Neural Comput & Applic (2019) 31 (Suppl 2):S931–S945
DOI 10.1007/s00521-017-3014-8

ORIGINAL ARTICLE

A chance-constrained portfolio selection model with random-rough variables

Madjid Tavana[1,2] · **Rashed Khanjani Shiraz**[3] · **Debora Di Caprio**[4,5]

Received: 7 December 2016 / Accepted: 11 April 2017 / Published online: 14 June 2017

Abstract Traditional portfolio selection (PS) models are based on the restrictive assumption that the investors have precise information necessary for decision-making. However, the information available in the financial markets is often uncertain. This uncertainty is primarily the result of unquantifiable, incomplete, imprecise, or vague information. The uncertainty associated with the returns in PS problems can be addressed using random-rough (Ra-Ro) variables. We propose a new PS model where the returns are stochastic variables with rough information. More precisely, we formulate a Ra-Ro mathematical programming model where the returns are represented by Ra-Ro variables and the expected future total return maximized against a given fractile probability level. The resulting change-constrained (CC) formulation of the PS optimization problem is a non-linear programming problem. The proposed solution method transforms the CC model in an equivalent deterministic quadratic programming problem using interval parameters based on optimistic and pessimistic trust levels. As an application of the proposed method and to show its flexibility, we consider a probability maximizing version of the PS problem where the goal is to maximize the probability that the total return is higher than a given reference value. Finally, a numerical example is provided to further elucidate how the solution method works.

Keywords Portfolio selection · Chance-constrained programming · Quadratic programming · Random-rough data · Rough set theory

✉ Madjid Tavana
tavana@lasalle.edu; http://tavana.us/

Rashed Khanjani Shiraz
Rashed.shiraz@gmail.com

Debora Di Caprio
dicaper@mathstat.yorku.ca; debora.dicaprio@istruzione.it

1 Business Systems and Analytics Department, Lindback Distinguished Chair of Information Systems and Decision Sciences, La Salle University, Philadelphia, PA 19141, USA

2 Business Information Systems Department, Faculty of Business Administration and Economics, University of Paderborn, 33098 Paderborn, Germany

3 School of Mathematics Science, University of Tabriz, Tabriz, Iran

4 Department of Mathematics and Statistics, York University, Toronto M3J 1P3, Canada

5 Polo Tecnologico IISS G. Galilei, Via Cadorna 14, 39100 Bolzano, Italy

1 Introduction

Portfolio selection (PS) is concerned with the allocation of capital among several securities in such a way that the investment yields the highest return at the lowest risk. Conventional PS models make the restrictive assumption that the investors have precise information that they can use for decision-making. However, the information available in the financial markets is often uncertain, this uncertainty being primarily the result of unquantifiable, incomplete, imprecise, or vague information. In addition, traditional portfolio optimization models assume that the future situation of the market can be accurately predicted with historical data, though this premise is not often applicable to the financial market due to the high volatility of the market environment.

Probabilistic approaches have been traditionally used to deal with the uncertainty in financial markets. See, for instance, the seminal work of Markowitz [1]. However, empirical studies show that there is a wide range of non-probabilistic factors affecting financial markets and, hence, limiting the use of probabilistic approaches [2].

The development of fuzzy and rough approaches to deal with the uncertainty and vagueness inherent to the mathematical formulations and data of real-life problems has open the way to fuzzy, rough, random-fuzzy (Ra-fuzzy), and random-rough (Ra-Ro) portfolio optimization models. Despite the increasing tendency to implement Ra-fuzzy and Ra-Ro variables in stochastic programming, in change-constrained (CC) programming problems, randomness keeps being treated separately from fuzziness and roughness [3, 4]. In particular, returns are mostly interpreted as random variables when formalizing PS optimization problems by means of CC models.

This fact expresses one novelty of the current paper. Indeed, while the applications of CC models to PS with fuzzy and Ra-fuzzy returns have been increasing over the last decade, to the best of the authors' knowledge, there is no research focusing on modeling and solving PS problems in a Ra-Ro CC programming environment.

Another merit of the paper is given by the fact that our model does not rely on a hybrid intelligent algorithm. Even though CC formulations have become a very common tool to model optimization problems under uncertainty, they often require the implementation of suitably designed algorithms that hybridize fuzzy simulation and real-coded genetic algorithms to provide an optimal solution [5, 6].

A substantial amount of the research on portfolio management relates to artificial intelligence (AI). AI approaches mostly propose hybrid intelligent algorithms to solve PS problems [7]. These algorithms are much more common than others in the literature due to their ability to account for the complexity of the return variables while providing either an exact solution or a heuristic one.

Thus, the main objective of the present paper is twofold:

1. to fill in the still considerable gap existing in the literature between chance-constrained programming and applications to real financial problems where uncertain data must be dealt with in a systematic manner;
2. to provide a meaningful alternative to artificial intelligence (AI) approaches such as artificial neural networks, expert systems, and hybrid intelligent systems.

   The solution method presented in this paper combines specific tools used to deal with rough data, namely, interval variables, optimistic/pessimistic trust levels, and lower/upper bound models. The advantage of using this alternative solution method is again twofold: it can be used both as an independent procedure and as a validation method for the design of new hybrid intelligent approaches.

   The technical contribution of the paper can be outlined as follows. We propose a new PS model where the returns are stochastic variables with rough information. More precisely, we formulate a Ra-Ro mathematical programming model where the returns are represented by Ra-Ro variables and the expected future total return maximized against a given fractile probability level. The resulting CC formulation of the PS optimization problem is a nonlinear programming problem. The proposed solution method transforms the CC model in an equivalent deterministic quadratic programming problem. The rough objective function is evaluated using lower and upper bound models obtained by means of interval parameters based on optimistic and pessimistic trust levels.

   As an application of the proposed method and to show its flexibility, we consider a probability maximizing version of the PS problem where the goal is to maximize the probability that the total return is higher than a given reference value. Finally, a numerical example is provided to further elucidate how the solution method works.

   The remainder of the paper is organized as follows. Section 2 provides a literature review addressing the different approaches that have been proposed so far with a focus on fuzzy/rough models. Section 3 describes some preliminaries and definitions regarding rough and Ra-Ro variables. Section 4 introduces the mathematical details of the PS problem with Ra-Ro returns considered in this study and develops the solution method for the resulting CC programming problem. Section 5 presents the probability maximization model associated with the PS problem and interprets it as an application of the CC formulation. Section 6 provides a numerical example to demonstrate how the solution method works in practice. Finally, Section 7 presents our conclusions.

2 Literature review

Markowitz [1] proposed the mean-variance analysis model by quantifying the investment return as the expected value of returns and the risk as the variance from the expected value. Apart from the traditional Markovitz model, several other approaches have been proposed in the literature to analyze PS problems: capital asset pricing models [8–10], mean-absolute-deviation models [11, 12], semi-variance models [13], safety-first models [14], and value at risk and conditional value at risk models [15, 16].

Traditional stochastic PS assumes that the returns are stochastic variables and propose mathematical approaches to solve the extended Markowitz's mean-variance model [17–22]. More recently, stochastic PS considers security returns as random variables and utilizes stochastic programming approaches to strike a balance between returns and risk in uncertain environments [23–30].

A major class of stochastic programming models is represented by the chance-constrained programming (CCP) models.

Charnes and Cooper [3] introduced CC programming requiring the objectives to be achieved under stochastic constraints held at least α of the times, where α is the safety margin required by the decision maker. Early applications of CC programming to portfolio analysis involve stochastic parameters and interpret returns as random variables [31–34]. The CC approaches to PS of the last decade consider fuzzy and Ra-fuzzy returns [5, 35, 36].

After the development of fuzzy set theory by Zadeh [37], researchers began using fuzzy set theory to formulate and solve PS problems in a fuzzy environment. Tanaka and Guo [38], Tanaka et al. [39], Parra et al. [40], and Carlsson et al. [41] used fuzzy set theory to replace the probability distributions for returns with possibility distributions. Other approaches used to model problems in uncertain environments include expectation models Liu [4], chance-constrained programming Charnes and Cooper [3]; Liu [4], and dependent-chance programming Liu [54]; Liu [55]. However, randomness and roughness are generally treated separately in all these models. Recently, Khanjani et al. [48] and Tavana et al. [49–51] have proposed fuzzy-stochastic and Ra-Ro decision models with uncertain data.

Wang and Zhu [2] gave an overview of the fuzzy PS models and showed that fuzzy set theory is a powerful tool when it comes to characterizing the uncertainties deriving from vagueness and ambiguity in financial markets. Ida [42] studied uncertainty in PS models with an interval objective function and used preference cones to show that robust efficient solutions can be identified by working with a finite subset of possible perturbations of the coefficients. Ammar and Khalifa [43] formulated fuzzy portfolio optimization problems as a convex quadratic programming approach presenting a method for an acceptable solution. Giove et al. [44] studied possibilistic portfolio models and treated the expected return rates of the securities as fuzzy or possibilistic variables, instead of random variables. While following a minimax regret approach, their possibilistic mathematical programming model describes the uncertainty in real-world problems as ambiguity and vagueness, rather than stochasticity. They treated the uncertainty in decision-making problems with interval data and adopted a minimax regret approach based on a regret function to solve their PS selection problem.

Gupta et al. [45] used mean-variance analysis and applied multi criteria decision-making via fuzzy mathematical programming to develop comprehensive models for PS and optimization. Hasuike et al. [36] considered PS problems with ambiguous expected returns treated as Ra-fuzzy variables. They formulated Ra-fuzzy PS problems as non-linear programming problems based on both stochastic and fuzzy programming approaches. The problem was solved by transforming it into an equivalent deterministic quadratic programming problem using probabilistic chance-constraints, possibility measures and fuzzy goals.

Liu [46, 47] combined the mean-absolute deviation risk function model with the extension principle of fuzzy set theory and formulated the portfolio optimization problems as two-level mathematical programs that allow to calculate the upper and lower bounds of the optimal return by applying the variable transformation and the duality theorem.

Khanjani Shiraz et al. [48] and Tavana et al. [49–51] have proposed fuzzy-stochastic and Ra-Ro decision models with uncertain data.

Finally, Mohagheghi et al. [52] introduced a practical model to select an optimal project portfolio taking into account project investment capital, return rate, and risk. In this model, interval type-2 fuzzy sets (IT2FSs) are used to formalize the extremely dynamic and highly uncertain setting where projects are developed and evaluated.

The most viable alternative to the fuzzy approach to uncertainty is given by rough set theory [53]. Rough sets handle inconsistent information using upper and lower approximations. Other approaches to model problems in uncertain environments include expectation models [4], CC programming [3, 4], and dependent-chance programming [54, 55].

The increasing implementation of fuzzy and rough approaches to deal with the uncertainty and vagueness inherent to real data, especially to financial ones, has made clear the need for decision makers to simultaneously consider random and ambiguous conditions in PS problems. That is, the classical onefold uncertain variables (random, fuzzy, and rough variables) do not always suffice to model complex real-world problems, especially in situations where randomness and roughness coexist.

Many authors have proposed the concept of twofold uncertain variables combining rough sets with fuzzy sets or rough sets with random variables. In particular, Liu [4, 55] and Xu and Yao [56, 57] presented the basic definitions and properties of the Ra-Ro variables incorporating them in the expected value, chance-constrained, dependent-chance, and the bi-level models. More precisely, Liu [4] introduced trust theory, a branch of mathematics that studies the behavior of rough events and can be considered the foundation of rough programming. Xu and Yao [57] studied a class of multiobjective programming problems with Ra-Ro coefficients and discussed the idea of transforming a constrained model with Ra-Ro variables into crisp equivalent model introducing an interactive algorithm to obtain a satisfying solution.

A substantial amount of the research related to portfolio management and, more in general, to real financial applications enters the realm of AI. The results reported by most of the research following AI approaches tend to show a higher accuracy of the artificial AI with respect to the traditional statistical methods when dealing with highly non-linear and time dependent financial problems. See, among others, Lam [58], West et al. [59], Fernandez and Gomez [60], Ko and Lin [61], Freitas et al. [62], and Hsu [63], where the authors focus the attention on the use of neural networks techniques in financial decision-making and PS optimization.

A comparative and quite widespread review of three famous AI techniques, i.e., artificial neural networks, expert systems, and hybrid intelligence systems in financial market, is presented by Bahrammirzaee [7].

Among the authors who acknowledged the importance of considering random and ambiguous conditions simultaneously for PS problems, Huang [64, 65] designed hybrid intelligent algorithms with security returns being represented by stochastic variables subject to fuzzy and rough information, respectively. In particular, Huang [65] utilized neural networks to calculate the expected and the chance return values while reducing the computational efforts and speeding up the solution process as compared with the existing Ra-fuzzy simulation approaches. In addition, a hybrid intelligent algorithm where fuzzy simulation and neural network techniques are applied to approximate the expected value of fuzzy returns was proposed by Li et al. [66].

3 Rough and Ra-Ro definitions and basic results

In this section, we present the definitions and theorems providing the basis of Ra-Ro variables and trust theory. The definitions relative to measurability concepts are taken from Halmos [67] while the notions of trust theory (i.e., those related to rough variables) follow Pawlak [53], Liu [4, 55], and Xu and Yao [57].

Definition 1: Let Ω be a nonempty set and $\mathcal{A}$ a σ-algebra of subsets of Ω. The pair $(\Omega, \mathcal{A})$ is called a *measurable space* and the elements in $\mathcal{A}$ are called *measurable sets* or *events*. A *probability measure* on $(\Omega, \mathcal{A})$ is a set function $\Pr = \Omega \to [0, 1]$ satisfying the following axioms:

Axiom 1.1. $\Pr\{\Omega\} = 1$;
Axiom 1.2. $\Pr\{A\} \geq 0$, for any $A \in \mathcal{A}$;
Axiom 1.3. $\Pr\left\{ \bigcup_{i=1}^{\infty} A_i \right\} = \sum_{i=1}^{\infty} \Pr\{A_i\}$, for every countable sequence $\{A_i\}_{i=1}^{\infty}$ of mutually disjoint events in $\mathcal{A}$.

Definition 2: Let Λ be a nonempty set, $\mathcal{A}$ a σ-algebra of subsets of Λ and Δ an element in $\mathcal{A}$. A *measure* on $\mathcal{A}$ is a set function π satisfying the following axioms:

Axiom 2.1. $\pi\{\varnothing\} = 0$;
Axiom 2.2. $\pi\{\Lambda\} < \infty$;
Axiom 2.3. $\pi\{\Delta\} > 0$;
Axiom 2.4. $\pi\{A\} \geq 0$, for any $A \in \mathcal{A}$;
Axiom 2.5. $\pi\left\{ \bigcup_{i=1}^{\infty} A_i \right\} = \sum_{i=1}^{\infty} \pi\{A_i\}$, for every countable sequence $\{A_i\}_{i=1}^{\infty}$ of mutually disjoint events in $\mathcal{A}$.

The tuple $(\Lambda, \mathcal{A}, \pi)$ is called a measure space.

Definition 3: Let Λ be a nonempty set, $\mathcal{A}$ a σ-algebra of subsets of Λ and Δ an element in $\mathcal{A}$. A *trust measure* on $(\Lambda, \mathcal{A})$ is a measure on $\mathcal{A}$, denoted by Tr, satisfying the following axioms:

Axiom 3.1. $\mathrm{Tr}\{\Lambda\} = 1$;
Axiom 3.2. $\mathrm{Tr}\{\varnothing\} = 0$;
Axiom 3.3. $\mathrm{Tr}\{A\} \leq \mathrm{Tr}\{B\}$, for every $A, B \in \mathcal{A}$ with $A \subseteq B$;
Axiom 3.4. $\mathrm{Tr}\{A\} + \mathrm{Tr}\{A^c\} = 1$, for every $A \in \mathcal{A}$.

The tuple $(\Lambda, \Delta, \mathcal{A}, \mathrm{Tr})$ is called a *rough space*.

Definition 4: Let $(\Lambda, \Delta, \mathcal{A}, \mathrm{Tr})$ be a rough space and K be an event. Then:

- the *upper trust* of K is defined by: $(\mathrm{Tr})^{\mathrm{Upper}}\{K\} = \frac{\pi\{K\}}{\pi\{\Lambda\}}$;
- the *lower trust* of K is defined by: $(\mathrm{Tr})^{\mathrm{Lower}}\{K\} = \frac{\pi\{K \cap \Delta\}}{\pi\{\Delta\}}$;
- the *trust* of K is defined by: $\mathrm{Tr}\{K\} = \frac{1}{2}\left[(\mathrm{Tr})^{\mathrm{Upper}}\{K\} + (\mathrm{Tr})^{\mathrm{Lower}}\{K\}\right]$.

Definition 5: Given a set X, the collection of all sets having the same lower and upper approximations is called a rough set and it is denoted by $\left(\underline{X}, \overline{X}\right)$. The lower approximation is a subset containing the objects *surely* belonging to the set X, whereas the upper approximation is a superset containing the objects *possibly* belonging to the set X, and $\underline{X} \subset X \subset \overline{X}$.

Definition 6: A rough variable ξ on a rough space $(\Lambda, \Delta, \mathcal{A}, Tr)$ is a function from Λ into the reals such that, for every Borel set O, $\{\lambda \in \Lambda | \xi(\lambda) \in O\} \in \mathcal{A}$. The lower and the upper approximations of the rough variable ξ are given respectively by:

$$\overline{\xi} = \{\xi(\lambda) | \lambda \in \Lambda\} \quad \text{and} \quad \underline{\xi} = \{\xi(\lambda) | \lambda \in \Delta\}.$$

Let $a, b, c, d \in \mathfrak{R}$ be such that $c \leq a < b \leq d$. If $\Lambda = \{\lambda = c \leq \lambda \leq d\}$, $\Delta = \{\lambda = a \leq \lambda \leq b\}$, $\mathcal{A}$ is the Borel algebra on Λ and Tr is the Lebesgue measure, the identity function $\xi(\lambda) = \lambda$ from the rough space $(\Lambda, \Delta, \mathcal{A}, Tr)$ into the set of real numbers is a rough variable. This rough variable is denoted by $([a,b],[c,d])$.

Definition 7: Let ξ and η be two rough variables on a rough space $(\Lambda, \Delta, \mathcal{A}, Tr)$.

- The *expected value* of ξ, denoted by $E(\xi)$, is defined by:

$$E(\xi) = \int_0^{+\infty} \mathrm{Tr}\{\xi \geq r\} \mathrm{d}r - \int_{-\infty}^{0} \mathrm{Tr}\{\xi \leq r\} \mathrm{d}r,$$

provided that at least one of the two integrals is finite.

- The *variance* of ξ, denoted by $\mathrm{Var}[\xi]$, is defined by:

$$\mathrm{Var}[\xi] = E[\xi - E[\xi]]^2.$$

- If $E[\xi]$ and $E[\eta]$ are finite, then the *covariance* of ξ and η is defined by:

$$\mathrm{Cov}[\xi, \eta] = E(\xi - E[\xi])E(\eta - E[\eta]) = E[\xi\eta] - E[\xi]E[\eta].$$

In particular, if ξ and η are independent rough variables, then $\mathrm{Cov}[\xi, \eta] = 0$; the inverse implication is not true.

- If $\alpha \in (0, 1]$, then the *α-optimistic* and the *α-pessimistic* values of ξ are given by $\xi^{\mathrm{Sup}(\alpha)} = \sup\{r = \mathrm{Tr}\{\xi \geq r\} \geq \alpha\}$ and $\xi^{\mathrm{Inf}(\alpha)} = \inf\{r = \mathrm{Tr}\{\xi \leq r\} \geq \alpha\}$, respectively.
- If ξ is an n-dimensional vector and $\{f_j = \mathfrak{R}^n \rightarrow \mathfrak{R}\}_{j=1,\ldots,m}$ is a set of continuous functions, then the upper trust, lower trust, and trust of the rough event characterized by $f_j(\xi) \leq 0, j = 1, \ldots, m$, are defined, respectively, by:

$$(\mathrm{Tr})^{\mathrm{Upper}}\left\{f_j(\xi)\leq 0, j = 1, \ldots, m\right\} = \frac{\pi\left\{\lambda\in\Lambda \middle| f_j(\xi)\leq 0\right\}}{\pi\{\Lambda\}},$$

$$(\mathrm{Tr})^{\mathrm{Lower}}\left\{f_j(\xi)\leq 0, j = 1, \ldots, m\right\} = \frac{\pi\left\{\lambda\in\Delta \middle| f_j(\xi)\leq 0\right\}}{\pi\{\Delta\}},$$

$$\mathrm{Tr}\left\{\begin{array}{l} f_j(\xi)\leq 0,\\ j = 1, \ldots, m \end{array}\right\} = (\mathrm{Tr})^{\mathrm{Lower}}\left\{\begin{array}{l} f_j(\xi)\leq 0,\\ j = 1, \ldots, m \end{array}\right\} + (\mathrm{Tr})^{\mathrm{Upper}}\left\{\begin{array}{l} f_j(\xi)\leq 0,\\ j = 1, \ldots, m \end{array}\right\}.$$

Definition 8: A Ra-Ro variable ξ is a random variable with an uncertain parameter represented by a rough set $X = \left(\underline{X}, \overline{X}\right)$, that is $\underline{X} \subset X \subset \overline{X}$.

Definition 9: Let ξ be a Ra-Ro variable with the uncertain parameter $\lambda = ([a, b], [c, d])$. The *expected value* of ξ is defined by

$$E(\xi) = \int_0^{+\infty}\mathrm{Tr}\{\lambda\in\Lambda | E[\xi(\lambda)]\geq r\}\mathrm{d}r - \int_{-\infty}^{0}\mathrm{Tr}\{\lambda\in\Lambda | E[\xi(\lambda)]\leq r\}\mathrm{d}r$$

provided that at least one of the two integrals is finite.

Definition 10: A Ra-Ro variable ξ is *normally distributed* if it is a normally distributed random variable whose density function is given by the function $f(x)$ defined below:

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\overline{\mu})^2}{2\sigma^2}}, -\infty < x < +\infty$$

where $\overline{\mu} = ([a, b], [c, d])$ is a rough variable.

To indicate that ξ is a normally distributed Ra-Ro variable, we write $\xi \sim N(\overline{\mu}, \sigma^2)$.

Proposition 1 (Liu [4], Xu and Yao [56, 57]):

Let ξ be a rough variable. Then, the following facts hold true.

(a) $\xi^{\mathrm{Inf}(\alpha)}$ is an increasing function of $\alpha \in [0, 1]$ and $\xi^{\mathrm{Sup}(\alpha)}$ is a decreasing function of $\alpha \in [0, 1]$.
(b) For $0 \leq \alpha \leq 1$, $\xi^{\mathrm{Inf}(\alpha)} = \xi^{\mathrm{Sup}(1-\alpha)}$ and $\xi^{\mathrm{Sup}(\alpha)} = \xi^{\mathrm{Inf}(1-\alpha)}$.
(c) For $0 \leq \alpha \leq 0.5$, $\xi^{\mathrm{Inf}(\alpha)} \leq \xi^{\mathrm{Sup}(\alpha)}$.
(d) For $0.5 \leq \alpha \leq 1$, $\xi^{\mathrm{Inf}(\alpha)} \geq \xi^{\mathrm{Sup}(\alpha)}$.
(e) If $\mathrm{Var}[\xi]$ is finite, then $\mathrm{Var}[\xi] = E[\xi^2] - (E[\xi])^2$.
(f) If $Var[\xi]$ is finite, then, for every $a, b \in \mathfrak{R}$, $Var[a\xi + b] = a^2 Var[\xi]$.
(g) If $\xi = ([a, b], [c, d])$, for some $a, b, c, d \in \mathfrak{R}$ such that $c \leq a < b \leq d$, then:

$$E(\xi) = \frac{1}{4}(a + b + c + d) \quad \text{and} \quad \mathrm{Var}[\xi] = \frac{1}{4}\left(a^2 + b^2 + c^2 + d^2\right) - \left(\frac{a + b + c + d}{4}\right)^2.$$

Proposition 2 [55]: If $\xi_1, \xi_2, \ldots, \xi_n$ are rough variables with finite expected values, then

$$\mathrm{Var}\left[\sum_{i=1}^{n}\xi_i\right] = \sum_{i=1}^{n}\mathrm{Var}[\xi_i] + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n}\mathrm{Cov}\left(\xi_i, \xi_j\right).$$

In particular, if $\xi_1, \xi_2, \ldots, \xi_n$ are independent, then

$$\mathrm{Var}\left[\sum_{i=1}^{n}\xi_i\right] = \sum_{i=1}^{n}\mathrm{Var}[\xi_i].$$

Proposition 3 [4, 57]: Let $f = \mathfrak{R}^n \rightarrow \mathfrak{R}$ be a continuous function and for every $i = 1, \ldots, m$, let ξ_i be a Ra-Ro variable defined on a rough space $(\Lambda_i, \Delta_i, A_i, \mathrm{Tr}_i)$. Then $\xi = f(\xi_1, \ldots, \xi_n)$ is a Ra-Ro variable on the rough product space $(\Lambda, \Delta, A, \mathrm{Tr})$, defined by $\xi(\lambda_1, \ldots, \lambda_n) = f(\xi_1(\lambda_1), \ldots, \xi_n(\lambda_n))$ for all $(\lambda_1, \ldots, \lambda_n) \in \Lambda$.

4 Chance-constrained PS model with Ra-Ro variables

Let us consider n securities. For every $j = 1, \ldots, n$, let r_j be the return rate of the jth security and denote by x_j the proportion of total amount of funds invested in the jth security. A PS problem consists in finding the investment rate vector $x = (x_1, x_2, \ldots, x_n)$ which maximizes the total portfolio return $\sum_{j=1}^{n} r_j x_j$.

In this paper, we center our attention on the PS problem with Ra-Ro returns. More precisely, we formulate a Ra-Ro CC programming model where the returns are represented by Ra-Ro variables and the expected future total return f maximized against a given fractile probability level β. That is, we consider the following possibility fractile optimization model with respect to the expected future return.

$$\max_{x} f \tag{1}$$

s.t.

$$\Pr\left\{\sum_{j=1}^{n} \tilde{\bar{r}}_j x_j \geq f\right\} \geq \beta$$

$$\sum_{j=1}^{n} a_j x_j \leq b,\ 0 \leq x_j \leq \hat{b}_j, j = 1, \dots, n.$$

where

- n is the total number of assets;
- x_j is the budget for the jth financial asset;
- $\tilde{\bar{r}}_j$ is the future return of the jth financial asset; each $\tilde{\bar{r}}_j$ is assumed to be a normally distributed Ra-Ro variable (see Definition 10), that is: $\tilde{\bar{r}}_j \sim N(\bar{r}_j, \sigma^2)$ where $\bar{r}_j = ([A_j, B_j]\ , [C_j, D_j])$ is a rough variable;
- a_j is the cost of investing in the jth financial asset;
- b is the upper bound value for the total budget;
- $\hat{b}_i$ is the upper bound value for the amount budgeted to the jth financial asset;
- β is a value between 0 and 1 representing the fixed fractile probability level.

In order for a CC programming model to be solved, the constraints of the model need to be converted into their respective crisp equivalents. Thus, the remainder of this section is devoted to show how this conversion is performed.

Let $\tilde{\bar{h}} = \sum_{j=1}^{n} \tilde{\bar{r}}_j x_j - f$ and denote by $E\left(\tilde{\bar{h}}\right)$ and $\mathrm{Var}\left(\tilde{\bar{h}}\right)$ the mean and the variance of the Ra-Ro variable $\tilde{\bar{h}}$, respectively. Due to the normal distribution of $\tilde{\bar{r}}_j$, $\tilde{\bar{h}}$ also has a normal distribution. In particular, the mean and the standard deviation of $\tilde{\bar{h}}$ are given by:

$$\bar{h} = E\left(\tilde{\bar{h}}\right) = \sum_{j=1}^{n} \bar{r}_j x_j - f$$

$$\sigma_{\tilde{\bar{h}}} = \sqrt{\mathrm{Var}\left(\sum_{j=1}^{n} \tilde{\bar{r}}_j x_j - f\right)} = \sqrt{\sum_{j=1}^{n} \sum_{i=1}^{n} \sigma_{ij}\ x_i x_j}$$

where $\sigma_{ij} = \mathrm{Cov}\left[\tilde{\bar{r}}_i, \tilde{\bar{r}}_j\right]$.

Thus, the chance-constraint $\Pr\left\{\sum_{j=1}^{n} \tilde{\bar{r}}_j x_j \geq f\right\} \geq \beta$ becomes

$$\Pr\left(\frac{\tilde{\bar{h}} - E\left(\tilde{\bar{h}}\right)}{\sigma_{\tilde{\bar{h}}}} \geq -\frac{E\left(\tilde{\bar{h}}\right)}{\sigma_{\tilde{\bar{h}}}}\right) \geq \beta,$$

and, hence:

$$\Pr\left(z \geq -\frac{E\left(\tilde{\bar{h}}\right)}{\sigma_{\tilde{\bar{h}}}}\right) \geq \beta$$

where $z = \frac{\tilde{\bar{h}} - E\left(\tilde{\bar{h}}\right)}{\sigma_{\tilde{\bar{h}}}}$ is the standard normal random variable with zero mean and unit variance.

It follows that:

$$\Pr\left(z \geq -\frac{E\left(\tilde{\bar{h}}\right)}{\sigma_{\tilde{\bar{h}}}}\right) \geq \beta \Rightarrow 1 - \Pr\left(z \leq -\frac{E\left(\tilde{\bar{h}}\right)}{\sigma_{\tilde{\bar{h}}}}\right) \geq \beta,$$

which implies:

$$-\Pr\left(z \leq -\frac{E\left(\tilde{\bar{h}}\right)}{\sigma_{\tilde{\bar{h}}}}\right) \geq \beta - 1 \Rightarrow \Pr\left(z \leq -\frac{E\left(\tilde{\bar{h}}\right)}{\sigma_{\tilde{\bar{h}}}}\right) \leq 1 - \beta \Rightarrow$$

$$\Phi\left(-\frac{E\left(\tilde{\bar{h}}\right)}{\sigma_{\tilde{\bar{h}}}}\right) \leq 1 - \beta \Rightarrow -\frac{E\left(\tilde{\bar{h}}\right)}{\sigma_{\tilde{\bar{h}}}} \leq -\Phi^{-1}(\beta) \Rightarrow \frac{E\left(\tilde{\bar{h}}\right)}{\sigma_{\tilde{\bar{h}}}} \geq \Phi^{-1}$$

$$(\beta) \Rightarrow E\left(\tilde{\bar{h}}\right) - \sigma_{\tilde{\bar{h}}} \Phi^{-1}(\beta) \geq 0$$

where Φ is the cumulative distribution function.

Hence, the chance-constraint $\Pr\left\{\sum_{j=1}^{n} \tilde{\bar{r}}_j x_j \geq f\right\} \geq \beta$ is equivalent to

$$\sum_{j=1}^{n} \bar{r}_j x_j - \Phi^{-1}(\beta) \sqrt{\sum_{j=1}^{n} \sum_{i=1}^{n} \sigma_{ij} x_i x_j \geq f}$$

and model (1) is converted into the following:

$$\max_{x} f \tag{2}$$

$s.t.$

$$\sum_{j=1}^{n} \bar{r}_j x_j - \Phi^{-1}(\beta) \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \geq f},$$

$$\sum_{j=1}^{n} a_j x_j \leq b, \quad 0 \leq x_j \leq b_j, \quad j = 1, \dots, n.$$

where $\bar{r}_j = ([A_j, B_j]\ , [C_j, D_j])$ with $C_j \leq A_j < B_j \leq D_j$.

Note that the constraint of model (2) is still uncertain since all the $\bar{r}_j$, $j = 1\ , \ \dots\ , n$, are rough variables. Thus, we must now introduce a crisp equivalent for this constraint. The above model is equivalent to the following model:

$$\max_x \sum_{j=1}^{n} \overline{r}_j x_j - \Phi^{-1}(\beta)\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}x_i x_j} \qquad (3)$$

s.t.

$$\sum_{j=1}^{n} a_j x_j \le b, \quad 0 \le x_j \le b_j, \quad j = 1, \ldots, n.$$

To find the rough objective value, it suffices to find the upper bound and lower bound of the objective values of model (3). In order to do so, we formulate the PS problem as the following programming problem with interval parameters:

$$\max_x \sum_{j=1}^{n} \left[r_j^{\mathrm{Sup}(\alpha)}, r_j^{\mathrm{Inf}(\alpha)}\right] x_j - \Phi^{-1}(\beta)\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}x_i x_j} \qquad (4)$$

s.t.

$$\sum_{j=1}^{n} a_j x_j \le b, \quad 0 \le x_j \le b_j, \quad j = 1, \ldots, n.$$

where $r_j^{\mathrm{Inf}(\alpha)}$ and $r_j^{\mathrm{Sup}(\alpha)}$ are the α-pessimistic and α-optimistic values, respectively, for the rough variable $\overline{r}_j$ when $0.5 \le \alpha \le 1$.

Let $S_\alpha = \left\{ v = (v_1, \ldots, v_n) | r_j^{\mathrm{Sup}(\alpha)} \le v_j \le r_j^{\mathrm{Inf}(\alpha)} \quad \forall j = 1, \ldots, n \right\}$.

For each $v \in S_\alpha$, denote by $Z_{\alpha,\beta}(v)$ the corresponding objective value obtained by using model (4). Also, let $Z^L_{\alpha,\beta}$ and $Z^U_{\alpha,\beta}$ be the *minimum* and the *maximum* of $Z_{\alpha,\beta}(v)$ on S_α, respectively. It follows that the lower and upper bound models that allow to evaluate model (4) are:

$$Z^L_{\alpha,\beta} = \min_{v \in S_\alpha} Z_{\alpha,\beta}(v) = \min_{v \in S_\alpha} \quad \max_x \sum_{j=1}^{n} v_j x_j - \Phi^{-1}(\beta)\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij} \; x_i x_j} \qquad (5)$$

s.t.

$$\sum_{j=1}^{n} a_j x_j \le b, \quad 0 \le x_j \le \hat{b}_j, \quad j = 1, \ldots, n.$$

and

$$Z^U_{\alpha,\beta} = \max_{v \in S_\alpha} Z_{\alpha,\beta}(v) = \max_{v \in S_\alpha} \quad \max_x \sum_{j=1}^{n} v_j x_j - \Phi^{-1}(\beta)\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij} \; x_i x_j} \qquad (6)$$

s.t.

$$\sum_{j=1}^{n} a_j x_j \le b, \quad 0 \le x_j \le \hat{b}_j, \quad j = 1, \ldots, n.$$

That is, the lower bound model is:

$$Z^L_{\alpha,\beta} = \min_x - \sum_{j=1}^{n} r_j^{\mathrm{Sup}(\alpha)} x_j + \Phi^{-1}(\beta)\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij} \; x_i x_j} \qquad (7)$$

s.t.

$$\sum_{j=1}^{n} a_j x_j \le b, \quad 0 \le x_j \le \hat{b}_j, \quad j = 1, \ldots, n.$$

while the upper bound model is:

$$Z^U_{\alpha,\beta} = \min_x - \sum_{j=1}^{n} r_j^{\mathrm{Inf}(\alpha)} x_j + \Phi^{-1}(\beta)\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij} \; x_i x_j} \qquad (8)$$

s.t.

$$\sum_{j=1}^{n} a_j x_j \le b, \quad 0 \le x_j \le \hat{b}_j, \quad j = 1, \ldots, n.$$

Remark: In this paper, we consider $\beta \ge 0.5$ due to the following facts:

(a) In practical decision-making, decision makers do not usually select a portfolio whose achievement probability is less than half of the expected total return.
(b) In the case when $\beta \ge 0.5$, both $Z^L_{\alpha,\beta} = \max_x \sum_{j=1}^{n} r_j^{\mathrm{Sup}(\alpha)} x_j - \Phi^{-1}(\beta)\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij} \; x_i x_j}$ and $Z^U_{\alpha,\beta} = \max_x \sum_{j=1}^{n} r_j^{\mathrm{Inf}(\alpha)} x_j - \Phi^{-1}(\beta)\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij} \; x_i x_j}$ are concave functions. Consequently, models (5) and (6) admit a solution.

Proposition 4: Let $\beta \in [0.5, 1]$ and $\alpha_1, \alpha_2 \in [0.5, 1]$ be such that $\alpha_1 \ge \alpha_2$. Then, we have:

$$Z^U_{\alpha_1,\beta} \ge Z^U_{\alpha_2,\beta}$$

$$Z^L_{\alpha_2,\beta} \ge Z^L_{\alpha_1,\beta}$$

Proof: Since $\alpha_1 \ge \alpha_2$, by proposition 1(a), we have $r_j^{\mathrm{Inf}(\alpha_1)} \ge r_j^{\mathrm{Inf}(\alpha_2)}$ and $r_j^{\mathrm{Sup}(\alpha_2)} \ge r_j^{\mathrm{Sup}(\alpha_1)}$. Then, considering the objective function of model (5) for the values α_1 and α_2, respectively, we get:

$$Z^U_{\alpha_1,\beta} = \max_x \sum_{j=1}^{n} r_j^{\mathrm{Inf}(\alpha_1)} x_j - \Phi^{-1}(\beta)\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij} \; x_i x_j} \ge \max_x \sum_{j=1}^{n} r_j^{\mathrm{Inf}(\alpha_2)} x_j - \Phi^{-1}(\beta)\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij} \; x_i x_j} = Z^U_{\alpha_2,\beta}$$

Thus, $Z^U_{\alpha_1,\beta} \ge Z^U_{\alpha_2,\beta}$ and (*a*) holds. Similarly, (*b*) can be established.

Proposition 5: Let $\beta_1, \beta_2 \in [0.5, 1]$ be such that $\beta_1 \ge \beta_2$ and let $\alpha \in [0.5, 1]$. Then, we have:

$$Z^U_{\alpha,\beta_2} \ge Z^U_{\alpha,\beta_1}$$

$Z^L_{\alpha,\beta_2} \geq Z^L_{\alpha,\beta_1}$

Proof:

Since $\beta_1 \geq \beta_2$, we have

$$\Phi^{-1}(\beta_1) \geq \Phi^{-1}(\beta_2) \Leftrightarrow -\Phi^{-1}(\beta_1) \leq -\Phi^{-1}(\beta_2).$$

Hence, considering the objective function of model (5) for the values β_1 and β_2, respectively, we have:

$$Z^U_{\alpha,\beta_2} = \max_x \sum_{j=1}^{n} r_j^{\mathrm{Inf}(\alpha)} x_j - \Phi^{-1}(\beta_2)\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}\ x_i x_j} \geq \max_x \sum_{j=1}^{n} r_j^{\mathrm{Inf}(\alpha)} x_j - \Phi^{-1}(\beta_1)\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}\ x_i x_j} = Z^U_{\alpha,\beta_1}$$

Consequently, (a) holds. Similarly, (b) can be obtained.

4.1 Solution approach

Let V be a $n \times n$ *positive definite matrix*, that is, a $n \times n$ symmetric matrix such that for all *column vectors* $x \in \mathfrak{R}^n$, $x^t V x > 0$. There exists a unique decomposition of V by a diagonal and an orthogonal matrix, namely:

$$V = QDQ^t$$
$$D = \mathrm{diag}(d_1, d_2, \ldots, d_n) = \begin{pmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{pmatrix}. \tag{9}$$

D is the *diagonal matrix* composed of the eigenvalues of V and Q is the *orthogonal matrix* (that is, $Q^t Q = I$) whose 1st, 2nd, …, nth column are the eigenvector associated with d_1, d_2, … , d_n, respectively. Eq. (9) is known as the *singular value decomposition* of V.

A variance-covariance matrix is a positive definite matrix. Thus, by letting V denote the variance-covariance matrix whose ijth element is σ_{ij}, we have:

$$\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}\ x_i x_j = x^t V x = (Q\ x)^t D (Q\ x).$$

From which it follows:

$$\Phi^{-1}(\beta)\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}\ x_i x_j} = \sqrt{\left(\Phi^{-1}(\beta)\ x\right)^t V \left(\Phi^{-1}(\beta)\ x\right)}$$
$$= \sqrt{\left(\Phi^{-1}(\beta)\ x\right)^t Q^t D Q \left(\Phi^{-1}(\beta)\ x\right)}.$$

Hence, by using the unique decomposition of D as $\sqrt{D}\sqrt{D}$, where:

$$\sqrt{D} = \mathrm{diag}\left(\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n}\right) = \begin{pmatrix} \sqrt{d_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{d_n} \end{pmatrix},$$

we have:

$$\Phi^{-1}(\beta)\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}\ x_i x_j}$$
$$= \sqrt{\left(\Phi^{-1}(\beta)\sqrt{D}\ Q\ x\right)^t \left(\Phi^{-1}(\beta)\sqrt{D}\ Q\ x\right)}.$$

Therefore, letting $y = \Phi^{-1}(\beta)\sqrt{D}\ Q\ x$, we have:

$$\Phi^{-1}(\beta)\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij} x_i x_j} = \Phi^{-1}(\beta)\sqrt{x^t V x} = \sqrt{y^t y}$$
$$= \sqrt{\sum_{j=1}^{n} y_j^2},$$

where y_j is the jth coordinate of the column vector y.

By the vector transformation $y = \Phi^{-1}(\beta)\sqrt{D}\ Q\ x$, we also have:

$$x = \frac{1}{\Phi^{-1}(\beta)} Q^{-1}\left(\sqrt{D}\right)^{-1} y,$$
$$a^t x = a^t\left(\frac{1}{\Phi^{-1}(\beta)} Q^{-1}\left(\sqrt{D}\right)^{-1} y\right)$$
$$= \left(\frac{1}{\Phi^{-1}(\beta)} a^t Q^{-1}\left(\sqrt{D}\right)^{-1}\right) y,$$
$$\left(r^{\mathrm{Inf}(\alpha)}\right)^t x = \left(r^{\mathrm{Inf}(\alpha)}\right)^t\left(\frac{1}{\Phi^{-1}(\beta)} Q^{-1}\left(\sqrt{D}\right)^{-1} y\right)$$
$$= \left(\frac{1}{\Phi^{-1}(\beta)}\left(r^{\mathrm{Inf}(\alpha)}\right)^t Q^{-1}\left(\sqrt{D}\right)^{-1}\right) y,$$
$$\left(r^{\mathrm{Sup}(\alpha)}\right)^t x = \left(r^{\mathrm{Sup}(\alpha)}\right)^t\left(\frac{1}{\Phi^{-1}(\beta)} Q^{-1}\left(\sqrt{D}\right)^{-1} y\right)$$
$$= \left(\frac{1}{\Phi^{-1}(\beta)}\left(r^{\mathrm{Sup}(\alpha)}\right)^t Q^{-1}\left(\sqrt{D}\right)^{-1}\right) y,$$

where a, $r^{\mathrm{Inf}(\alpha)}$ and $r^{\mathrm{Sup}(\alpha)}$ stand for the column vectors whose jth coordinate is a_j (the cost of investing in the jth financial asset), $r_j^{\mathrm{Inf}(\alpha)}$, and $r_j^{\mathrm{Sup}(\alpha)}$, respectively.

Note that if the new variables of the problem change from x_j to y_j, then the upper bound for the amount budgeted to the jth financial asset must also change by the same vector transformation. Thus, the upper bound for y_j is given by the jth coordinate of the following vector:

$$\hat{b}^* = \Phi^{-1}(\beta)\sqrt{D}\ Q\ \hat{b}.$$

Finally, let:

$$a^{*t} = \frac{1}{\Phi^{-1}(\beta)} a^t\ Q^{-1}\left(\sqrt{D}\right)^{-1},$$

$$\hat{b}^* = \Phi^{-1}(\beta)\sqrt{D}\ Q\ \hat{b},$$

$$b^* = b,$$

$$\left(r^{*\mathrm{Inf}(\alpha)}\right)^t = \frac{1}{\Phi^{-1}(\beta)}\left(r^{\mathrm{Inf}(\alpha)}\right)^t Q^{-1}\left(\sqrt{D}\right)^{-1},$$

$$\left(r^{*\mathrm{Sup}(\alpha)}\right)^t = \frac{1}{\Phi^{-1}(\beta)}\left(r^{\mathrm{Sup}(\alpha)}\right)^t Q^{-1}\left(\sqrt{D}\right)^{-1}.$$

Then, we can transform models (7) and (8) into the following:

$$Z_\alpha^L = \min_y - \sum_{j=1}^n r_j^{*\mathrm{Sup}(\alpha)} y_j + \sqrt{\sum_{j=1}^n y_j^2} \tag{10}$$

s.t.

$$\sum_{j=1}^n a_j^* y_j \le b, \quad 0 \le y_j \le \hat{b}_j^*, j = 1, \ldots, n$$

and

$$Z_\alpha^U = \min_y - \sum_{j=1}^n r_j^{*\mathrm{Inf}(\alpha)} y_j + \sqrt{\sum_{j=1}^n y_j^2} \tag{11}$$

s.t.

$$\sum_{j=1}^n a_j^* y_j \le b, \quad 0 \le y_j \le \hat{b}_j^*, j = 1, \ldots, n$$

By the variable change $u^2 = \sum_{j=1}^n y_j^2$, models (10) and (11) become:

$$Z_\alpha^L = \min_y - \sum_{j=1}^n r_j^{*\mathrm{Sup}(\alpha)} y_j + u \tag{12}$$

s.t.

$$\sum_{j=1}^n a_j^* y_j \le b, \quad 0 \le y_j \le \hat{b}_j^*, j = 1, \ldots, n$$

$$u^2 = \sum_{j=1}^n y_j^2, u \ge 0, \ \ y_j \ge 0, \ \ j = 1, \ \ldots, \ n.$$

and

$$Z_\alpha^U = \min_y - \sum_{j=1}^n r_j^{*\mathrm{Inf}(\alpha)} y_j + u \tag{13}$$

s.t.

$$\sum_{j=1}^n a_j^* y_j \le b, \quad 0 \le y_j \le \hat{b}_j^*, j = 1, \ldots, n$$

$$u^2 = \sum_{j=1}^n y_j^2, u \ge 0, \ \ y_j \ge 0, \ \ j = 1, \ldots, n.$$

The last two models are linear programming models with one quadratic constraint.

Figure 1 summarizes the method proposed by this study to solve the CC PS model with Ra-Ro returns (i.e., model (1)).

Fig. 1 Scheme of the solution method for the chance-constrained model with Ra-Ro returns

Chance-constrained maximization problem

max f

s.t.

$Pr\left\{\sum_{j=1}^n \tilde{\bar{r}}_j x_j \ge f\right\} \ge \beta$

$\sum_{j=1}^n a_j x_j \le b, \quad 0 \le x_j \le \hat{b}_j, \ j = 1, \ldots, n$

f is the expected total return

$\beta \in [0.5, 1]$ is fixed fractile probanility level

Step 1.	Convert the Ra-Ro constraint into an uncertain constraint with rough returns $\bar{r}_j$, $j = 1, \ldots, n$. ➢ Obtain Model (2) and Model (3).
Step 2.	Fix $\alpha \in [0.5, 1]$. For all $j = 1, \ldots, n$, find the α-optimistic value $r_j^{Sup(\alpha)}$ and the α-pessimistic value $r_j^{Inf(\alpha)}$. Convert the model with rough returns into a model with interval parameters. ➢ Obtain Model (4).
Step 3.	Create the lower and upper bound models to evaluate the model with interval parameters. ➢ Obtain Models (5) and (6) and their equivalents, i.e. Models (7) and (8).
Step 4.	Change variables and parameters following the vector transformation: $y = \Phi^{-1}(\beta)\sqrt{D}\, Q\, x$ where $x = (x_1, x_2, ..., x_n)$, D and Q are the diagonal and orthogonal matrices decomposing the variance-covariance matrix of the returns, and Φ is the cumulative distribution function of the standard normal random variable.
Step 5.	Formulate the upper bound and lower bound models based upon the variable/parameter changes of Step 4. ➢ Obtain Models (10) and (11) and their equivalents, i.e. Models (12) and (13).
Step 6.	Solve the models of Step 5 using familiar software such as *Matlab* and *GAMS*.

5 Probability maximization model with Ra-Ro variables

In the previous section, we considered the optimization problem of a decision maker who needs to maximize his total future return f when the returns of the financial assets are assumed to be Ra-Ro variables. Suppose that instead of maximizing f, a decision maker sets a target value for the total future return f. Then, the decision maker will need to consider a PS problem maximizing the probability that the future return is greater than the target value.

In this section, we consider such a maximization model as an application of the proposed CC PS model with Ra-Ro returns. Since the probability maximization model considered in this section is an extension of the PS CC model proposed in Section 4, the relative solution method is the same as the one developed in Section 4, even though suitably expanded and adapted to the different goals required by the probability model. This contributes to show the flexibility of the main CC model and the strength of the solution method.

Let f be the target value with respect to which the decision maker compares his total future return. We propose the following formalization of the PS problem faced by the decision maker.

$$\begin{aligned} &\max \quad \tilde{P} = \Pr\left\{ \sum_{j=1}^{n} \tilde{\overline{r}}_j x_j \geq f \right\} \\ &\text{s.t.} \\ &\sum_{j=1}^{n} a_j x_j \leq b, 0 \leq x_j \leq \hat{b}_j, j = 1, \ldots, n. \end{aligned} \tag{14}$$

Since this objective function includes Ra-Ro variables, it is not a well-defined problem. Thus, we introduce the probability chance-constraint to shift the uncertainty from the objective function to the constraints. We obtain the following:

$$\begin{aligned} &\max \ \beta \\ &\text{s.t.} \\ &\Pr\left\{ \sum_{j=1}^{n} \tilde{\overline{r}}_j x_j \geq f \right\} \geq \beta, \\ &\sum_{j=1}^{n} a_j x_j \leq b, 0 \leq x_j \leq \hat{b}_j, j = 1, \ldots, n. \end{aligned} \tag{15}$$

Proceeding as in Section 4, we can convert model (15) into the following:

$$\begin{aligned} &\max_x \beta \\ &s.t. \\ &\Phi^{-1}(\beta) \leq \frac{\sum_{j=1}^{n} \overline{r}_j x_j - f}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j}}, \sum_{j=1}^{n} a_j x_j \leq b, 0 \leq x_j \leq \hat{b}_j, j = 1, \ldots, n. \end{aligned} \tag{16}$$

Since $\Phi^{-1}(\beta)$ is increasing with respect to β, this problem is equivalent to the following:

$$\begin{aligned} &\max_x \quad \frac{\sum_{j=1}^{n} \overline{r}_j x_j - f}{\sqrt{\sum_{j=1}^{n} \sum_{i=1}^{n} \sigma_{ij} \ x_i x_j}} \\ &\text{s.t.} \\ &\sum_{j=1}^{n} a_j x_j \leq b, 0 \leq x_j \leq \hat{b}_j, j = 1, \ldots, n. \end{aligned} \tag{17}$$

The above model is equivalent to the lower and upper bound models obtained by using the α-pessimistic value $r_j^{\mathrm{Inf}(\alpha)}$ and α-optimistic value $r_j^{\mathrm{Sup}(\alpha)}$ of the rough variable $\overline{r}_j$ with $0.5 \leq \alpha \leq 1$. The lower bound model is:

$$\begin{aligned} &Z_\alpha^L = \max_x \frac{\sum_{j=1}^{n} r_j^{\mathrm{Sup}(\alpha)} x_j - f}{\sqrt{\sum_{j=1}^{n} \sum_{i=1}^{n} \sigma_{ij} \ x_i x_j}} \\ &\text{s.t.} \\ &\sum_{j=1}^{n} a_j x_j \leq b, 0 \leq x_j \leq \hat{b}_j, j = 1, \ldots, n. \end{aligned} \tag{18}$$

The upper bound model is:

$$\begin{aligned} &Z_\alpha^U = \max_x \frac{\sum_{j=1}^{n} r_j^{\mathrm{Inf}(\alpha)} x_j - f}{\sqrt{\sum_{j=1}^{n} \sum_{i=1}^{n} \sigma_{ij} \ x_i x_j}} \\ &\text{s.t.} \\ &\sum_{j=1}^{n} a_j x_j \leq b, 0 \leq x_j \leq \hat{b}_j, j = 1, \ldots, n. \end{aligned} \tag{19}$$

Furthermore, applying transformations similar to those described in Section 4, models (18) and (19) are transformed into the following:

$$\begin{aligned} &Z_\alpha^L = \max_y \quad \frac{\sum_{j=1}^{n} r_j^{*\mathrm{Sup}(\alpha)} y_j - f}{\sqrt{\sum_{j=1}^{n} y_j^2}} \\ &\text{s.t.} \\ &\sum_{j=1}^{n} a_j^* y_j \leq b, \quad 0 \leq y_j \leq \hat{b}_j^*, j = 1, \ldots, n \end{aligned} \tag{20}$$

and

$$\begin{aligned} &Z_\alpha^U = \max_y \quad \frac{\sum_{j=1}^{n} r_j^{*\mathrm{Inf}(\alpha)} y_j - f}{\sqrt{\sum_{j=1}^{n} y_j^2}} \\ &\text{s.t.} \\ &\sum_{j=1}^{n} a_j^* y_j \leq b, \quad 0 \leq y_j \leq \hat{b}_j^*, j = 1, \ldots, n \end{aligned} \tag{21}$$

The above models are equivalent to following two models, respectively.

The lower bound model is as follows:

$$Z_\alpha^L = \min_y \quad \frac{\sqrt{\sum_{j=1}^n y_j^2}}{\sum_{j=1}^n r_j^{*\mathrm{Sup}(\alpha)} y_j - f} \tag{22}$$

s.t.

$$\sum_{j=1}^n a_j^* y_j \le b, \quad 0 \le y_j \le \hat{b}_j^*, j = 1, \ldots, n$$

The upper bound model is as follows:

$$Z_\alpha^U = \min_y \quad \frac{\sqrt{\sum_{j=1}^n y_j^2}}{\sum_{j=1}^n r_j^{*\mathrm{Inf}(\alpha)} y_j - f} \tag{23}$$

s.t.

$$\sum_{j=1}^n a_j^* y_j \le b, \quad 0 \le y_j \le \hat{b}_j^*, j = 1, \ldots, n$$

Using the following transformation of Charnes and Cooper [68],

$$\delta = \frac{1}{\sum_{j=1}^n r_j^{*\mathrm{Inf}(\alpha)} y_j - f},$$

we obtain:

$$\delta \left(\sum_{j=1}^n r_j^{*\mathrm{Inf}(\alpha)} y_j - f \right) = 1.$$

Also, we have:

$$\delta \sqrt{\sum_{j=1}^n y_j^2} = \sqrt{\delta^2 \sum_{j=1}^n y_j^2} = \sqrt{(\delta y)^t (\delta y)}.$$

Thus, using the following variable changes

$$\delta \ y_j = z_j, \ \ddot{f} = f \ \delta, \ \ddot{b} = b \ \delta, \ \ddot{\hat{b}}_j^* = \hat{b}_j^* \ \delta$$

and substituting them in models (22) and (23), the lower and upper bound models become, respectively:

$$Z_\alpha^L = \min_z \quad \sqrt{\sum_{j=1}^n z_j^2} \tag{24}$$

s.t.

$$\sum_{j=1}^n r_j^{*Inf(\alpha)} z_j - \ddot{f} = 1 \ \sum_{j=1}^n a_j^* z_j \le \ddot{b}, \quad 0 \le z_j \le \ddot{\hat{b}}_j^*, j = 1, \ldots, n$$

and

$$Z_\alpha^U = \min_z \quad \sqrt{\sum_{j=1}^n z_j^2} \tag{25}$$

s.t.

$$\sum_{j=1}^n r_j^{*\mathrm{Sup}(\alpha)} z_j - \ddot{f} = 1 \ \sum_{j=1}^n a_j^* z_j \le \ddot{b}, \quad 0 \le z_j \le \ddot{\hat{b}}_j^*, j = 1, \ldots, n$$

Finally, as in Section 4, let $u^2 = \sum_{j=1}^n z_j^2$. The lower bound model becomes:

$$Z^L = \min_z \quad u \tag{26}$$

s.t.

$$\sum_{j=1}^n r_j^{*\mathrm{Inf}(\alpha)} z_j - \ddot{f} = 1 \ \sum_{j=1}^n a_j^* z_j \le \ddot{b}, \quad 0 \le z_j \le \ddot{\hat{b}}_j^*, j = 1, \ldots, n$$

$$u^2 = \sum_{j=1}^n z_j^2 \ddot{f} \ge 0, \quad u \ge 0, \quad z_j \ge 0, j = 1, \ldots, n$$

while the upper bound model becomes:

$$Z^U = \min_z \quad u \tag{27}$$

s.t.

$$\sum_{j=1}^n r_j^{*\mathrm{Sup}(\alpha)} z_j - \ddot{f} = 1 \ \sum_{j=1}^n a_j^* z_j \le \ddot{b}, \quad 0 \le z_j \le \ddot{\hat{b}}_j^*, j = 1, \ldots, n$$

$$u^2 = \sum_{j=1}^n z_j^2 \ddot{f} \ge 0, \quad u \ge 0, \quad z_j \ge 0, j = 1, \ \ldots, \ n$$

The probability maximization model and the corresponding solution method are summarized in Fig. 2.

6 Numerical example

Let us consider seven securities and let their returns be described by the Ra-Ro variables reported in Table 1. Thus, for every $j = 1, \ldots, 7$, the return of the jth financial asset is a normally distributed Ra-Ro variable, $\tilde{\bar{r}}_j \sim N(\bar{r}_j, 0.2)$, whose mean is a rough variable $\bar{r}_j = ([A_j, B_j], [C_j, D_j])$ with $C_j \le A_j < B_j \le D_j$.

We also assume that the returns of different *securities* are independent from each other. This implies that $\mathrm{Cov}(\tilde{\bar{r}}_j, \tilde{\bar{r}}_i) = 0$, whenever $j \ne i$, and, consequently, that the variance-covariance matrix V is already a diagonal matrix.

For every $j = 1, \ldots, 7$, we fix the upper bound for the budget x_j to invest in the jth financial asset to be $\hat{b}_j = 0.2$ and the cost of investing in the jth financial asset to be $a_j = 1$. Finally, as it is usually the case in practical PS situations, we set the sum of all budget proportions invested in the portfolio to be equal to 1, that is, $b = 1$ and $\sum_{j=1}^7 x_j = 1$.

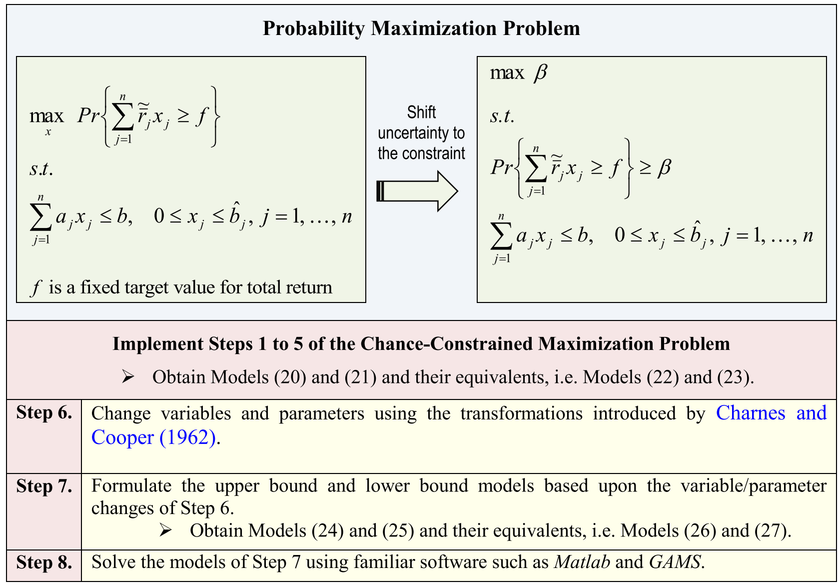

Fig. 2 Scheme of the solution method for the probability maximization model with Ra-Ro returns

We solved the PS problem just described applying the solving method introduced in Section 4. We implemented the six steps of the proposed method by using the following pairs of values for the trust level α and the fractile probability level β:

$(\alpha = 0.6, \beta = 0.90)$; $(\alpha = 0.7, \beta = 0.90)$;
$(\alpha = 0.8, \beta = 0.90)$; $(\alpha = 0.9, \beta = 0.90)$; and
$(\alpha = 1.0, \beta = 0.90)$.

In particular, for the specific PS problem under consideration, models (7) and (8), obtained by the decision maker in step 3, can be written as follows:

Table 1 Random-rough returns

Security	Return
1	$\tilde{\bar{r}}_1 \sim N(\bar{r}_1, 0.2)$ with $\bar{r}_1 = ([0.3, 0.5], [0.2, 0.8])$
2	$\tilde{\bar{r}}_2 \sim N(\bar{r}_2, 0.2)$ with $\bar{r}_2 = ([0.3, 0.4], [0.2, 0.7])$
3	$\tilde{\bar{r}}_3 \sim N(\bar{r}_3, 0.3)$ with $\bar{r}_3 = ([0.5, 0.6], [0.4, 0.8])$
4	$\tilde{\bar{r}}_4 \sim N(\bar{r}_4, 0.2)$ with $\bar{r}_4 = ([0.4, 0.5], [0.3, 1])$
5	$\tilde{\bar{r}}_5 \sim N(\bar{r}_5, 0.3)$ with $\bar{r}_5 = ([0.4, 0.6], [0.3, 0.8])$
6	$\tilde{\bar{r}}_6 \sim N(\bar{r}_6, 0.3)$ with $\bar{r}_6 = ([0.4, 0.5], [0.3, 0.7])$
7	$\tilde{\bar{r}}_7 \sim N(\bar{r}_7, 0.3)$ with $\bar{r}_7 = ([0.6, 0.7], [0.5, 0.9])$

$$Z^L_{\alpha,\beta} = \min_x - \sum_{j=1}^{7} r_j^{\mathrm{Sup}(\alpha)} x_j + \Phi^{-1}(\beta)\sqrt{\sum_{j=1}^{7}\sum_{i=1}^{7} \sigma_{ij} \;\; x_i x_j} \tag{28}$$

s.t.

$$\sum_{j=1}^{7} x_j = 1, \quad 0 \leq x_j \leq 0.2, j = 1, \ldots, 7$$

$$Z^U_{\alpha,\beta} = \min_x - \sum_{j=1}^{7} r_j^{\mathrm{Inf}(\alpha)} x_j + \Phi^{-1}(\beta)\sqrt{\sum_{j=1}^{7}\sum_{i=1}^{7} \sigma_{ij} \;\; x_i x_j} \tag{29}$$

s.t.

$$\sum_{j=1}^{7} x_j = 1, \quad 0 \leq x_j \leq 0.2, j = 1, \ldots, 7$$

Tables 2 and 3 report the computational results derived by solving the upper and lower bound models (12) and (13), obtained in step 5, for each of the pairs of values (α, β) listed above.

For instance, the numerical results show that, from an optimistic viewpoint (upper bound model), if the investor wants

Table 2 Optimal values for the decision variables (upper bound model) when $\beta = 0.90$

Trust level (α)	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0.6	0.158	0.012	0.186	0.200	0.154	0.09	0.200
0.7	0.174	0.000	0.181	0.200	0.161	0.084	0.200
0.8	0.152	0.161	0.134	0.200	0.101	0.051	0.200
0.9	0.189	0.159	0.117	0.200	0.101	0.034	0.200
1.0	0.200	0.195	0.101	0.200	0.101	0.022	0.181

Table 3 Optimal values for the decision variables (lower bound model) when $\beta = 0.90$

Trust level (α)	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0.6	0.112	0.073	0.187	0.200	0.133	0.095	0.200
0.7	0.090	0.099	0.189	0.200	0.124	0.098	0.200
0.8	0.069	0.124	0.191	0.200	0.115	0.100	0.200
0.9	0.053	0.174	0.177	0.200	0.107	0.089	0.200
1.0	0.037	0.147	0.200	0.200	0.108	0.108	0.200

to maximize the investment return at a probability greater than $\beta = 0.90$ with trust levels $\alpha = 0.6$, then the optimal budget proportions that he must invest in the seven stocks under consideration are given by:

$$x_1^* = 0.158, x_2^* = 0.012, x_3^* = 0.186, x_4^* = 0.200, x_5^* = 0.154, x_6^* = 0.09, \text{ and } x_7^* = 0.200.$$

Based upon these optimal values, the maximum portion of the portfolio corresponds to $x_4^* = 0.200$ and $x_7^* = 0.200$, while the minimum portion of the portfolio is given by $x_2^* = 0.012$.

In order to make the due comparisons, we also solved the specific PS problem of this example for the following (α, β) pairs:

$$(\alpha = 0.6, \beta = 0.95); (\alpha = 0.7, \beta = 0.95); (\alpha = 0.8, \beta = 0.95); (\alpha = 0.9, \beta = 0.95); \text{ and } (\alpha = 1.0, \beta = 0.95).$$

The computational results obtained for these pairs of values implementing again the method provided in Section 4 and, hence, using the upper and lower bound models (12) and (13), are presented in Tables 4 and 5.

Table 4 Optimal values for the decision variables (upper bound model) when $\beta = 0.95$

Trust level (α)	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0.6	0.139	0.126	0.143	0.196	0.121	0.076	0.200
0.7	0.155	0.102	0.142	0.200	0.128	0.074	0.200
0.8	0.154	0.200	0.118	0.200	0.093	0.056	0.179
0.9	0.186	0.200	0.107	0.200	0.095	0.046	0.167
1.0	0.200	0.200	0.100	0.200	0.100	0.040	0.160

Table 5 Optimal values for the decision variables (lower bound model) when $\beta = 0.95$

Trust level (α)	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0.6	0.107	0.161	0.145	0.199	0.107	0.08	0.200
0.7	0.091	0.179	0.147	0.200	0.101	0.082	0.200
0.8	0.076	0.196	0.149	0.200	0.095	0.084	0.200
0.9	0.075	0.200	0.147	0.200	0.095	0.082	0.200
1.0	0.058	0.200	0.158	0.200	0.092	0.092	0.200

7 Conclusion

The existing literature on portfolio selection (PS) is considerably large and the related research focuses on the definition of solution methods versatile enough to allow for applications to different settings. In particular, future returns are usually treated as random variables with fixed expected values and variances. However, in most of the real-life PS problems, the observed values of data and parameters are imprecise or vague. The uncertainty associated with the returns in PS problems can be represented by random-rough (Ra-Ro) variables, but, to the best of the authors' knowledge, there are currently no attempts to model and solve PS problems via a chance-constrained (CC) approach in a Ra-Ro environment.

The present paper has a twofold objective: (1) filling in the gap still existing in the literature between CC programming and applications to real financial problems; (2) providing an insightful alternative to artificial intelligence (AI) approaches such as artificial neural networks, expert system and hybrid intelligent systems.

The current study has proposed a new CC model for PS optimization where the returns are stochastic variables with rough information. We have formulated a Ra-Ro mathematical programming model where the returns are represented by Ra-Ro variables and the expected future total return maximized against a given fractile probability level.

The solution method proposed for the resulting non-linear CC formulation transforms the CC model in an equivalent deterministic quadratic programming problem. The rough objective function is evaluated through lower and upper bound models obtained using interval parameters based on optimistic and pessimistic trust levels. As an application of the proposed method, we have considered a probability maximizing version of the PS problem where the goal becomes to maximize the probability that the total return is higher than a given reference value.

We have shown that the proposed method can be adapted to different formulations of the PS problem by applying suitable parameter and variable transformations and provided an elucidative numerical example.

Acknowledgements The authors would like to thank the anonymous reviewers and the editor for their insightful comments and suggestions. The second author, Dr. Khanjani Shiraz, is grateful to the Iran National Science Foundation for the support he received through grant number 93040776.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Markowitz H (1952) Portfolio selection. J Financ 7:77–91
2. Wang S, Zhu S (2002) On fuzzy portfolio selection problem. Fuzzy Optim Decis Making 1:361–377
3. Charnes A, Cooper WW (1959) Chance-constrained programming. Manag Sci 6(1):73–79
4. Liu B (2002) Theory and practice of uncertain programming, 1st edn. Physica-Verlag, Heidelberg
5. Mehlawat MK, Gupta P (2014) Fuzzy chance-constrained multiobjective portfolio selection model. IEEE Trans Fuzzy Syst 22(3):653–671
6. Omidi F, Abbasi B, Nazemi A (2017) An efficient dynamic model for solving a portfolio selection with uncertain chance constraint models. J Comput Appl Math 319:43–55
7. Bahrammirzaee A (2010) A comparative survey of artificial intelligence applications in finance: artificial neural networks, expert system and hybrid intelligent systems. Neural Comput & Applic 19:1165–1195
8. Lintner BJ (1965) Valuation of risky assets and the selection of risky investments in stock portfolios and capital budgets. Review of Economics and Statistics 47:13–37
9. Mossin J (1966) Equilibrium in capital asset markets. Econometrica 34(4):768–783
10. Sharpe WF (1964) Capital asset prices: a theory of market equivalent under conditions of risk. J Financ 19(3):425–442
11. Konno H (1990) Piecewise linear risk functions and portfolio optimization. Journal of Operations Research Society of Japan 33:139–159
12. Konno H, Shirakawa H, Yamazaki H (1993) A mean-absolute deviation-skewness portfolio optimization model. Ann Oper Res 45: 205–220
13. Bawa VS, Lindenberg EB (1977) Capital market equilibrium in a mean-lower partial moment framework. J Financ Econ 5:189–200
14. Elton EJ, Gruber MJ (1955) Modern portfolio theory and investment analysis. Wiley, New York
15. Rockafellar RT, Uryasev S (2000a) Optimization of conditional value-at-risk. Journal of Risk 2(3):1–21
16. Rockafellar RT, Uryasev S (2000b) Optimization of conditional value-at-risk. Journal of Risk 2(3):1–21
17. Best MJ, Hlouskova J (2000) The efficient frontier for bounded assets. Mathematical Methods of Operations Research 52:195–212
18. Merton R (1972) An analytic derivation of the efficient frontier. J Financ Quant Anal 7:1851–1872
19. Perold AF (1984) Large-scale portfolio optimization. Manag Sci 30:1143–1160
20. Sharp WF (1963) A simplified model for portfolio analysis. Manag Sci 9:277–293
21. Vörös J (1986) Portfolio analysis - an analytic derivation of the efficient portfolio frontier. Eur J Oper Res 23:294–300
22. Yoshimoto A (1996) The mean-variance approach to portfolio optimization subject to transaction costs. Journal of Operations Research Society of Japan 39:99–117
23. Chang J, Shi P (2011) Using investment satisfaction capability index based particle swarm optimization to construct a stock portfolio. Inf Sci 181(14):2989–2999
24. Chen L, He S, Zhang S (2011) Tight bounds for some risk measures, with applications to robust portfolio selection. Oper Res 59(4):847–855
25. Fatah K, Shi P, Ameen J, Wiltshire R (2009) A preference ranking model based on both mean–variance analysis and cumulative distribution function using simulation. International Journal of Operational Research 5(3):311–327
26. Fatah K, Shi P, Ameen J, Wiltshire R (2010) Risk averse preference models for normalised lotteries based on simulation. International Journal of Operational Research 8(2):189–207
27. Gregory C, Darby-Dowman K, Mitra G (2011) Robust optimization and portfolio selection: the cost of robustness. Eur J Oper Res 212(2):417–428
28. Huang D, Zhu S, Fabozzi F, Fukushima M (2010) Portfolio selection under distributional uncertainty: a relative robust CvaR approach. Eur J Oper Res 203:185–194
29. Li X, Shou B, Qin X (2012) An expected regret minimization portfolio selection model. Eur J Oper Res 218(2):484–492
30. Yu M, Takahashi S, Inoue H, Wang S (2010) Dynamic portfolio optimization with risk control for absolute deviation model. Eur J Oper Res 201:349–364
31. Brockett PL, Charnes A, Cooper WW, Kwon KH, Ruefli TW (1992) Chance constrained programming approach to empirical analysis of mutual fund investment strategies. Decis Sci 23:385–408
32. Li SX (1995) An insurance and investment portfolio model using chance constrained programming. Omega 23:577–585
33. Li SX, Huang Z (1996) Determination of the portfolio selection for a property-liability insurance company. Eur J Oper Res 88:257–268
34. Williams JO (1997) Maximizing the probability of achieving investment goals. J Portf Manag 24:77–81
35. Gupta P, Inuiguchi M, Mehlawat MK, Mittal G (2013) Multiobjective credibilistic portfolio selection model with fuzzy chance-constraints. Inf Sci 229:1–17
36. Hasuike T, Katagiri H, Ishii H (2009) Portfolio selection problems with random fuzzy variable returns. Fuzzy Sets Syst 160(18):2579–2596
37. Zadeh LA (1965) Fuzzy sets. Inf Control 8(3):338–353
38. Tanaka H, Guo P (1999) Portfolio selection based on upper and lower exponential possibility distributions. Eur J Oper Res 114(1): 115–126
39. Tanaka H, Guo P, Zimmermann HJ (2000) Possibility distributions of fuzzy decision variables obtained from possibilistic linear programming problems. Fuzzy Sets Syst 113(2):323–332
40. Parra MA, Terol AB, Urıa MR (2001) A fuzzy goal programming approach to portfolio selection. Eur J Oper Res 133(2):287–297
41. Carlsson C, Fullér R, Majlender P (2002) A possibilistic approach to selecting portfolios with highest utility score. Fuzzy Sets Syst 131(1):13–21
42. Ida M (2003) Portfolio selection problem with interval coefficients. Appl Math Lett 16:709–713
43. Ammar E, Khalifa HA (2003) Fuzzy portfolio optimization: a quadratic programming approach. Chaos, Solutions & Fractals 18: 1045–1054
44. Giove S, Funari S, Nardelli C (2006) An interval portfolio selection problem based on regret function. Eur J Oper Res 170:253–264
45. Gupta P, Mehlawat MK, Saxena A (2008) Asset portfolio optimization using fuzzy mathematical programming. Inf Sci 178(6): 1734–1755
46. Liu ST (2011a) The mean-absolute deviation portfolio selection problem with interval-valued returns. J Comput Appl Math 235: 4149–4157

47. Liu ST (2011b) A fuzzy modeling for fuzzy portfolio optimization. Expert Syst Appl 38:13803–13809
48. Khanjani Shiraz R, Charles V, Jalalzadeh L (2014) Fuzzy rough DEA model: a possibility and expected value approaches. Expert System with Applications 41(2):434–444
49. Tavana M, Khanjani Shiraz R, Hatami-Marbini A, Agrell P, Paryab K (2012) Fuzzy stochastic data envelopment analysis with application to base realignment and closure (BRAC). Expert Syst Appl 39(15):12247–12259
50. Tavana M, Khanjani Shiraz R, Hatami-Marbini A, Agrell P, Paryab K (2013) Chance-constrained DEA models with random fuzzy inputs and outputs. Knowl-Based Syst 52(2013):32–52
51. Tavana M, Khanjani Shiraz R, Hatami-Marbini A (2014) A new chance-constrained DEA model with Birandom input and output data. J Oper Res Soc. doi:10.1057/jors.2013.157
52. Mohagheghi V, Mousavi SM, Vahdani B, Shahriari MR (2016) R&D project evaluation and project portfolio selection by a new interval type-2 fuzzy optimization approach. Neural Comput & Applic. doi:10.1007/s00521-016-2262-3
53. Pawlak Z (1982) Rough sets. International Journal of Information and Computer Science 11:341–356
54. Liu B (1997) Dependent-chance programming: a class of stochastic programming. Computers and Mathematics with Applications 34(12):89–104
55. Liu B (2004) Uncertainty theory: an introduction to its axiomatic foundations. Springer Verlag, Berlin
56. Xu J, Yao L (2009a) A class of expected value multi-objective programming problems with random rough coefficients. Mathematical and Computer Modeling 50(1–2):141–158
57. Xu J, Yao L (2009b) A class of multiobjective linear programming models with random rough coefficients. Mathematical and Computer Modeling 49(1–2):189–206
58. Lam M (2004) Neural network techniques for financial performance prediction, integrating fundamental and technical analysis. Decision Support System 37:567–581
59. West D, Scott D, Qian J (2005) Neural network ensemble strategies for financial decision applications. Comput Oper Res 32(10):2543–2559
60. Fernandez A, Gomez S (2007) Portfolio selection using neural networks. Comput Oper Res 34:1177–1191
61. Ko PC, Lin PC (2008) Resource allocation neural network in portfolio selection. Expert Syst Appl 35(1–2):330–337
62. Freitas FD, De Souza AF, de Almeida AR (2009) Prediction-based portfolio optimization model using neural networks. Neurocomputing 72(10–12):2155–2170
63. Hsu VM (2013) A hybrid procedure with feature selection for resolving stock/futures price forecasting problems. Neural Comput & Applic 22:651–671
64. Huang X (2007a) Two new models for portfolio selection with stochastic returns taking fuzzy information. Eur J Oper Res 180(1):396–405
65. Huang X (2007b) A new perspective for optimal portfolio selection with random fuzzy returns. Inf Sci 177(23):5404–5414
66. Li X, Zhang Y, Wong HS, Qin Z (2009) A hybrid intelligent algorithm for portfolio selection problem with fuzzy returns. J Comput Appl Math 233(2):264–278
67. Halmos P (1974) Naive set theory. Springer, New York
68. Charnes A, Cooper WW (1962) Programming with linear fractional functionals. Naval Research Logistics Quarterly 9:181–186