{L}(1-d_l)}{m+s+L}$.

By summation of all the above four relations we have:

$$\begin{aligned}
&\sum_{r=1}^{s} \hat{u}_r \left\{\max_j y_{rj}\right\} - \sum_{r=1}^{s} \hat{u}_r y_{rp} + \sum_{l=1}^{L} \hat{w}_l^{out} \left\{\max_j z_{lj}\right\} \\
&\quad - \sum_{l=1}^{L} \hat{w}_l^{out} z_{lp} + \sum_{i=1}^{m} \hat{v}_i x_{ip} - \sum_{i=1}^{m} \hat{v}_i \left\{\min_j x_{ij}\right\} + \sum_{l=1}^{L} \widehat{w_l^{in}} z_{lp} \\
&\quad - \sum_{l=1}^{L} \hat{w}_l^{in} \left\{\min_j z_{lj}\right\} = 1
\end{aligned}$$

And now, we define $\hat{u}_0$ as follows:

$$\begin{aligned}
\hat{u}_0 = &-\sum_{r=1}^{s} \hat{u}_r \left\{\max_j y_{rj}\right\} - \sum_{l=1}^{L} \hat{w}_l^{out} \left\{\max_j z_{lj}\right\} + \sum_{i=1}^{m} \hat{v}_i \left\{\min_j x_{ij}\right\} \\
&+ \sum_{l=1}^{L} \hat{w}_l^{in} \left\{\min_j z_{lj}\right\}
\end{aligned}$$

Finally, if for each $l \in \{1, \ldots, L\}$ we set $\hat{\varphi}_l^{out} = \hat{w}_l^{out}$ and $\hat{\varphi}_l^{in} = \hat{w}_l^{in}$. Then, we obtain the following relation:

$$\sum_{r=1}^{s} \hat{u}_r y_{rp} + \sum_{l=1}^{L} \hat{\varphi}_l^{out} z_{lp} - \sum_{i=1}^{m} \hat{v}_i x_{ip} - \sum_{l=1}^{L} \hat{\varphi}_l^{in} z_{lp} + \hat{u}_0 = -1$$

Therefore, according to the above relations, the vector $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{w}}^{\boldsymbol{out}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}^{\boldsymbol{in}}, \hat{u}_0, \hat{\boldsymbol{h}}, \hat{\boldsymbol{q}}^{\boldsymbol{out}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{q}}^{\boldsymbol{in}}, \hat{\boldsymbol{\chi}}^{\boldsymbol{out}}, \hat{\boldsymbol{\chi}}^{\boldsymbol{in}}, \hat{\boldsymbol{\varphi}}^{\boldsymbol{out}}, \hat{\boldsymbol{\varphi}}^{\boldsymbol{in}})$ is a feasible solution for Model (16) with the objective function value $\hat{\bar{\xi}}_p = 0$.

Case (*ii*): There is at least an index t such that $R^+_{tp}=0$, and/or an index h such that $R^-_{hp}=0$, and/or an index η such that $\bar{R}^+_{\eta p}=0$, and/or an index ζ such that $\bar{R}^-_{\zeta p}=0$. Without loss of generality, we consider that there is at least an index t such that $R^+_{tp}=0$. We set:

$$\tilde{h}_i=\begin{cases}\frac{1}{m+s+L}, & i=t\\ \hat{h}_i, & \forall i; i\neq t\end{cases}$$

$$\tilde{u}_r=\begin{cases}0, & r=t\\ \hat{u}_r, & \forall r\neq t\end{cases}$$

$$\tilde{u}_0=\hat{u}_0+\frac{y_{tp}}{m+s+L}$$

And define:

$$\left(\widetilde{\boldsymbol{w}^{out}},\tilde{\boldsymbol{v}},\widetilde{\boldsymbol{w}^{in}},\widetilde{\boldsymbol{q}^{out}},\tilde{\boldsymbol{w}},\widetilde{\boldsymbol{q}^{in}},\widetilde{\boldsymbol{\chi}^{out}},\widetilde{\boldsymbol{\chi}^{in}},\widetilde{\boldsymbol{\varphi}^{out}},\widetilde{\boldsymbol{\varphi}^{in}}\right)$$
$$=\left(\widehat{\boldsymbol{w}^{out}},\hat{\boldsymbol{v}},\widehat{\boldsymbol{w}^{in}},\widehat{\boldsymbol{q}^{out}},\hat{\boldsymbol{w}},\widehat{\boldsymbol{q}^{in}},\widehat{\boldsymbol{\chi}^{out}},\widehat{\boldsymbol{\chi}^{in}},\widehat{\boldsymbol{\varphi}^{out}},\widehat{\boldsymbol{\varphi}^{in}}\right)$$

Hence, the vector $(\tilde{\boldsymbol{u}},\widetilde{\boldsymbol{w}^{out}},\tilde{\boldsymbol{v}},\widetilde{\boldsymbol{w}^{in}},u_0,\tilde{\boldsymbol{h}},\widetilde{\boldsymbol{q}^{out}},\tilde{\boldsymbol{w}},\widetilde{\boldsymbol{q}^{in}},\widetilde{\boldsymbol{\chi}^{out}},\widetilde{\boldsymbol{\chi}^{in}},\widetilde{\boldsymbol{\varphi}^{out}},\widetilde{\boldsymbol{\varphi}^{in}})$ is another feasible solution with the objective function $\bar{\xi}_p=0$.

Finally, since the objective function is a maximum form, then we have $\bar{\xi}^*_p\geq 0$. As a result, in optimality we have $\bar{\xi}^*_p\in[0,1]$.□

Theorem 5. $\bar{\xi}^*_p=\max\{e^*_k k=1,\ldots,2^L\}$.

Proof.

Let $I=\{l:d^*=0\}$ and $R=\{l:d^*=1\}$ and consider the following model:

$$\bar{\xi}^*_p=\max\ 1+\sum_{r=1}^{s}\eta_r y_{rp}+\sum_{l\in R}\overline{w^{out}_l}z_{lp}-\sum_{i=1}^{m}\mu_i x_{ip}-\sum_{l\in I}\overline{w^{in}_l}z_{lp}+\pi_0$$
$$-M\left(\sum_{r=1}^{s}\delta_r R^+_{rp}+\sum_{l\in R}\overline{q^{out}_l}\bar{R}^+_{lp}+\sum_{r=1}^{s}\sigma_i R^-_{ip}+\sum_{l\in I}\overline{q^{in}_l}\bar{R}^-_{lp}\right)$$

s.t.

$$\begin{aligned}
&\sum_{r=1}^{s}\eta_r y_{rj}+\sum_{l\in R}\overline{w^{out}_l}z_{lj}-\sum_{i=1}^{m}\mu_i x_{ij}-\sum_{l\in I}\overline{w^{in}_l}z_{lj}+\pi_0\leq 0, && \forall j,\\
&\eta_r R^+_{rp}+\delta_r\geq\tfrac{1}{m+s+L}, && \forall r,\\
&\overline{w^{out}_l}\bar{R}^+_{lp}+\overline{q^{out}_l}\geq\tfrac{1}{m+s+L}, && l\in R,\\
&\mu_i R^-_{ip}+\sigma_i\geq\tfrac{1}{m+s+L}, && \forall i,\\
&\overline{w^{in}_l}\bar{R}^-_{lp}+\overline{q^{in}_l}\geq\tfrac{1}{m+s+L}, && l\in I,\\
&\eta_r,\ \mu_i,\ \delta_r,\ \sigma_i\geq 0, && \forall i,\ \forall r,\\
&\overline{w^{out}_l},\ \overline{w^{in}_l},\overline{q^{out}_l},\ \overline{q^{in}_l}\geq 0, && \forall l.
\end{aligned}\tag{17}$$

Assume that $(\eta^*_r,\mu^*_i,\delta^*_r,\sigma^*_i,\overline{w^{out*}_l},\ \overline{w^{in*}_l},\overline{q^{out*}_l},\ \overline{q^{in*}_l},\pi^*_0)$ is an optimal solution for Model (17). Clearly, we have $\eta^*_r=u^*_r,\ \forall r;\mu^*_i=v^*_i,\ \forall i;\ \delta^*_r=h^*_r,\ \forall r;\ w^*_i=\sigma^*_i,\ \forall i;\overline{w^{out*}_l}=w^{out*}_l,\ l\in R;\ \overline{w^{in*}_l}=w^{in*}_l,\ l\in I;\overline{q^{out*}_l}=q^{out*}_l,\ l\in R;\ \overline{q^{in*}_l}=q^{in*}_l,\ l\in I;$. Thus, we conclude that $\bar{\xi}^*_p=\xi^*_p$.

If we consider, z_{lj} $(\forall l\in R,)$ as outputs and $z_{lj}(\forall l\in I)$ as outputs, then Model (17) will be equal to Model (14). Hence, $\bar{\xi}^*_p\in\{\xi^*_1,\ldots,\xi^*_{2^N}\}$ and we have to prove that $\bar{\xi}^*_p=\max\{\xi^*_1,\ldots,\xi^*_{2^N}\}$. Suppose, contrary to our claim, that $\bar{\xi}^*_p<\xi^{**}_p$, where ξ^{**}_p is the optimal solution of the following model

$$\xi^{**}_p=\max\ 1+\sum_{r=1}^{s}\eta_r y_{rp}+\sum_{l\in OUT}\overline{w^{out}_l}z_{lp}-\sum_{i=1}^{m}\mu_i x_{ip}-\sum_{l\in IN}\overline{w^{in}_l}z_{lp}+\pi_0$$
$$-M\left(\sum_{r=1}^{s}\delta_r R^+_{rp}+\sum_{l\in OUT}\overline{q^{out}_l}\bar{R}^+_{lp}+\sum_{r=1}^{s}\sigma_i R^-_{ip}+\sum_{l\in IN}\overline{q^{in}_l}\bar{R}^-_{lp}\right)$$

s.t.

$$\begin{aligned}
&\sum_{r=1}^{s}\eta_r y_{rj}+\sum_{l\in OUT}\overline{w^{out}_l}z_{lj}-\sum_{i=1}^{m}\mu_i x_{ij}-\sum_{l\in IN}\overline{w^{in}_l}z_{lj}+\pi_0\leq 0, && \forall j,\\
&\eta_r R^+_{rp}+\delta_r\geq\tfrac{1}{m+s+L}, && \forall r,\\
&\overline{w^{out}_l}\bar{R}^+_{lp}+\overline{q^{out}_l}\geq\tfrac{1}{m+s+L}, && l\in OUT,\\
&\mu_i R^-_{ip}+\sigma_i\geq\tfrac{1}{m+s+L}, && \forall i,\\
&\overline{w^{in}_l}\bar{R}^-_{lp}+\overline{q^{in}_l}\geq\tfrac{1}{m+s+L}, && l\in IN,\\
&\eta_r,\ \mu_i,\ \delta_r,\ \sigma_i\geq 0, && \forall i,\ \forall r,\\
&\overline{w^{out}_l},\ \overline{w^{in}_l},\overline{q^{out}_l},\ \overline{q^{in}_l}\geq 0, && \forall l.
\end{aligned}\tag{18}$$

Let $(\eta^{**}_r,\mu^{**}_i,\delta^{**}_r,\sigma^{**}_i,\overline{w^{out**}_l},\ \overline{w^{in**}_l},\overline{q^{out**}_l},\ \overline{q^{in**}_l},\pi^{**}_0)$ be an optimal solution for Model (18). If we define $\breve{\eta}_r=\eta^{**}_r,\ \forall r;\ \breve{\mu}_i=\mu^{**}_i;\ \breve{\delta}_r=\delta^{**}_r;\breve{\sigma}_i=\sigma^{**}_i;\ \breve{\pi}_0=\pi^{**}_0;\ \breve{w}^{out}_l=\begin{cases}\overline{w^{out**}_l}, & l\in OUT\\ 0, & \text{Otherwise}\end{cases};$ $\breve{w}^{in}_l=\begin{cases}\overline{w^{in**}_l}, & l\in IN\\ 0, & \text{Otherwise}\end{cases};$ $\breve{q}^{out}_l=\begin{cases}\overline{q^{out**}_l}, & l\in OUT\\ 0, & \text{Otherwise}\end{cases};$ $\breve{q}^{in}_l=\begin{cases}\overline{q^{in**}_l}, & l\in IN\\ 0, & \text{Otherwise}\end{cases};$ and $\breve{d}_l=\begin{cases}1, & l\in OUT\\ 0, & l\in IN\end{cases}$. Then, the vector $(\breve{\eta}_r,\breve{\mu}_i,\breve{\delta}_r,\breve{\sigma}_i,\breve{w}^{out}_l,\breve{w}^{in}_l,\breve{q}^{out}_l,\breve{q}^{in}_l,\breve{\pi}_0,\breve{d}_l)$ is a feasible solution to Model (16) with the objective function ξ^{**}_p, that is larger than the optimal solution of Model (16), that is a contradiction. □

Theorem 5 shows that the proposed Model (16) provides the maximum values of all the possible cases for considering flexible measures. This fact indicates the optimistic property of our proposed Model (16).

As it said before, there are 2^L different cases for considering flexible factors that we show them with the index set $S=\{1,2,\ldots,2^L\}$. The objective function of evaluating DMU_P by considering case $t\in S$ whether by solving Model (5) or Model (14) is denoted by ξ^t_p. The following corollary is a straight forward result of Theorems 2 and 5.

Corollary 1. For each $t\in S$, we have $\underline{\xi}^*_p\leq\xi^t_p\leq\bar{\xi}^*_p$.

Fig. 1 depicts the flowchart of our approach containing three steps:

(I) ***Formulation step*:** This step involves gathering data, including inputs, outputs, and flexible measures that can play the role of inputs or outputs.

(II) ***Solution step*:** This step involves solving Model (13) for a pessimistic approach or Model (16) for an optimistic approach.

(III) ***Classification step*:** This step involves classifying the flexible measures and making a decision on the optimal solution obtained from the selected model in the previous step.

Table 1
Supplier selection factors.

Factors	Notations	Definitions	Status	Units
Inputs	x_1	Employees	Non-Negative	Number of employees
	x_2	Unit price	Non-Negative	Price per unit in US Dollars
Outputs	y_1	Production capacity	Non-Negative	Number of units in 1000,000s
	y_2	On-time orders	Non-Negative	Order Quantity
	y_3	Operating income	Free	Operating income in 10,000 US Dollars
Flexible	z_1	ISO certifications	Non-Negative	Number of Certifications
measure	z_2	R&D expenditure	Non-Negative	R & D expenditure in 1000 US Dollars

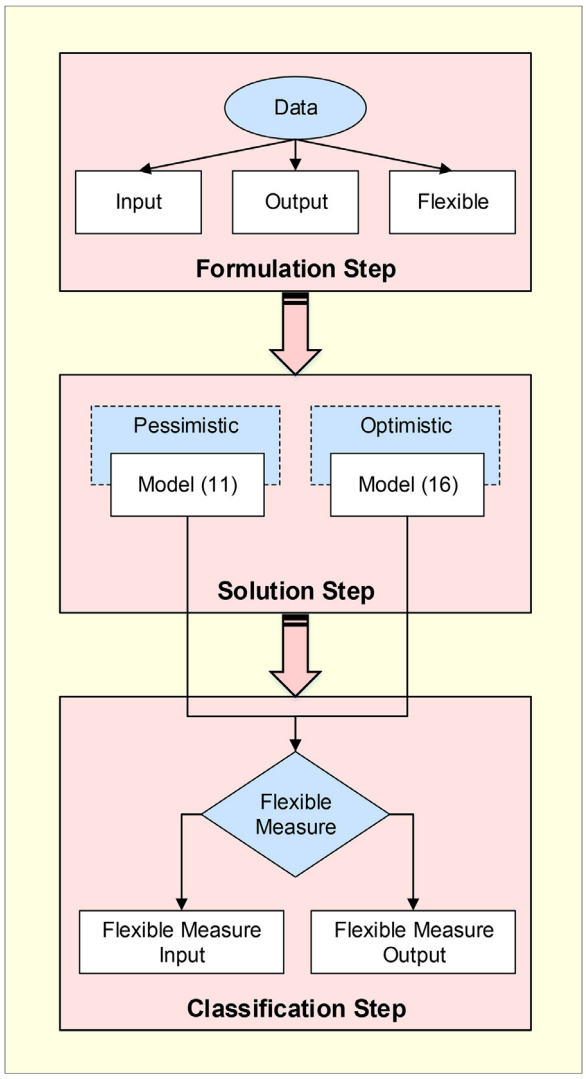

Fig. 1. Flowchart of the proposed method.

5. Case study: general machines international[4]

General Machines International (GMI) is an independent manufacturer, distributor, and marketer of replacement parts for motor vehicles in the automotive industry. Headquartered in Florida, GMI has 8500 employees working in nine manufacturing and engineering facilities located in Alabama, Florida, Georgina, Kansas, Kentucky, Louisiana, Mississippi, Oklahoma, and Texas.

In this study, we evaluated 34 suppliers of steel for GMI in 2018. Table 1 presents the criteria, and Fig. 2 graphically depicts the supplier selection model used at GMI to assess the steel suppliers.

As is shown in Table 1, the inputs are the number of employees and the unit price. The outputs are the production capacity, the number of on-time orders, and the operating income. Note that the number of ISO certifications and R&D expenditure are flexible measures. In addition, the operating income output may have negative or non-negative values. The descriptions of these criteria are presented as follows:

- ***Number of employees*** **(x_1):** represents the number of employees working for the supplier. Employee shortage forces the employees to undertake more responsibilities. An overworked employee is more likely to make mistakes and can negatively affect the operations of the supplier [3,82,84].
- ***Unit price*** **(x_2):** represents the supplier's price per unit. Higher prices could be attributed to higher quality products, better reputation, and ultimately higher productivity and profitability [28,44].
- ***Production capacity*** **(y_1):** represents the volume of products that can be produced by a supplier using current resources. The additional production volume does not increase the suppliers' fixed costs. In addition, the higher suppliers' capacity utilization will result in lower variable costs per unit and higher productivity [26,37].
- ***Number of on-time orders*** **(y_2):** represents the number of on-time order deliveries by the supplier, signifying reputation, and accountability. Poor on-time delivery by a supplier impacts more than customers - it is often a sign of suppliers' poor production efficiency and materials handling procedures [21,38].
- ***Operating income*** **(y_3):** represents the operating income of the supplier. Operating income is considered a good indicator of how efficiently suppliers are managing their expenses. Goo operating income reveals the amount of revenue returned to a supplier once it has covered virtually all of its fixed and variable costs except for taxes and interest. Operating income can be negative when suppliers' expenses are more than their revenues [4,26].
- ***Number of ISO certifications*** **(z_1):** represents the number of ISO certificates awarded to the supplier. This factor represents the training and educational achievements of the supplier. Terlaak and King [76] show that certified suppliers grow faster post-certification because buyers are more interested in certified suppliers. Häversjö [40] and Zakuan et al. [87] found that there is a positive relationship between the ISO certificate and the firm's performance. The number of ISO certifications also plays a dual role (pre- and post-contract) for a supplier. On the one hand, this factor can be treated as an input when it is considered as an expenditure, and on the other hand, this factor can be treated as an output when it is regarded as an achievement [42,43,59].
- ***R&D expenditure*** **(z_2):** represents the amount of R&D spending by the supplier. This factor plays the role of both input and output [57]. This factor is an expenditure in nature and hence can

[4] The name and the data are changed to protect the anonymity of this automotive part manufacturer.

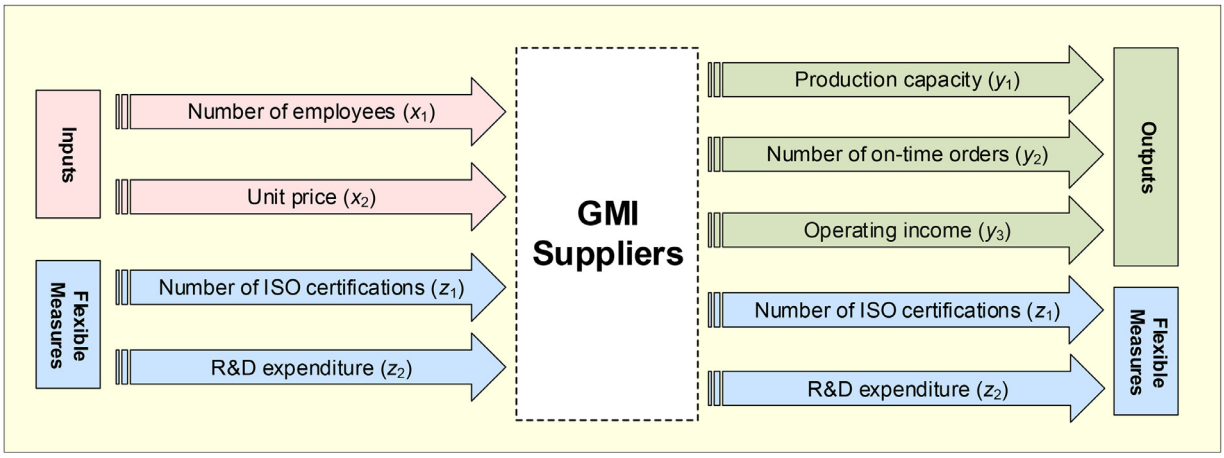

Fig. 2. Proposed supplier selection model.

Table 2
Supplier selection dataset.

Supplier Number	Inputs		Flexible		Outputs		
	x_1	x_2	z_1	z_2	y_1	y_2	y_3
1	1320	16,100	5	2450	6.2	445	−472
2	950	16,600	4	1850	2.8	380	6646
3	280	17,100	1	2100	1.05	250	3038
4	260	15,083	1	950	0.4	120	2882
5	1350	15,783	8	1500	4.8	345	5721
6	850	15,050	6	2050	6.1	380	5654
7	1182	15,600	5	2100	2	420	944
8	420	15,383	3	1400	2.5	130	6345
9	720	14,883	1	900	1.7	125	−1044
10	310	15,000	4	2250	0.18	145	14
11	120	15,100	3	800	0.75	160	4942
12	220	15,700	1	700	0.08	80	−143
13	130	15,700	1	750	0.12	95	4974
14	860	16,100	3	1950	1.2	110	469
15	1245	16,000	6	2300	3.7	285	3027
16	560	16,200	8	2320	0.35	255	6384
17	450	15,500	2	2100	0.175	310	2145
18	350	10,480	2	1450	0.24	140	871
19	420	10,850	3	1540	0.8	135	1739
20	1650	11,900	7	1850	1.5	335	748
21	500	16,500	1	1750	0.4	135	1269
22	1320	15,500	12	2100	5.4	375	−944
23	105	14,933	1	650	0.84	95	−243
24	320	16,400	5	750	0.3	115	5240
25	94	14,917	1	700	0.06	85	6907
26	112	15,800	1	800	0.047	80	−1066
27	85	14,867	1	700	0.5	90	739
28	1200	16,700	2	1350	0.8	235	−1723
29	95	15,633	1	850	0.15	115	−2849
30	180	16,100	3	950	0.42	125	−2350
31	150	15,500	1	650	0.34	95	−182
32	250	15,700	4	750	0.8	105	5756
33	350	15,900	4	1100	0.072	155	5587
34	750	15,700	3	1250	0.5	185	1442

be treated as an input [60]. Nevertheless, R&D plays a critical role in the innovation process and investigating R&D results in better goods and services and efficient operations. Hence, this factor can play the role of output [55].

Table 2 presents the supplier selection dataset for GMI, which includes the input (x_1 and x_2), flexible (z_1 and z_2), and output (y_1, y_2, and y_3) variables.

We first consider the following four possible statuses for two flexible measure w_1 and w_2:

- Status 1: Both z_1 and z_2 are inputs ($d_1 = d_2 = 0$).
- Status 2: Both z_1 and z_2 are outputs ($d_1 = d_2 = 1$).
- Status 3: z_1 and z_2 are input and output measures, respectively, ($d_1 = 0,\ d_2 = 1$).
- Status 4: z_1 and z_2 are output and input measures, respectively, ($d_1 = 1,\ d_2 = 0$).

Table 3 presents the input/output status of flexible measures. According to the pessimistic (optimistic) scenario, the ISO certificate measure should play an output role for 22 (26) DMUs and an input role for 12 (8) DMUs. We use the simple majority decision rule to classify inputs and outputs (see [25]). Therefore, the ISO certificate measure is designated as an output in both pessimistic and optimistic scenarios. On the other hand, the R&D expenditure measure is treated as an output for 22 (26) out of 34 DMUs in both pessimistic and optimistic scenarios. In summary, we can use conventional DEA models to evaluate the performance of the DMUs since the ISO certificate and R&D expenditure are classified as output measures in both pessimistic and optimistic scenarios.

We solve Model (8) (or equivalently Model (14)) for each of the four statuses and report the obtained efficiency scores in the first four columns of Table 3. The results show that: (i) 14 DMUs are efficient for all possible statuses of the flexible measures, (ii) 9 DMUs are inefficient for any status of the flexible measures, (iii) 11 DMUs are efficient for at least one status and also inefficient for at least one other statuses of the flexible measures. We can conclude that 14 DMUs are efficient from the pessimistic point of view. Meanwhile, there are 25 efficient DMUs from an optimistic standpoint. This can be confirmed by considering the last two columns of Table 3, which are labeled as 'optimistic' and 'pessimistic.' These columns are obtained by solving Models (11) and (16), respectively. All models have been solved by the GAMS software. One can practically validate Corollary 1 when the obtained efficiency scores of Model (11) is less than or equal to those obtained by Model (16). Moreover, solving Models (11) and (16) lead to the minimum and maximum efficiency scores among the four possible statuses for flexible measures, respectively, which confirms the correctness of Theorems 2 and 5. The columns labeled as 'd_1^*' and 'd_2^*' in Table 3 display the optimal value of the binary variables corresponding to the flexible measures z_1 and z_2, respectively. For example, solving Model (11) for Supplier #1 gives $d_1^* = d_2^* = 1$, which means both flexible measures are treated as outputs. Next, when Supplier #24 is under evaluation with Model (16), then we obtain $d_1^* = 1$ and $d_2^* = 0$ and hence z_1 and z_2 play an output and input role, respectively.

Fig. 3 depicts the minimum and maximum possible efficiency scores obtained by Models (11) and (16), respectively. It demonstrates the efficiency score intervals for all possible statuses of the

Table 3
Results for different approaches.

Supplier Number	Direct substitution				Pessimistic			Optimistic		
	Case 1	Case 2	Case 3	Case 4	$\underline{\xi}_p^*$	d_1^*	d_2^*	$\bar{\xi}_p^*$	d_1^*	d_2^*
1	1.0000	1.0000	1.0000	1.0000	1.0000	0	0	1.0000	1	1
2	1.0000	1.0000	1.0000	1.0000	1.0000	1	1	1.0000	1	1
3	1.0000	1.0000	1.0000	1.0000	1.0000	1	1	1.0000	1	1
4	1.0000	0.7850	1.0000	0.6549	0.6549	1	0	1.0000	0	0
5	1.0000	1.0000	0.6432	1.0000	0.6432	0	1	1.0000	1	1
6	1.0000	1.0000	1.0000	1.0000	1.0000	1	1	1.0000	1	1
7	1.0000	1.0000	1.0000	1.0000	1.0000	1	1	1.0000	1	1
8	1.0000	1.0000	1.0000	0.9016	0.9016	1	0	1.0000	1	1
9	1.0000	0.5399	1.0000	0.8348	0.5399	1	1	1.0000	0	0
10	0.5617	1.0000	1.0000	0.7221	0.5617	0	0	1.0000	1	1
11	1.0000	1.0000	1.0000	1.0000	1.0000	1	1	1.0000	1	1
12	0.6930	0.6936	0.7036	0.6930	0.6930	1	0	0.7036	1	1
13	1.0000	0.7636	0.7752	0.6959	0.6959	1	1	1.0000	0	0
14	0.5202	0.5251	0.6587	0.5678	0.5202	1	1	0.6587	0	1
15	0.5864	0.8210	0.8269	0.5864	0.5864	0	0	0.8269	1	1
16	0.7267	1.0000	1.0000	1.0000	0.7267	0	0	1.0000	1	1
17	1.0000	1.0000	1.0000	1.0000	1.0000	1	1	1.0000	1	1
18	1.0000	1.0000	1.0000	1.0000	1.0000	1	1	1.0000	1	1
19	1.0000	1.0000	1.0000	1.0000	1.0000	1	1	1.0000	1	1
20	1.0000	1.0000	1.0000	1.0000	1.0000	1	1	1.0000	1	1
21	0.6720	0.6506	0.8250	0.5734	0.5734	1	0	0.8250	0	1
22	0.5947	1.0000	0.5639	1.0000	0.5639	0	1	1.0000	1	1
23	1.0000	1.0000	1.0000	1.0000	1.0000	1	1	1.0000	1	1
24	0.6300	0.8223	0.6009	1.0000	0.6009	0	1	1.0000	1	0
25	1.0000	1.0000	1.0000	1.0000	1.0000	1	1	1.0000	1	1
26	0.6407	0.8091	0.8091	0.6407	0.6407	0	0	0.8091	1	1
27	1.0000	1.0000	1.0000	1.0000	1.0000	1	1	1.0000	1	1
28	1.0000	0.4082	0.5685	0.6394	0.4082	1	1	1.0000	0	0
29	1.0000	1.0000	1.0000	1.0000	1.0000	0	0	1.0000	1	1
30	0.5767	0.7849	0.6615	0.6766	0.5767	0	0	0.7849	1	1
31	1.0000	0.7328	0.7847	1.0000	0.7328	1	1	1.0000	0	0
32	1.0000	1.0000	0.6702	1.0000	0.6702	0	1	1.0000	1	1
33	0.6558	0.7990	0.7273	0.8787	0.6558	0	0	0.8787	1	0
34	0.6544	0.5374	0.6275	0.6721	0.5374	1	1	0.6544	0	0
Average	0.8680	0.8727	0.8661	0.8746	0.7789	–	–	0.9453	–	–

flexible measures. In addition, the last row in Table 3 shows the average efficiency score for each status. Fig. 3 shows the differences between these four cases, and graphically exhibits the maximum efficiency of the DMUs in descending order.

The results presented in Table 3 shows the average efficiency scores in the proposed Models (11) and (16). The results indicate that the average efficiency score of Model (11)is less than those of Model (16). Also, according to the pessimistic viewpoint, 14 out of 34 DMUs are efficient by solving Model (11), while based on the optimistic viewpoint, 26 out of 34 DMUs are efficient by solving Model (16). As we expected, the number of efficient DMUs in Model (11)is less than Model (16), which confirms the

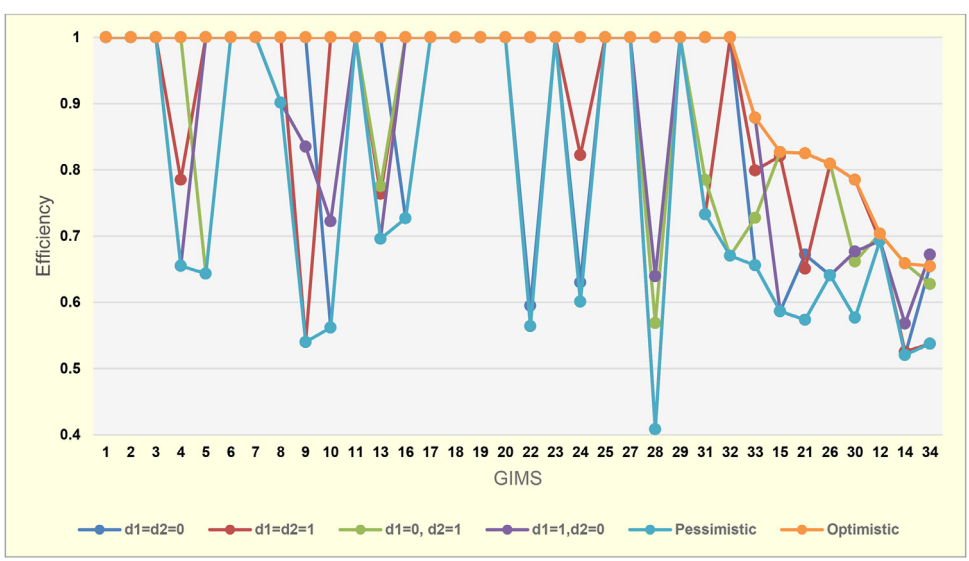

Fig. 3. Status of the flexible measures.

discriminating power of Model (11) is higher than the discriminating power of Model (16).

6. Conclusion and future research directions

In this study, we developed a new non-radial DDF model for handling negative data. The well-known RDM model has two significant drawbacks. The first one is the failure to guarantee projections on the Pareto efficient frontier, as happens with the classical radial DEA model, and the second one is the failure to spot the difference between the strong- and weak-efficient (inefficient) DMUs. The non-radial DDF model proposed in this study allows for the evaluation of the weak-efficient (inefficient) DMUs and project-inefficient DMUs on the strong-efficient frontier.

There are often flexible measures in the real-world problems that can play the roles of input and output variables. Our secondary goal was to extend our proposed model to handle flexible measures. We proved some useful properties of the proposed model and presented a real-world case study to demonstrate the efficacy of the method proposed in this study. We assessed the efficiency of 34 suppliers at an automotive part manufacturer. We took into account two inputs, five outputs, and two flexible measures. Twenty-one suppliers were found to be efficient.

The direct substitution approach can theoretically solve the pessimistic MBLP Model (11) and the optimistic MBLP Model (16) proposed in this study. However, there are practical implications and computational burden as the number of flexible measures and DMUs increases in real-world performance evaluation problems. The number of required problems in the direct substitution approach depends on the number of flexible measures and the number of DMUs. For example, Cook and Zhu [25] evaluated 137 bank branches with three flexible measures of *deposits, open account*, and *withdrawals*. The direct substitution approach in their problem requires 1096 ($2^3 \times 137$) linear programs to solve. In contrast, the mathematical modeling approach proposed in this study requires 137 MBLPs. The number of solution models needed in our approach is significantly less than those required in the direct substitution method. This requirement is particularly problematic and has enormous practical implications and computational burden in today's real-world problems involving big data [49]. Other researchers have also addressed the computational burden of the direct substitution approach. For example, Navas et al. [58] compared the computational burden of their mathematical models with direct substitution approach and concluded "In general, there are 2^L various combinations for L flexible measures. In other words, in order to solve all possible combinations for flexible measures, one must solve $n2^L$ LPs. However, our classifier approach needs only solving n MBLPs. Therefore, in our study, where $n = 157$ and $L = 1$, the problem is reduced by half (from 314 LPs to 157 MBLPs). For other applications, as the number of flexible measures increases, the problem reduction is greater."

Developing a new approach for finding a single (most) efficient supplier in the presence of flexible measures is an interesting further research direction [81,85]. We hope DEA researchers find our work inspiring and apply our model to other DEA models such as two-stage and network DEA models. In addition, incorporating fuzzy and stochastic data into our proposed models could be a promising line of research. While we used our model for supplier evaluation, the proposed method could be used for efficiency measurement in banks, hospitals, and other private and public sector organizations.

Declaration of Competing Interest

None.

CRediT authorship contribution statement

Madjid Tavana: Conceptualization, Formal analysis, Methodology, Writing - original draft, Writing - review & editing, Visualization. **Mohammad Izadikhah:** Formal analysis, Methodology, Validation, Writing - original draft. **Mehdi Toloo:** Conceptualization, Formal analysis, Methodology, Funding acquisition, Writing - original draft. **Razieh Roostaee:** Investigation, Resources, Data curation, Software.

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