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A robust cross-efficiency data envelopment analysis model with undesirable outputs

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ABSTRACT

Degenerate optimal weights and uncertain data are two challenging problems in conventional data envelopment analysis (DEA). Cross-efficiency and robust optimization are commonly used to handle such problems. We develop two DEA adaptations to rank decision-making units (DMUs) characterized by uncertain data and undesirable outputs. The first adaptation is an interval approach, where we propose lower- and upper-bounds for the efficiency scores and apply a robust cross-efficiency model to avoid problems of non-unique optimal weights and uncertain data. We initially use the proposed interval approach and categorize DMUs into fully efficient, efficient, and inefficient groups. The second adaptation is a robust approach, where we rank the DMUs, with a measure of cross-efficiency that extends the traditional classification of efficient and inefficient units. Results show that we can obtain higher discriminatory power and higher-ranking stability compared with the interval models. We present an example from the literature and a real-world application in the banking industry to demonstrate this capability.

1. Introduction

Data envelopment analysis (DEA), originated by Charnes, Cooper, and Rhodes (1978), is a non-parametric data-driven mathematical method that has been developed for evaluating the efficiency of a set of similar decision-making units (DMUs) with multiple inputs and multiple outputs. The Charnes, Cooper, and Rhodes (CCR) model assesses the technical (radial) efficiencies of DMUs under the constant returns to scale assumption. Banker, Charnes, and Cooper (1984) formulated the BCC model as an extension of the CCR model to consider variable returns to scale. The conventional DEA models assign a weight for each input and output and measure the performance of a DMU, known as a unit under evaluation, as the ratio of the weighted sum of outputs (virtual output) to the weighted sum of inputs (virtual input). Each DMU is evaluated by a set of weights that have been optimized in favor of the DMU.

Traditional DEA classifies DMUs as being efficient or inefficient but fails to discriminate between efficient DMUs. A common problem in DEA literature is to rank the DMUs linearly. A wide range of studies has been undertaken to address the ranking problem in DEA. Cook, Kress, and Seiford (1992) developed a prioritization approach for ranking efficient units in DEA by imposing various conditions on the weights (multipliers) in a DEA analysis. Andersen and Petersen (1993) introduced the super-efficiency model, where the DMU under evaluation is removed from the production possibility set. Liu and Peng (2008) extended a common-weight model to improve the discriminatory power of the ranking of the performance indices of efficient DMUs. To improve the discriminatory power between efficient and inefficient DMUs, Charles, Aparicio, and Zhu (2019) have proposed a simple method using the well-known pure DEA model, which considers either inputs only or outputs only.

Sexton, Silkman, and Hogan (1986) introduced the cross-efficiency concept in DEA, which allows the overall efficiency of a DMU to be

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evaluated through self-evaluation or peer-evaluation. In contrast to conventional DEA, where a DMU is evaluated by its optimal weights (self-evaluation), cross-efficiency appraises a DMU by a set of weights that are optimally obtained in favor of all other DMUs (peer-evaluation). Optimal weights for efficient DMU are not always unique, which decreases the usefulness of cross-efficiency evaluation. However, different secondary goals, including aggressive and benevolent models, have been introduced to tackle this issue (Doyle & Green, 1994; Sexton et al., 1986). Lin, Chen, and Xiong (2016) introduced an iterative method that provides a unique set of weights for positive input and output data, which decreases the number of zero weights without applying any prior weight restriction. Ruiz (2013) formulated a directional distance cross-efficiency model to assess the performance of 28 European, North American, and Asian–Australian airlines. Oral, Amin, and Oukil (2015) suggested a maximum resonated appreciative model to address the problem of multiple optimal solutions in the cross-efficiency approach. Lim (2012) suggested a pair of minimax and maximin formulations as aggressive and benevolent cross-efficiency DEA models, respectively.

Li, Zhu, and Liang (2018) consider inconsistency and unbalancing in the cross-efficiency methods to (i) suggest a practical adjustment measure to rectify the traditional cross-efficiency and (ii) propose a game-like iterative procedure to obtain an optimal balanced cross-efficiency. Kao and Liu (2019) propose a decomposable cross-efficiency model to measure two basic structures of network systems (i.e., series and parallel). Seyedalizadeh Ganji, Rassafi, and Xu (2019) apply the standard and the inverted input-oriented CCR models and develop a novel double-frontier cross-efficiency scheme that accounts for both optimistic and pessimistic perspectives simultaneously. Abolghasem, Toloo, and Amézquita (2019) propose a cross-efficiency approach in the presence of flexible measures, which simultaneously plays the role of input and output, to evaluate the performance of healthcare systems in 120 countries (for more details on the flexible measure, see Toloo, Keshavarz, & Hatami-Marbini, 2018).

Some production processes have undesirable outputs whose reduction results in better performance. For instance, overdue debts are an undesirable output and should be reduced (not increased). In contrast to desirable inputs and outputs in conventional DEA, undesirable inputs and outputs should be increased and decreased, respectively. Färe, Grosskopf, Lovell, and Pasurka (1989) provided a non-linear DEA model to deal with desirable and undesirable outputs. Seiford and Zhu (2002) preserved convexity and linearity to devise an alternative method for considering undesirable factors in DEA under variable returns to scale. Liu, Meng, Li, and Zhang (2010) launched a systematic investigation into the construction of DEA models without transferring undesirable data. Liu, Chu, Yin, and Sun (2017) introduced a technique for DEA cross-efficiency evaluation in the presence of undesirable outputs and suggested an equitable evaluation model for efficiency evaluation. Toloo and Hančlová (2020) model real-world problems in which one may encounter *multi-valued* measures, which are measured by various standards, and only one of their values is to be selected. They formulate two individual and summative selective directional distance models in the presence of undesirable outputs, and they developed a pair of the multiplier- and envelopment-based selection approaches.

The DEA method offers benefits compared to other statistical approaches for efficiency measurement. However, one crucial issue with DEA is its sensitivity to perturbation, and various uncertain methods have been proposed to address this concern. Non-deterministic DEA approaches can be classified into four general streams: (i) stochastic (Charles & Cornillier, 2017; Olesen & Petersen, 2016), (ii) fuzzy (Aghayi, 2016; Hatami-Marbini, Emrouznejad, & Tavana, 2011), (iii) interval (Despotis & Smirlis, 2002; Toloo, Aghayi, & Rostamy-Malkhalifeh, 2008), and (iv) robust optimization (RO) (Toloo & Mensah, 2019). We limit our focus on RO and interval approaches herein.

An interval approach was developed by Cooper, Park, and Yu (1999) in which data lies within bounded intervals. One of the most challenging issues with this approach is that the evaluation of efficiency scores is based on the lower- and upper-bounds of the relative efficiencies of DMUs. Despite this problem, many researchers have focused on developing variations (e.g., Despotis & Smirlis, 2002; Entani, Maeda, & Tanaka, 2002; Wang, Greatbanks, & Yang, 2005; Kao, 2006). The objective of most interval approaches, such as Entani et al. (2002) and Despotis and Smirlis (2002), is to calculate lower- and upper-bounds of the relative efficiencies of the DMUs. Regardless of scale transformations on the data suggested in Cooper et al. (1999) study, Despotis and Smirlis (2002) drew on variable alterations and developed a new pair of DEA models to estimate lower- and upper limits of efficiency values. Considering Despotis and Smirlis (2002) and Wang et al. (2005), researchers have proposed different radial and non-radial DEA models to handle uncertainty (e.g., Hatami-Marbini, Emrouznejad, & Agrell, 2014; Toloo et al., 2018; Hatami-Marbini, Ghelej Beigi, Hougaard, & Gholami, 2018; Ebrahimi & Toloo, 2020; Ye, Yang, & Wang, 2019).

RO provides an alternative approach to handle imprecise data, which assumes data belongs to an uncertainty set. Soyster (1973) introduced the RO approach and proposed a robust model for linear programming problems in which the constraints are satisfied under all possible perturbations of model parameters. Ben-Tal and Nemirovski (2000) revisited RO and showed that a small perturbation on data might make the usual optimal solution completely meaningless and even infeasible. The authors suggested a new idea for modeling uncertain data based on ellipsoidal uncertainty sets and proposed a second-order cone programming model. Bertsimas and Sim (2004) considered a polyhedral uncertainty set and proposed an RO approach whose level of robustness is adjustable. In comparison with the RO approaches, where the objective is to find solutions that are immune to all perturbations of the data in an uncertainty set, a branch of RO called adjustable RO has evolved where some of the decision variables are adjusted after some portion of the uncertain data reveals itself. Yanıkoğlu, Gorissen, and den Hertog (2018) provided a recent state-of-the-art literature review of the applications and theoretical/methodological aspects of adjustable RO.

Robust DEA has drawn the attraction of a wide range of researchers. Sadjadi and Omrani (2008), for the first time, applied the RO approaches presented by Ben-Tal and Nemirovski (2000) and Bertsimas and Sim (2004) to develop a pair of DEA models for assessing the performance of electricity distribution companies under data uncertainty. Shokouhi, Hatami-Marbini, Tavana, and Saati (2010) presented a robust DEA model by applying the RO approach to Despotis and Smirlis (2002) interval DEA models and analyzed the proposed model using Monte-Carlo simulation. Omrani (2013) introduced an RO approach for achieving a common set of weights in DEA with data uncertainty in inputs and outputs. Aghayi, Tavana, and Raayatpanah (2016) suggested a robust DEA model with a conventional common set of weights and different levels of conservatism and data uncertainty and used goal programming to calculate the relative efficiency scores of DMUs. To address the uncertainty problem, Aghayi and Maleki (2016) combined an output-oriented version of the CCR model with the existing interval uncertainty in both desirable and undesirable outputs, which resulted in two evaluation approaches. The RO approach also provides the basis for a study by Arabmaldar, Jablonsky, and Hosseinzadeh Saljooghi (2017), where two linear robust super-efficiency models are presented for ranking DMUs under uncertainty in both input and output spaces. Zahedi-seresht, Jahanshahloo, and Jablonsky (2017) present an approach for scenario-based RO by considering data uncertainty in DEA models. Ehrgott, Holder, and Nohadani (2018) propose an uncertain DEA model for which an optimal solution obtains the maximum possible efficiency score by considering the minimal amount of uncertainty required to achieve this efficiency score.

Toloo and Mensah (2019) suggest a reduced robust DEA approach with nonnegative decision variables to decrease computational burden. Salahi, Toloo, and Hesabirad (2019) propose equivalent formulations of the robust Russell measure and its enhanced models under interval and ellipsoidal uncertainties in their best- and worst-cases. The authors show that the suggested formulations stay convex for both best- and worst-cases under interval uncertainty as well as worst-case with ellipsoidal uncertainty. Lu, Tao, An, and Lai (2019) address the validity of performance evaluation in the presence of imprecise and negative data and propose a second-order cone based robust DEA model. Yousefi, Alizadeh, Hayati, and Baghery (2018) review the shortcomings of the most widely used methods for health, safety, and environment management system, and propose an integrated robust DEA approach to evaluate and prioritize the health, safety, and environment management risks in various industries. Amirkhan, Didehkhani, Khalili-Damghani, and Hafezalkotob (2018) consider mixed fuzzy-robust uncertainties in DEA and propose scenario-based robust DEA models under different return to scale conditions. Salahi, Toloo, and Torabi (2020) develop the robust counterpart for the envelopment form of the CCR model using the Bertsimas and Sim (2004)'s robust approach and then extend the robust CCR solutions to find a robust common set of weights. Shirazi and Mohammadi (2020) develop a robust slacks-based measure in the presence of undesirable output and use it to evaluate the efficiency in the airline industry.

We focus in this paper on ranking DMUs in the presence of interval input data, and we permit desirable and undesirable interval outputs through a cross-efficiency evaluation. However, DEA can have degenerate optimal weights, and we propose a model to circumvent this issue. Our model results in efficiency measurements from both optimistic and pessimistic viewpoints. The RO approach assesses DMUs for which the efficiency value is more certain than the interval approach.

The main contributions of this study are twofold. First, we develop two DEA adaptations (an interval approach and a robust approach) to rank DMUs characterized by uncertain data and undesirable outputs. Second, we present an example from the literature and a real-world application to compare our method with an interval method. This example demonstrates the ability of our approach to improving discernibility among DMUs.

The remainder of this paper is organized as follows. We review DEA models in the presence of undesirable outputs in Section 2. We propose a pair of interval models based on interval DEA approaches in the presence of imprecise data and undesirable outputs in Section 3. We also use cross-efficiency evaluation and develop a model for enhancing discrimination between DMUs. Section 4 introduces a new cross-efficiency evaluation method by integrating the proposed models in Section 3 with an RO approach. A case study and a comparative example are presented in Section 5 to illustrate the practical application of the proposed methods. Section 6 presents our conclusions and future research directions.

2. DEA cross-efficiency evaluation considering the undesirable output

We assume there are DMUs, indexed by j, each producing s semi-positive outputs, $y_{rj}(r=1,\ldots,s)$, by using m semi-positive inputs, $x_{ij}(i=1,\ldots,m)$. The CCR model assesses the input-oriented technical efficiency of the unit being evaluated, DMU_d:

$$
\begin{aligned}
E_{dd} &= \max \sum_{r=1}^{s} u_{rd} y_{rd} \\
s.t. & \\
&\sum_{i=1}^{m} v_{id} x_{id} = 1 \\
&\sum_{i=1}^{m} v_{id} x_{ij} - \sum_{r=1}^{s} u_{rd} y_{rj} \geq 0, j=1,\ldots,n \qquad (1) \\
&v_{id} \geq 0, i=1,\ldots,m \\
&u_{rd} \geq 0, r=1,\ldots,s
\end{aligned}
$$

where v_{id} and μ_{rd} are the respective i^{th} input and r^{th} output weights. The value of E_{dd} is the CCR-efficiency score of DMU_d.

Let $\left(v_d^*, u_d^*\right) \in \mathbb{R}^{m+s}$ be an optimal solution of the CCR model (1), then the cross-efficiency score of DMU_j corresponding to DMU_d is:

$$E_{dj} = \frac{\sum_{r=1}^{s} u_{rd}^* y_{rj}}{\sum_{i=1}^{m} v_{id}^* x_{ij}} \qquad (2)$$

The CCR model does not allow undesirable outputs, and we divided outputs into those that are desirable, $y_{rj}(r=1,\ldots,s)$, and those that are undesirable, $b_{tj}(t=1,\ldots,k)$. Liu et al. (2017) suggest the following model to evaluate the performance of DMU_d in the presence of undesirable outputs:

$$
\begin{aligned}
E_{dd} &= \max \sum_{r=1}^{s} u_{rd} y_{rd} + \sum_{t=1}^{k} w_{td} \bar{b}_{td} \\
s.t. & \\
&\sum_{i=1}^{m} v_{id} x_{id} = 1 \\
&\sum_{i=1}^{m} v_{id} x_{ij} - \sum_{r=1}^{s} u_{rd} y_{rj} - \sum_{t=1}^{k} w_{td} \bar{b}_{tj} \geq 0, j=1,\ldots,n \qquad (3) \\
&\sum_{r=1}^{s} u_{rd} y_{rd} \geq \alpha\varepsilon \\
&\sum_{t=1}^{k} w_{td} \bar{b}_{td} \geq \beta\varepsilon \\
&v_{id} \geq 0, i=1,\ldots,m \\
&u_{rd} \geq 0, r=1,\ldots,s \\
&w_{td} \geq 0, t=1,\ldots,k,
\end{aligned}
$$

where w_{td} is the weight assigned to the t^{th} undesirable output and $\bar{b}_{tj} = -b_{tj} + w > 0, (t=1,\ldots,k)$ is the t^{th} undesirable output of $\mathrm{DMU}_j(j=1,\ldots,n)$. More specifically, at first, each undesirable output is multiplied by (-1). Then by finding the k-dimensional vector $w=(w_1,\ldots,w_k)$, the t^{th} negative undesirable outputs of DMU_j is transformed into a positive value. We refer the reader to Seiford and Zhu (2002) for a more in-depth discussion of this transformation. The current study adopts a linear monotone, decreasing transformation for dealing with undesirable outputs in Model (3). Moreover, ε is the non-Archimedean infinitesimal that prevents each term of the objective function from being zero, and α and β denote the lowest percentages of ε that which are the lower-bounds for the weighted sum of desirable outputs and the weighted sum of undesirable outputs of the DMU under evaluation, respectively. In other words, α and β vary between 0 and 1. If one sets $\alpha=\beta=0$, then it means the model can assign zero value to some of the weights of desirable and undesirable outputs. In addition, $\alpha=\beta=1$ means that $\varepsilon>0$ is a lower bound of the weighted sum of desirable outputs and the weighted sum of undesirable outputs of the unit under evaluation. Appendix A provides an approach for finding a suitable value for ε in Model (3). Running Model (A.1) for each DMU and $\varepsilon^* = \min\left\{\varepsilon_1^*,\ldots,\varepsilon_n^*\right\}$ provides the most suitable value for epsilon. Moreover, the epsilon (ε) plays an important role in the selection-based problems, and therefore finding the best value for ε is essential in such problems (see Toloo, 2014). Mehrabian, Jahanshahloo, Alirezaee, and Amin (2000) introduced the *assurance interval* $[0,\varepsilon^*]$ and *assurance value* $\varepsilon \in [0,\varepsilon^*]$ in which the maximum ε^* is a positive number such that the model is feasible for all $\varepsilon \in [0,\varepsilon^*]$ and is infeasible for all $\varepsilon > \varepsilon^*$. In addition, choosing a small assurance value keeps the epsilon-based DEA models feasible (Amin & Toloo, 2004), and selecting a larger assurance value improves their discriminatory power (Cook, Kress, & Seiford, 1996).

Model (3) prevents the weights of desirable and undesirable outputs from being zero. In other words, the model produces a set of strictly positive weights, which means all desirable and undesirable outputs have been considered in the evaluation process. Liu et al. (2017) (Theorem 1, p. 880) proves Model (3) has strictly positive weights and better discrimination power. These properties inspired us to use Model (3) and develop our models.

Assume that $(v_d^*, u_d^*, w_d^*) \in \mathbb{R}^{m+s+k}$ is an optimal solution of Model (3), then, the cross-efficiency of DMU_j corresponding to DMU_d is:

$$E_{dj} = \frac{\sum_{r=1}^{s} u_{rd}^* y_{rj} + \sum_{t=1}^{k} w_{td}^* \bar{b}_{tj}}{\sum_{i=1}^{m} v_{id}^* x_{ij}} \tag{4}$$

The cross-efficiency score E_j measures the average of $E_{dj}(d = 1, ..., n)$. Finally, the resulting cross-efficiency scores are defined as new efficiency measures for all DMUs.

3. Proposed cross-efficiency models with undesirable outputs based on interval approach

Some DEA models have been extended to measure interval efficiency in the presence of interval data and undesirable outputs. We extend the methods proposed by Despotis and Smirlis (2002) and Liu et al. (2017) to measure cross-efficiency with undesirable outputs where the lower-bound of the weighted sum of desirable and the weighted sum of undesirable outputs of the unit under evaluation is maximized.

Suppose uncertain inputs, desirable outputs, and undesirable outputs are known to lie within intervals $[x_{ij}^L, x_{ij}^U]$, $[y_{rj}^L, y_{rj}^U]$, and $[b_{tj}^L, b_{tj}^U]$, respectively, where $x_{ij}^L \geq 0$, $y_{rj}^L \geq 0$, and $b_{tj}^L \geq 0$. Upper- and lower-bounds are suggested for the efficiency values of DMUs. The calculation of the upper-bound presents the DMU under evaluation in the best-case scenario and the other DMUs in the worst-case scenario (optimistic perspective). So, we consider the lower-bound of inputs and the upper-bounds of desirable and undesirable outputs, and for the other DMUs, we consider the upper-bound of inputs and the lower-bounds of desirable and undesirable outputs. The lower-bound of the DMU under evaluation in the worst-case scenario and the other DMUs in the best-case scenario. So we consider the upper-bound of inputs and the lower-bounds of desirable and undesirable outputs for the DMU under evaluation, and the lower-bound of inputs and the upper-bounds of desirable and undesirable outputs for the other DMUs. We then extend the proposed method by Despotis and Smirlis (2002) and modify Model (3) to evaluate optimistic and pessimistic perspectives, respectively:

$$e_{dd}^U = \max \sum_{r=1}^{s} u_{rd} y_{rd}^U + \sum_{t=1}^{k} w_{td} \bar{b}_{td}^U$$

$$s.t.$$

$$\sum_{i=1}^{m} v_{id} x_{id}^L = 1$$

$$\sum_{i=1}^{m} v_{id} x_{ij}^U - \sum_{r=1}^{s} u_{rd} y_{rj}^L - \sum_{t=1}^{k} w_{td} \bar{b}_{tj}^L \geq 0, j = 1, ..., n, j \neq d$$

$$\sum_{i=1}^{m} v_{id} x_{id}^L - \sum_{r=1}^{s} w_{rd} y_{rd}^U - \sum_{t=1}^{k} w_{td} \bar{b}_{td}^U \geq 0 \tag{5}$$

$$\sum_{r=1}^{s} u_{rd} y_{rd}^U \geq \alpha \varepsilon_d^{*U}$$

$$\sum_{t=1}^{k} w_{td} \bar{b}_{td}^U \geq \beta \varepsilon_d^{*U}$$

$$v_{id} \geq 0, i = 1, ..., m$$

$$u_{rd} \geq 0, r = 1, ..., s$$

$$w_{td} \geq 0, t = 1, ..., k$$

and

$$e_{dd}^L = \max \sum_{r=1}^{s} u_{rd} y_{rd}^L + \sum_{t=1}^{k} w_{td} \bar{b}_{td}^L$$

$$s.t.$$

$$\sum_{i=1}^{m} v_{id} x_{id}^U = 1$$

$$\sum_{i=1}^{m} v_{id} x_{ij}^L - \sum_{r=1}^{s} u_{rd} y_{rj}^U - \sum_{t=1}^{k} w_{td} \bar{b}_{tj}^U \geq 0, j = 1, ..., n, j \neq d$$

$$\sum_{i=1}^{m} v_{id} x_{id}^U - \sum_{r=1}^{s} u_{rd} y_{rd}^L - \sum_{t=1}^{k} w_{td} \bar{b}_{td}^L \geq 0 \tag{6}$$

$$\sum_{r=1}^{s} u_{rd} y_{rd}^L \geq \alpha \varepsilon_d^{*L}$$

$$\sum_{t=1}^{k} w_{td} \bar{b}_{td}^L \geq \beta \varepsilon_d^{*L}$$

$$v_{id} \geq 0, i = 1, ..., m$$

$$u_{rd} \geq 0, r = 1, ..., s$$

$$w_{td} \geq 0, t = 1, ..., k$$

Model (5) evaluates DMU_d in favor of the unit and aggressively against the other units, i.e., $\text{DMU}_d = (x_d^L, y_d^U, \bar{b}_d^U)$ and $\text{DMU}_j = \left(x_j^U, y_j^L, \bar{b}_j^L\right) \forall j \neq k$. Conversely, in Model (6), the levels of inputs and outputs are adjusted unfavorably for the unit under evaluation and in favor of the other units, i.e., i.e. $\text{DMU}_d = (x_d^U, y_d^L, \bar{b}_d^L)$ and $\text{DMU}_j = \left(x_j^L, y_j^U, \bar{b}_j^U\right) \forall j \neq k$.

The efficiency score of DMU_d from Models (5) and (6) is not certain and belongs to the interval $[e_{dd}^L, e_{dd}^U]$, and the DMUs are allocated to one of the following categories based on these intervals:

- Fully efficient: $E^{++} = \{DMU_j | e_{dd}^L = 1\}$,
- Efficient: $E^{+} = \{DMU_j | e_{dd}^L < 1, e_{dd}^U = 1\}$, or
- Inefficient: $E^{-} = \{DMU_j | e_{dd}^U < 1\}$.

Appendix B suggests a method for finding a suitable value for ε in Models (5) and (6). Since more than one DMU is likely to be identified as efficient by Model (5), we prefer to improve the discrimination power between DMUs. Let us assume that (v_d^*, u_d^*, w_d^*) is an optimal solution to Model (5). Then the cross-efficiency of DMU_j corresponding to DMU_d is:

calculated as follows:

$$E_{dj}^U = \frac{\sum_{r=1}^{s} u_{rd}^* y_{rj}^U + \sum_{t=1}^{k} w_{td}^* \bar{b}_{tj}^U}{\sum_{i=1}^{m} v_{id}^* x_{ij}^L}, d = 1, ..., n. \tag{7}$$

The cross-efficiency score E_j^U of each DMU_j is then the average of $E_{dj}^U(d = 1, ..., n)$. Due to the existence of alternative optimal solutions of Model (5), the cross-efficiency score of DMUs may not be unique, but the following mixed binary linear programming model ranks the DMUs to increase our evaluation power:

$$R_d^U = \min \sum_{j=1}^{n} z_j$$

$$s.t.$$

$$\sum_{i=1}^{m} v_{id} x_{id}^L = 1$$

$$\sum_{r=1}^{s} u_{rd} y_{rd}^U + \sum_{t=1}^{k} w_{td} \bar{b}_{td}^U = e_{dd}^U$$

$$\sum_{i=1}^{m} v_{id} x_{ij}^U - \sum_{r=1}^{s} u_{rd} y_{rj}^L - \sum_{t=1}^{k} w_{td} \bar{b}_{tj}^L \geq 0, j = 1, ..., n, j \neq d$$

$$e_{dd}^U \sum_{i=1}^{m} v_{id} x_{ij}^U - \sum_{r=1}^{s} u_{rd} y_{rj}^L - \sum_{t=1}^{k} w_{td} \bar{b}_{tj}^L + s_j = 0, j = 1, ..., n, j \neq d$$

$$\sum_{r=1}^{s} u_{rd} y_{rd}^{U} \geq \alpha \varepsilon_{d}^{*U} \quad (9)$$

$$\sum_{t=1}^{k} w_{td} \overline{b}_{td}^{U} \geq \beta \varepsilon_{d}^{*U}$$

$$s_j \leq M z_j, j = 1, \ldots, n$$

$$v_{id}, u_{rd}, w_{td} \geq 0, i = 1, \ldots, m, r = 1, \ldots, s, t = 1, \ldots, k$$

$$z_j \in \{0, 1\}, j = 1, \ldots, n$$

$$s_j \text{free in sign}, \ j = 1, \ldots, n$$

The value of e_{dd}^{U} is the upper-bound efficiency of DMU_d from Model (5), and the value of M is arbitrarily large. The first two constraints imply that the upper-bound efficiency score of DMU_d is preserved at its level resulting from Model (5). The first and second constraints also result in $s_d = 0$. The third and seventh constraints ensure that all DMUs belong to the production possibility set. The fourth constraint ensures:

$$e_{dd}^{U} \sum_{i=1}^{m} v_{id} x_{ij}^{U} - \sum_{r=1}^{s} u_{rd} y_{rj}^{L} - \sum_{t=1}^{k} w_{td} \overline{b}_{tj}^{L} = -s_j, j = 1, \ldots, n, j \neq d \quad (9)$$

Two cases arise from equation (9):

(i) If $s_j > 0$, then

$$e_{dd}^{U} \sum_{i=1}^{m} v_{id} x_{ij}^{U} - \sum_{r=1}^{s} u_{rd} y_{rj}^{L} - \sum_{t=1}^{k} w_{td} \overline{b}_{tj}^{L} < 0, j = 1, \ldots, n, j \neq d \quad (10)$$

$$e_{dd}^{U} < \frac{\sum_{r=1}^{s} u_{rd} y_{rj}^{L} + \sum_{t=1}^{k} w_{td} \overline{b}_{tj}^{L}}{\sum_{i=1}^{m} v_{id} x_{ij}^{U}} \leq \frac{\sum_{r=1}^{s} u_{rd} y_{rj}^{U} + \sum_{t=1}^{k} w_{td} \overline{b}_{tj}^{U}}{\sum_{i=1}^{m} v_{id} x_{ij}^{L}} = e_{dj}^{U}, j = 1, \ldots, n, j \neq d$$

In this case, the optimistic view of the cross-efficiency score of DMU_j corresponding to DMU_d is larger than the CCR-efficiency score of DMU_d.

(ii) If $s_j \leq 0$, then

$$e_{dd}^{U} \sum_{i=1}^{m} v_{id} x_{ij}^{U} - \sum_{r=1}^{s} u_{rd} y_{rj}^{L} - \sum_{t=1}^{k} w_{td} \overline{b}_{tj}^{L} \geq 0, j = 1, \ldots, n, j \neq d \quad (11)$$

$$\frac{\sum_{r=1}^{s} u_{rd} y_{rj}^{L} + \sum_{t=1}^{k} w_{td} \overline{b}_{tj}^{L}}{\sum_{i=1}^{m} v_{id} x_{ij}^{U}} \leq e_{dd}^{U} \leq \frac{\sum_{r=1}^{s} u_{rd} y_{rj}^{U} + \sum_{t=1}^{k} w_{td} \overline{b}_{tj}^{U}}{\sum_{i=1}^{m} v_{id} x_{ij}^{L}} = e_{dj}^{U}, j = 1, \ldots, n, j \neq d$$

In this case, the cross efficiency of DMU_j corresponding to DMU_d is greater than the upper-bound efficiency of DMU_d and the set of optimal weights selected by DMU_d ranks DMU_d behind DMU_j.

In Model (8), z_j is a binary variable and therefore, from the seventh constraint, if $z_j = 0$, then the constraint $s_j \leq M z_j$ turns to $s_j \leq 0$ which shows $e_{dd}^{U} \leq e_{dj}^{U}$. On the other hand, if $z_j = 1$, then considering the fact that Model (8) minimizes $\sum_{j=1}^{n} z_j$ we can conclude that $s_j > 0$. In this case, in fact, Model (8) minimizes the number of DMUs which rank before DMU_d, i.e., $e_{dd}^{U} \leq e_{dj}^{U}$.

Model (8) determines the best ranking of DMU_d based on its own optimal weights. If R_d^U is the optimal objective function value of Model (8) corresponding to DMU_d, then the best ranking for DMU_d is $R_d^U + 1$. Unfortunately, an optimal solution to Model (8) may not be unique. We solve this problem by considering the constraint $\sum_{j=1}^{n} z_j = R_d^U$. Notice that $\sum_{r=1}^{s} u_{rd} y_{rd}^{U} + \sum_{t=1}^{k} w_{td} \overline{b}_{td}^{U} = e_{dd}^{U}$ shows that the cross-efficiency score measured from the optimistic perspective is maintained. We subsequently calculate:

$$\overline{R}_d^U = max \sum_{j=1}^{n} \xi_j$$

$$s.t.$$

$$\sum_{i=1}^{m} v_{id} x_{id}^{L} = 1$$

$$\sum_{r=1}^{s} u_{rd} y_{rd}^{U} + \sum_{t=1}^{k} w_{td} \overline{b}_{td}^{U} = e_{dd}^{U}$$

$$\sum_{i=1}^{m} v_{id} x_{ij}^{U} - \sum_{r=1}^{s} u_{rd} y_{rj}^{L} - \sum_{t=1}^{k} w_{td} \overline{b}_{tj}^{L} - \xi_j = 0, j = 1, \ldots, n, j \neq d$$

$$e_{dd}^{U} \sum_{i=1}^{m} v_{id} x_{ij}^{U} - \sum_{r=1}^{s} u_{rd} y_{rj}^{L} - \sum_{t=1}^{k} w_{td} \overline{b}_{tj}^{L} + s_j = 0, j = 1, \ldots, n, j \neq d \quad (12)$$

$$\sum_{r=1}^{s} u_{rd} y_{rd}^{U} \geq \alpha \varepsilon_{d}^{*U}$$

$$\sum_{t=1}^{k} w_{td} \overline{b}_{td}^{U} \geq \beta \varepsilon_{d}^{*U}$$

$$s_j \leq M z_j, j = 1, \ldots, n$$

$$\sum_{j=1}^{n} z_j = R_d^U$$

$$v_{id}, u_{rd}, w_{td} \geq 0, i = 1, \ldots, m, r = 1, \ldots, s, t = 1, \ldots, k$$

$$\xi_j \geq 0, j = 1, \ldots, n$$

$$z_j \in \{0, 1\}, j = 1, \ldots, n$$

$$s_j \text{ free in sign}, \ j = 1, \ldots, n$$

Model (12) reveals that the model is aggressive, meaning that it maintains the efficiency score of DMU_d at its upper-bound preserves its best ranking.

Assume that $\left(v_d^*, u_d^*, w_d^*, z^*, s^*\right) \in \mathbb{R}^{m+s+k+2n}$ is an optimal solution obtained by solving Models (8) and (12), then, the cross-efficiency score of DMU_j corresponding to DMU_d can be calculated as follows:

$$CE_{dj}^{U} = \frac{\sum_{r=1}^{s} u_{rd}^{*} y_{rj}^{U} - \sum_{t=1}^{k} w_{td}^{*} \overline{b}_{tj}^{U}}{\sum_{i=1}^{m} v_{id} x_{ij}^{L}} \quad (13)$$

and the cross-efficiency score CE_j^U of each DMU_j is the average of $CE_{dj}^{U} (d = 1, \ldots, n)$. Pessimistic counterparts are calculated similarly and are denoted with a superscript L.

4. Cross-efficiency evaluation considering undesirable outputs and the RO approach

We now propose a model based on the RO approach for ranking DMUs. Let us consider $\mathrm{DMU}_j (j = 1, \ldots, n)$ with interval inputs. Assume that I_j^x, O_j^g, and O_j^b respectively represent the set of inputs, desirable outputs, and undesirable outputs of DMUs that are subject to uncertainty. Let us further consider parameters γ_j^x, γ_j^g, and γ_j^b (not necessarily integers) which are assumed to be valued in bounded intervals $\gamma_j^x \in \left[0, \left|I_j^x\right|\right]$, $\gamma_j^g \in \left[0, \left|O_j^g\right|\right]$, and $\gamma_j^b \in \left[0, \left|O_j^b\right|\right]$. The role of γ_j^x, γ_j^g, and γ_j^b is to adjust the demanded conservatism level in the proposed RO formulation. The level of conservatism is a user-selectable degree that identifies the high, medium, or low-risk groups. These numbers are applied to the upper bound on the number of uncertain parameters in which their worst-case values can be obtained. Here, we aim to ensure that the solution remains feasible against changes in all $\left[\gamma_j^x\right]$, $\left[\gamma_j^g\right]$, and $\left[\gamma_j^b\right]$ combinations, and changes in x_{tj}, y_{tj}, and b_{tj} are at most $\left(\gamma_j^x - \left[\gamma_j^x\right]\right)\left(x_{tj}^{U} - x_{tj}^{L}\right)$, $\left(\gamma_j^g - \left[\gamma_j^g\right]\right)\left(y_{tj}^{U} - y_{tj}^{L}\right)$ and $\left(\gamma_j^b - \left[\gamma_j^b\right]\right)\left(b_{tj}^{U} - b_{tj}^{L}\right)$, respectively. So we consider the conditions that only a subset of inputs, desirable outputs, and undesirable outputs should change to immunize the solution against the worst cases (Bertsimas & Sim, 2004). The robust cross-efficiency evaluation is:

$$e_{dd}^{R} = \max \sum_{r=1}^{s} u_{rd} y_{rd}^{U} + \sum_{t=1}^{k} w_{td} \widehat{b}_{td}^{U} - \alpha_d^g - \alpha_d^b$$

$s.t.$

$$\sum_{i=1}^{m} v_{id}x_{id}^{L} + \alpha_d^x = 1$$

$$\sum_{i=1}^{m} v_{id}x_{ij}^{U} - \sum_{r=1}^{s} u_{rd}y_{rj}^{L} - \sum_{t=1}^{k} w_{td}\overset{-L}{b}_{tj} - \alpha_j^x - \alpha_j^g - \alpha_j^b \geq 0, j = 1,...,n, j \neq d$$

$$\sum_{i=1}^{m} v_{id}x_{id}^{L} - \sum_{r=1}^{s} u_{rd}y_{rd}^{U} - \sum_{t=1}^{k} w_{td}\overset{-U}{b}_{td} + \alpha_d^x + \alpha_d^g + \alpha_d^b \geq 0 \tag{14}$$

$$\sum_{r=1}^{s} u_{rd}y_{rd}^{U} - \alpha_d^g \geq \alpha\varepsilon_d^{*R}$$

$$\sum_{t=1}^{k} w_{td}\overset{-U}{b}_{td} - \alpha_d^b \geq \beta\varepsilon_d^{*R}$$

$$v_{id} \geq 0, i = 1,...,m$$

$$u_{rd} \geq 0, r = 1,...,s$$

$$w_{td} \geq 0, t = 1,...,k$$

The values of α_j^x, α_j^g, and α_j^b correspond with the uncertain inputs and outputs of DMU$_j$, respectively,

$$\alpha_j^x = \max_{c_j^x}\left\{\sum_{i\in s_j^x} v_{id}\left(x_{ij}^U - x_{ij}^L\right) + \left(\gamma_j^x - \left[\gamma_j^x\right]\right)v_{t_j^x}\left(x_{ij}^U - x_{ij}^L\right)\right\}, \forall j$$

$$\alpha_j^g = \max_{c_j^g}\left\{\sum_{r\in s_j^g} u_{rd}\left(y_{rj}^U - y_{rj}^L\right) + \left(\gamma_j^g - \left[\gamma_j^g\right]\right)u_{t_j^g}\left(y_{rj}^U - y_{rj}^L\right)\right\}, \forall j$$

$$\alpha_j^b = \max_{c_j^b}\left\{\sum_{k\in s_d^b} w_{td}\left(\overset{-U}{b}_{pj} - \overset{-L}{b}_{pj}\right) + \left(\gamma_j^b - \left[\gamma_j^b\right]\right)w_{t_j^b}\left(\overset{-U}{b}_{tj} - \overset{-L}{b}_{tj}\right)\right\}, \forall j, \text{and}$$

where the maximums are defined over

$$c_j^x = \left\{s_j^x \cup \left\{t_j^x\right\} | s_j^x \subseteq I_j^x, \left|s_j^x\right| = \left[\gamma_j^x\right], t_j^x \in \left(I_j^x - s_j^x\right)\right\}, \forall j,$$

$$c_j^g = \left\{s_j^g \cup \left\{t_j^g\right\} | s_j^g \subseteq O_j^g, \left|s_j^g\right| = \left[\gamma_j^g\right], t_j^g \in \left(O_j^g - s_j^g\right)\right\}, \forall j, \text{and}$$

$$c_j^b = \left\{s_j^b \cup \left\{t_j^b\right\} | s_j^b \subseteq O_j^b, \left|s_j^b\right| = \left[\gamma_j^b\right], t_j^b \in \left(O_j^b - s_j^b\right)\right\}, \forall j.$$

The set s_j denotes the collection of uncertain data that can reach their maximum perturbations, while t_j denotes another uncertain data set that perturbs in the amount of $\left(\gamma_j^{x,g,b} - \left[\gamma_j^{x,g,b}\right]\right)$. Appendix C suggests a model for finding a suitable value for ε in Model (14).

The RO approach serves as a basis for Model (14), where α_j^x, α_j^g, and α_j^b are introduced to keep the constraints feasible, and move from the optimistic to the pessimistic point of view; it moves to a pessimistic view. Indeed, these variables are introduced to protect the constraints in Model (14) against existing uncertainty in inputs and outputs. If the values of γ_j^x, γ_j^g, and γ_j^b are integers, then α_j^x, α_j^g, and α_j^b become:

$$\alpha_j^x = \max_{c_j^x}\left\{\sum_{i\in s_j^x} v_{id}\left(x_{ij}^U - x_{ij}^L\right)\right\}, c_j^x = \left\{s_j^x | s_j^x \subseteq I_j^x, \left|s_j^x\right| = \left[\gamma_j^x\right]\right\}, \forall j,$$

$$\alpha_j^g = \max_{c_j^g}\left\{\sum_{r\in s_j^g} u_{rd}\left(y_{rj}^U - y_{rj}^L\right)\right\}, c_j^g = \left\{s_j^g | s_j^g \subseteq O_j^g, \left|s_j^g\right| = \left[\gamma_j^g\right]\right\}, \forall j, \text{ and}$$

$$\alpha_j^b = \max_{c_j^b}\left\{\sum_{t\in s_d^b} w_{td}\left(\overset{-U}{b}_{tj} - \overset{-L}{b}_{tj}\right)\right\}, c_j^b = \left\{s_j^b | s_j^b \subseteq O_j^b, \left|s_j^b\right| = \left[\gamma_j^b\right]\right\}, \forall j$$

If we assume that $\gamma_j^x = \gamma_j^g = \gamma_j^b = 0$, so that the inputs and outputs in Model (5) are fixed, then we obtain $\alpha_j^x = \alpha_j^g = \alpha_j^b = 0$ and Model (14) reduces to Model (5). Similarly, if $\gamma_j^x = \left|I_i^x\right|$, $\gamma_j^g = \left|O_j^g\right|$, and $\gamma_j^b = \left|O_j^b\right|$, then $\alpha_j^x = \sum_{i=1}^{m} v_{id}\left(x_{ij}^U - x_{ij}^L\right)$, $\alpha_j^g = \sum_{r=1}^{s} u_{rd}\left(y_{rj}^U - y_{rj}^L\right)$ and $\alpha_j^b = \sum_{t=1}^{k} w_{td}\left(\widehat{b}_{tj}^U - \widehat{b}_{tj}^L\right)$. Model (14) becomes Model (6).

Varying γ_j^x, γ_j^g, and γ_j^b, within their intervals, adjust the degree of robustness in an optimal solution. Since Model (14) is non-linear, using the proposition of Bertsimas and Sim (2004), we can transform Model (14) into a linear model. To this end, consider the following definitions for the inputs, desirable outputs, and undesirable outputs, in the second group of constraints of Model (14):

$$\alpha_j^x = \max_{c_j^x}\left\{\sum_{i=1}^{m} v_{id}\left(x_{ij}^U - x_{ij}^L\right) + \left(\gamma_j^x - \left[\gamma_j^x\right]\right)v_{t_j^x}\left(x_{ij}^U - x_{ij}^L\right)\right\} \tag{15}$$

$s.t.$

$$\sum_{j\in\left|O_j^g\right|} \widehat{z}_j^g \leq \gamma_j^g$$

$$0 \leq \widehat{z}_j \leq 1$$

$$\alpha_j^g = \max_{c_j^g}\left\{\sum_{r=1}^{s} u_{rd}\left(y_{rj}^U - y_{rj}^L\right) + \left(\gamma_j^g - \left[\gamma_j^g\right]\right)u_{t_j^g}\left(y_{rj}^U - y_{rj}^L\right)\right\} \tag{16}$$

$s.t.$

$$\sum_{j\in\left|O_j^g\right|} \widehat{z}_j^g \leq \gamma_j^g$$

$$0 \leq \widehat{z}_j \leq 1$$

$$\alpha_j^b = \max_{c_j^b}\left\{\sum_{t=1}^{k} w_{td}\left(\overset{-U}{b}_{tj} - \overset{-L}{b}_{tj}\right) + \left(\gamma_j^b - \left[\gamma_j^b\right]\right)w_{t_j^b}\left(\overset{-U}{b}_{tj} - \overset{-L}{b}_{tj}\right)\right\} \tag{17}$$

$s.t.$

$$\sum_{j\in\left|O_j^b\right|} \widehat{z}_j^b \leq \gamma_j^b$$

$$0 \leq \widehat{z}_j \leq 1$$

where $\widehat{z}_j$ indicates the total sum of data perturbations. By considering R_{ij}, P_{rj}, and q_{pj} as dual variables for the constraints in Models (15), (16), and (17), the second constraint of Model (14) can be written as follows:

$$\sum_{i=1}^{m} v_{id}x_{ij}^{U} - \widehat{z}_j^x\gamma_j^x - \sum_{i=1}^{m} R_{ij}$$

$$-\sum_{r=1}^{s} u_{rd}y_{rj}^{L} - \widehat{z}_j^g\gamma_j^g - \sum_{r=1}^{s} p_{rj}$$

$$-\sum_{t=1}^{k} w_{td}\widehat{b}_{tj}^{L} - \widehat{z}_j^b\gamma_j^b - \sum_{t=1}^{k} q_{tj} \geq 0, j = 1,...,n, j \neq d$$

$$\widehat{z}_j^x + R_{ij} \geq v_{id}\left(x_{ij}^U - x_{ij}^L\right), i = 1,...,m, j = 1,...,n$$

$$\widehat{z}_j^g + P_{rj} \geq u_{rd}\left(y_{rj}^U - y_{rj}^L\right), r = 1,...,s, j = 1,...,n$$

$$\widehat{z}_j^b + q_{tj} \geq w_{td}\left(\overset{-U}{b}_{tj} - \overset{-L}{b}_{tj}\right), t = 1,...,k, j = 1,...,n$$

$$\widehat{z}_j^x, \widehat{z}_j^g, \widehat{z}_j^b \geq 0, j = 1,...,n$$

$R_{ij}, P_{rj}, q_{pj} \geq 0, \forall i, r, t, j$

$v_{id}, u_{rd}, w_{td} \geq 0, i = 1, \ldots, m, r = 1, \ldots, s, t = 1, \ldots, k$

Models (15), (16), and (17) are feasible and bounded for all γ_j^x, γ_j^g, and γ_j^b. Therefore, the strong duality theorem guarantees their dual problems are feasible and bounded. Consequently, we can now rewrite Model (14) as a linear:

$$E_{dd}^R = \max \sum_{r=1}^{s} u_{rd} y_{rd}^U + \sum_{t=1}^{k} w_{td} \widehat{b}_{td}^U - \widehat{z}_d^g \gamma_d^g - \sum_{r=1}^{s} p_{rd} - \widehat{z}_d^b \gamma_d^b - \sum_{t=1}^{k} q_{td}, \tag{18}$$

$s.t.$

$$\sum_{i=1}^{m} v_{id} x_{id}^L + \widehat{z}_d^x \gamma_d^x + \sum_{i=1}^{m} R_{id} = 1 \tag{18a}$$

$$\sum_{i=1}^{m} v_{id} x_{ij}^U - \widehat{z}_j^x \gamma_j^x - \sum_{i=1}^{m} R_{ij} - \sum_{r=1}^{s} u_{rd} y_{rj}^L - \widehat{z}_j^g \gamma_j^g - \sum_{r=1}^{s} p_{rj} - \sum_{t=1}^{k} w_{td} \overline{b}_{tj}^L - \widehat{z}_j^b \gamma_j^b - \sum_{t=1}^{k} q_{tj} \geq 0, \forall j, j \neq d \tag{18b}$$

$$\sum_{i=1}^{m} v_{id} x_{id}^L + \widehat{z}_d^x \gamma_d^x + \sum_{i=1}^{m} R_{id} - \sum_{r=1}^{s} u_{rd} y_{rd}^U + \widehat{z}_d^g \gamma_d^g + \sum_{r=1}^{s} p_{rd} - \sum_{t=1}^{k} w_{td} \overline{b}_{td}^U + \widehat{z}_d^b \gamma_d^b + \sum_{p=1}^{k} q_{pd} \geq 0 \tag{18c}$$

$$\sum_{r=1}^{s} u_{rd} y_{rd}^U - \widehat{z}_d^g \gamma_d^g - \sum_{r=1}^{s} p_{rd} \geq \alpha \varepsilon_d^{*U} \tag{18d}$$

$$\sum_{t=1}^{k} w_{td} \overline{b}_{td}^U - \widehat{z}_d^b \gamma_d^b - \sum_{t=1}^{k} q_{td} \geq \beta \varepsilon_d^{*U} \tag{18e}$$

$$\widehat{z}_j^x + R_{ij} \geq v_{id}\left(x_{ij}^U - x_{ij}^L\right), i = 1, \ldots, m, j = 1, \ldots, n \tag{18f}$$

$$\widehat{z}_j^g + P_{rj} \geq u_{rd}\left(y_{rj}^U - y_{rj}^L\right), r = 1, \ldots, s, j = 1, \ldots, n \tag{18g}$$

$$\widehat{z}_j^b + q_{tj} \geq w_{td}\left(\overline{b}_{tj}^U - \overline{b}_{tj}^L\right), t = 1, \ldots, k, j = 1, \ldots, n \tag{18h}$$

$$\widehat{z}_j^x, \widehat{z}_j^g, \widehat{z}_j^b \geq 0, j = 1, \ldots, n \tag{18i}$$

$$R_{ij}, P_{rj}, q_{tj} \geq 0, \forall i, r, t, j \tag{18j}$$

$$v_{id}, u_{rd}, w_{td} \geq 0, i = 1, \ldots, m, r = 1, \ldots, s, t = 1, \ldots, k \tag{18k}$$

Model (18) promotes a definition of robust-efficiency.

Definition 2.. *If $E_{dd}^R = 1$, then DMU_d is robust-efficient; otherwise, it is robust-inefficient. We use the following model to rank DMUs based on their robust efficiency scores.*

$$R_d^R = \min \sum_{j=1}^{n} z_j \tag{19}$$

$s.t.$

$(18a - 18k)$

$$\sum_{r=1}^{s} u_{rd} y_{rd}^U + \sum_{p=1}^{k} w_{td} \overline{b}_{td}^U - \widehat{z}_d^g \gamma_d^g - \sum_{r=1}^{s} p_{rd} - \widehat{z}_d^b \gamma_d^b - \sum_{t=1}^{k} q_{td} = E_{dd}^R \tag{19a}$$

$$E_{dd}^R \left(\sum_{i=1}^{m} v_{id} x_{ij}^U - \widehat{z}_j^x \gamma_j^x - \sum_{i=1}^{m} R_{ij} \right) - \sum_{r=1}^{s} u_{rd} y_{rj}^L - \widehat{z}_j^g \gamma_j^g - \sum_{r=1}^{s} p_{rj} - \sum_{t=1}^{k} w_{td} \overline{b}_{tj}^L - \widehat{z}_j^b \gamma_j^b - \sum_{t=1}^{k} q_{tj} + s_j = 0, \forall j, j \neq d \tag{19b}$$

$$E_{dd}^R \left(\sum_{i=1}^{m} v_{id} x_{id}^L + \widehat{z}_d^x \gamma_d^x + \sum_{i=1}^{m} R_{id} \right) - \sum_{r=1}^{s} u_{rd} y_{rd}^U + \widehat{z}_d^g \gamma_d^g + \sum_{r=1}^{s} p_{rd} - \sum_{t=1}^{k} w_{td} \overline{b}_{td}^U + \widehat{z}_d^b \gamma_d^b + \sum_{t=1}^{k} q_{td} + s_d = 0 \tag{19c}$$

$$s_j \leq M z_j, j = 1, \ldots, n \tag{19d}$$

$$z_j \in \{0, 1\}, j = 1, \ldots, n \tag{19e}$$

$$s_j \text{free in sign}, \ j = 1, \ldots, n \tag{19e}$$

Model (19) preserves the robust efficiency of DMU_d and maintains the most favorable efficiency rating for it. To explore the possible existence of alternative solutions to Model (19), we propose the following model:

$$\overline{R}_d^R = max \sum_{j=1}^{n} \xi_j \tag{20}$$

$s.t.$

$$\sum_{i=1}^{m} v_{id} x_{ij}^U - \widehat{z}_j^x \gamma_j^x - \sum_{i=1}^{m} R_{ij} - \sum_{r=1}^{s} u_{rd} y_{rj}^L - \widehat{z}_j^g \gamma_j^g - \sum_{r=1}^{s} p_{rj} - \sum_{t=1}^{k} w_{td} \overline{b}_{tj}^L - \widehat{z}_j^b \gamma_j^b - \sum_{t=1}^{k} q_{tj} - \xi_j = 0, \forall j, j \neq d \tag{20a}$$

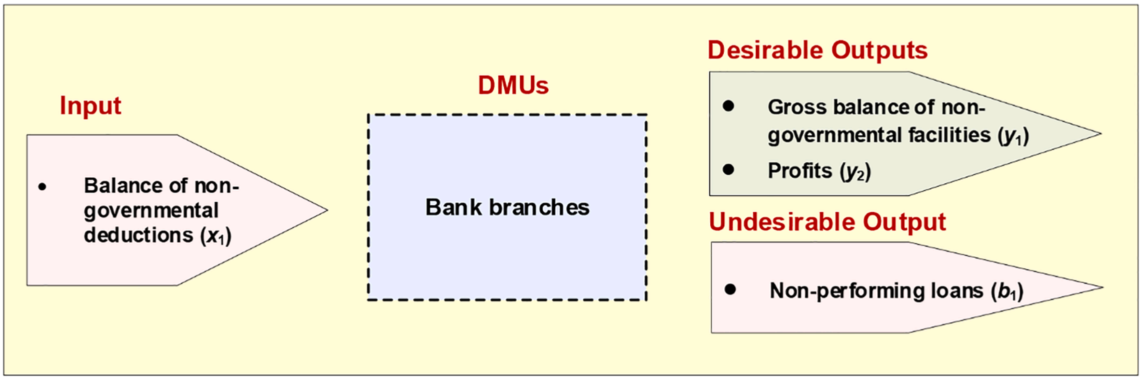

Fig. 1. Performance measures for the banking case study.

Table 1
Descriptive statistics of the interval inputs and outputs variables for 50 bank branches.

Data	Input		Desirable outputs				Undesirable output	
	x_1^L	x_1^U	y_1^L	y_1^U	y_2^L	y_2^U	b_1^L	b_2^U
Min	14191.00	17662.00	600754258.00	4522860624.00	316.00	2450.00	18351.00	20779.00
Max	2038982.00	2039982.00	294507045133.00	326419360882.00	513181.00	636158.00	528121.00	656716.00
Mean	112052.04	124455.40	18044229592.38	22705523858.58	19112.90	25767.54	72450.788	95216.26
Std. Dev.	284059.92	284261.91	40953143627.59	45353419909.56	71773.42	88793.72	79388.28	98759.09

$(18a)$

$(18d - 18k)$

$(19a - 19f)$

$$\sum_{j=1}^{n} z_j = R_d^R \tag{20b}$$

s_j free in sign, $j = 1, ..., n$

$\xi_j \geq 0, j = 1, ..., n$

Suppose that (v_d^*, u_d^*, w_d^*) is an optimal solution, then the cross-efficiency of DMU$_j$ corresponding to DMU$_d$ is:

$$CE_{dj}^R = \frac{\sum_{r=1}^{s} u_{rd}^* y_{rd}^U + \sum_{t=1}^{k} w_{td}^* \overline{b}_{td}^{U}}{\sum_{i=1}^{m} v_{id}^* x_{id}^L}, d = 1, ..., n \tag{21}$$

The cross-efficiency score CE_j^R of each DMU$_j$ is defined as the average of $CE_{dj}^R (d = 1, ..., n)$. Finally, the cross-efficiency scores obtained are defined as new efficiency measures (hereafter called RCE) for all DMUs. Appendix D provides two theorems for proving that Model (20) is feasible and $\overline{R}_d^R \leqslant \overline{R}_d^U$.

5. Experimental results

We consider in this section a case study and a comparative example to demonstrate the applicability of the models proposed above. The first example in Subsection 5.1 deals with a case study of about 50 Iranian

Table 2
The Efficiency and ranking of the banks based on Models (5), (6), and (20).

Bank	e_{dd}^L	e_{dd}^U	RCE	Ranking	Bank	e_{dd}^L	e_{dd}^U	RCE	Ranking
1	0.4615	0.9473	0.2128	33	26	0.7103	1	0.3837	4
2	0.4401	0.9122	0.3209	9	27	0.4519	0.9194	0.2663	18
3	0.4189	0.8810	0.2986	12	28	0.3896	0.6902	0.2139	32
4	0.3425	0.9236	0.3555	8	29	0.5353	0.8315	0.1482	44
5	1	1	0.5502	2	30	0.3865	0.8930	0.2888	15
6	0.7782	0.9500	0.3021	10	31	0.6178	0.8033	0.2064	35
7	0.3521	0.7639	0.2602	19	32	0.7026	0.8859	0.2339	25
8	0.4571	0.8078	0.2464	22	33	0.5813	0.7114	0.2335	27
9	0.5326	1	0.2342	24	34	0.7332	1	0.5412	3
10	0.5531	0.9524	0.2894	14	35	0.7549	0.8013	0.1530	43
11	0.5400	0.8818	0.2206	29	36	0.4376	0.8975	0.1091	48
12	0.5260	0.8379	0.2047	36	37	0.6204	0.7019	0.2184	30
13	0.4459	1	0.3664	6	38	0.5589	0.8100	0.1170	46
14	0.2393	0.9633	0.2873	16	39	0.6432	0.8469	0.2108	34
15	0.5060	0.9343	0.2592	20	40	0.4570	0.8193	0.1704	40
16	0.4238	0.8092	0.1873	39	41	0.5084	0.6535	0.1990	37
17	0.4369	0.9012	0.2713	17	42	0.6251	0.7619	0.1598	42
18	0.4761	0.8119	0.2441	23	43	0.5200	0.7935	0.0953	49
19	0.2554	0.9275	0.2895	13	44	0.6545	0.6641	0.2165	31
20	0.5024	0.8781	0.2484	21	45	0.5217	0.7062	0.1399	45
21	1	1	0.5530	1	46	0.5643	0.7596	0.1111	47
22	0.4226	1	0.2992	11	47	0.5232	0.6281	0.1891	38
23	0.7121	1	0.3775	5	48	0.4861	0.8374	0.0805	50
24	0.5660	1	0.3631	7	49	0.5737	0.6626	0.2246	28
25	0.4015	0.8956	0.2337	26	50	0.4983	0.6708	0.1675	41

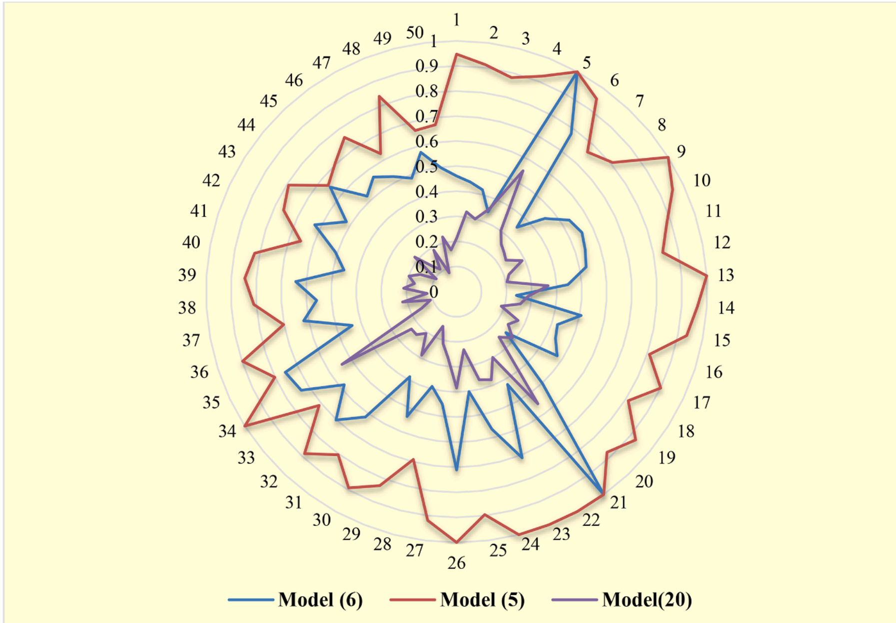

Fig. 2. The efficiency scores for models (5), (6), and (20) for Banks.

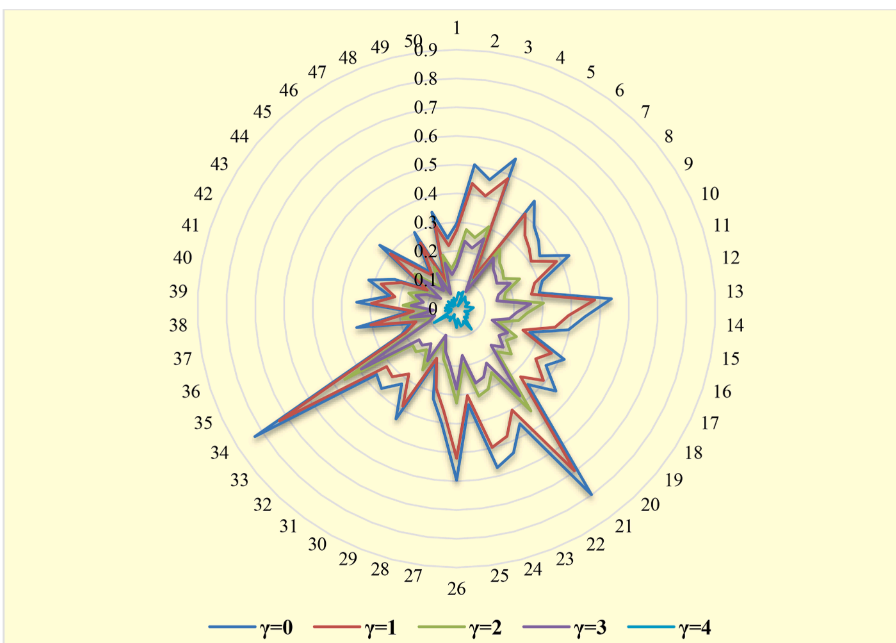

Fig. 3. Cross-efficiency scores obtained by Model (20) for different γ's levels.

National Bank branches across the Ardabil province in Iran. Subsection 5.2 describes a comparative example among the proposed models in this paper and with suggested methods in the relevant literature.

5.1. Iranian National bank case study

We consider real-world data from 50 Iranian National Bank branches across the Ardabil province over three years from 2011 to 2014. These bank branches consume one input, the balance of non-governmental deductions (x_1), to produce two desirable outputs, the gross balance of non-governmental facilities (y_1) and the profit of each branch (y_2), and one undesirable output, the non-performing loans (b_1) (see Fig. 1).

The descriptive statistics of the interval inputs and outputs are presented in Table 1 shows the maximum, minimum, mean, and standard deviation values of the input, desirable outputs, and undesirable output variables for the 50 Iranian National bank branches from 2011 to 2014 in Iranian Rials.

We employ Models (5) and (6) to calculate the upper- and lower-bounds of efficiency at the bank branches. We also set $w_1 = 850000$ for transforming the data of undesirable output. Note that the value of ε in Model (20) is 6.5×10^{-4}, which is obtained by solving Model (A.1). Model (20) obtains a deterministic robust-efficiency score for each bank branch for ranking the DMUs.

Given the results of the second, third, seventh, and eighth columns in Table 1, some banks have an efficiency score upper-bound of one and a lower-bound less than one. So, their efficiencies and ranking cannot be determined precisely. The fourth and the ninth columns in Table 2 include the means of cross-efficiency scores for Model (20). We calculate

Table 3
Interval inputs and outputs of the combined-cycle power plants during the six-year study.

DMU	Input – Fuel (M^3)		Desirable output - electricity power (1000kw/Hr)		Undesirable output (Gas/Ton)							
	x^L_{1j}	x^U_{1j}	y^L_{1j}	y^U_{1j}	NO_x		SO_2		CO_2		SO_3	
					b^L_{1j}	b^U_{1j}	b^L_{2j}	b^U_{2j}	b^L_{3j}	b^U_{3j}	b^L_{4j}	b^U_{4j}
1	1,002,243	1,534,381	4,663,820	5,948,123	3.6	5	1.8	6.6	2338	3015	0	0.1
2	971,509	1,298,112	4,821,296	5,657,392	3.7	4.4	2.1	4.7	2367	2727	0	0.1
3	1,331,457	1,831,098	7,220,851	7,699,512	5.3	5.8	2.8	6.7	3478	3631	0	0.1
4	766,658	1,117,322	3,781,843	4,628,520	2.8	3.7	1.7	4.8	1779	2250	0	0.1
5	24,213	1,060,942	356,963	3,184,631	0.6	3.2	0.4	2.2	318	2119	0	0
6	1,045,455	1,283,541	5,339,780	5,975,686	3.8	4.4	1.3	3	2545	2806	0	0
7	412,442	758,142	1,925,856	2,631,210	1.7	2.3	0.1	1.1	1052	1557	0	0
8	446,094	1,017,339	1,836,793	4,289,004	1.8	3.6	1.1	4.2	1089	2229	0	0.1
9	1,244,520	1,820,737	4,222,796	7,935,571	4.3	8	0.2	12.5	2806	4788	0	0.2
10	1,056,182	1,410,680	5,126,256	6,213,138	3.4	4.4	0.2	3	2262	2802	0	0
11	311,239	635,257	1,820,209	2,106,015	1.6	1.9	1	3.3	979	1091	0	0.1
12	204	796,605	515	2,128,410	0	2.6	0	5	3209	1595	0	0.1
13	1,234,922	2,303,468	4,500,169	9,886,102	5.2	8.3	4.1	11.4	1222	4993	0.1	0.2
14	422,191	905,874	1,770,332	2,761,553	1.9	2.9	0.4	2.4	3546	1848	0	0
15	147,683	2,769,634	5,008,772	1,030,008	5.5	8.5	3	9.2	985	5535	0	0.1
16	161,614	928,637	1,258,570	2,678,996	1.9	4.5	1.4	7.7	3382	2661	0	0.1
17	1,298,688	1,961,314	4,785,753	5,898,717	5.3	5.7	0.5	4	2338	3828	0	0.1

Table 4
Efficiency and ranking the power plants using Models θ^*_L and θ^*_U, and Models (5), (6) and (20).

DMU	θ^*_L	θ^*_U	Class	e^L_{dd}	e^U_{dd}	Class	RCE	Rank
1	0.000727	1	E^+	0.000291	0.7939	E^-	0.9495	9
2	0.000831	1	E^+	0.000356	0.7938	E^-	0.9529	8
3	0.000934	1	E^+	0.000378	0.7086	E^-	0.8743	14
4	0.000790	1	E^+	0.000324	0.8926	E^-	0.9610	4
5	0.000079	1	E^+	0.000037	1	E^+	0.9330	11
6	0.000894	1	E^+	0.000399	0.5089	E^-	0.8763	13
7	0.000581	1	E^+	0.000243	0.5614	E^-	0.9539	7
8	0.000387	1	E^+	0.000173	1	E^+	0.9804	1
9	0.000414	1	E^+	0.000222	1	E^+	0.8634	16
10	0.000860	1	E^+	0.000348	0.4965	E^-	0.8890	12
11	0.000784	1	E^+	0.000275	1	E^+	0.9578	6
12	0.000037	1	E^+	0.000007	1	E^+	0.9581	5
13	0.000423	1	E^+	0.000187	1	E^+	0.8677	15
14	0.000450	1	E^+	0.000187	0.6361	E^-	0.9637	3
15	0.000425	1	E^+	0.000173	1	E^+	0.9422	10
16	0.000222	1	E^+	0.000144	1	E^+	0.9647	2
17	0.000587	1	E^+	0.000234	0.6095	E^-	0.8246	17

these columns with $\gamma \in [0, 4]$ and a step length of 0.2 to solve (20).

According to the three proposed classes in Section 2, $Bank_5$ and $Bank_{21}$ are in the first position order because these branches are in E^{++} and are efficient for all cases. $Bank_{13}$, $Bank_{22}$, $Bank_{23}$, $Bank_{24}$, $Bank_{26}$, and $Bank_{34}$ are E^+. All other branches are in the E^-. So we used robust Model (20) to identify the unknown ranking of the branches. Fig. 2

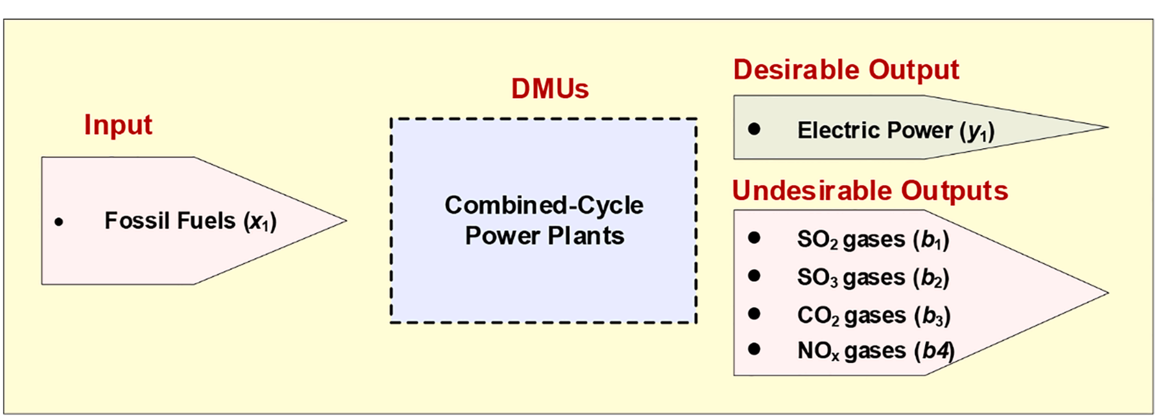

Fig. 4. Performance measures for the combined-cycle power plant example.

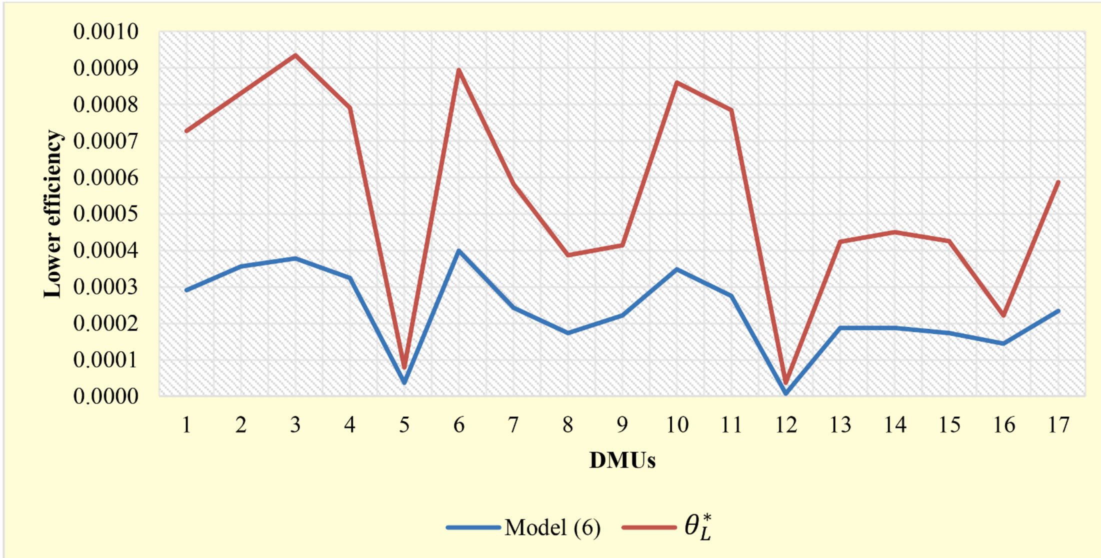

Fig. 5. Efficiency scores obtained from θ_L^* and Model (6).

presents the results for the pessimistic, optimistic, and robust cross-efficiency models.

Fig. 3 illustrates the cross-efficiency scores of the 50 bank branches under evaluation for $\gamma = 0, 1, 2, 3, \text{and} 4$. The constraint $\gamma = \gamma^x + \gamma^g + \gamma^b$ is added to Model (20) to control the uncertainties in the parameters since the complexity in Model (20) arises from the level of conservatism in input, desirable, and undesirable output data. We also add the constraints $\gamma^x \leq m$, $\gamma^g \leq s$ and $\gamma^b \leq k$ to Model (20) to control the number of imprecise inputs, and desirable and undesirable outputs. Model (20) is a non-linear model solved by branch-and-reduce optimization navigator (BARON) through GAMS. We can separately specify the values of γ^x, γ^g and γ^b in large problems to keep the model linear. Fig. 3 shows that the cross-efficiency scores decrease as the level of γ increases. As a result, when we exchange the optimality with performance, the efficiency of bank branches reduces. Therefore, by increasing the perturbation of uncertain data from $\gamma = 0$ to $\gamma = 4$, the efficiency score of a few branches remains close to one. In the banking industry, when the uncertainty level increases, a higher price should be paid for robustness (Bertsimas & Sim, 2004).

5.2. *Tavanir example*

We now consider calculating efficiency metrics for 17 combined-cycle power plants (Khalili-Damghani, Tavana, & Haji-Saami, 2015). The input variable is fossil fuels (x_1). The output variables are desirable electric power (y_1), and undesirable SO_2 gases (b_1), SO_3 gases (b_2), CO_2 gases (b_3), and the NO_x gases (b_4). Table 3 shows our data intervals for the 17 power plants during the six years of study.

Khalili-Damghani et al. (2015) employ the envelopment form of DEA models, under non-increasing and non-decreasing assumptions to evaluate the performance of all DMUs in the presence of interval data and undesirable outputs. Since the proposed interval models in this paper are under the CRS assumption, we apply Models (5), (6), and (20) with a transformation vector $(w_1, w_2, w_3, w_4) = (25, 35, 20000, 0.8)$ to analyze the 17 power plants. Solving Model (A.1) leads to $\varepsilon^* = 1.5 \times 10^{-5}$. We then compare the obtained results from our formulated models with those yielded by the interval CRS model 1 of Khalili-Damghani et al. (2015), denoted by $[\theta_L^*, \theta_U^*]$. Table 4 reports the obtained results by Models (5), (6), and (20). Fig. 4 illustrates the assumed structure of

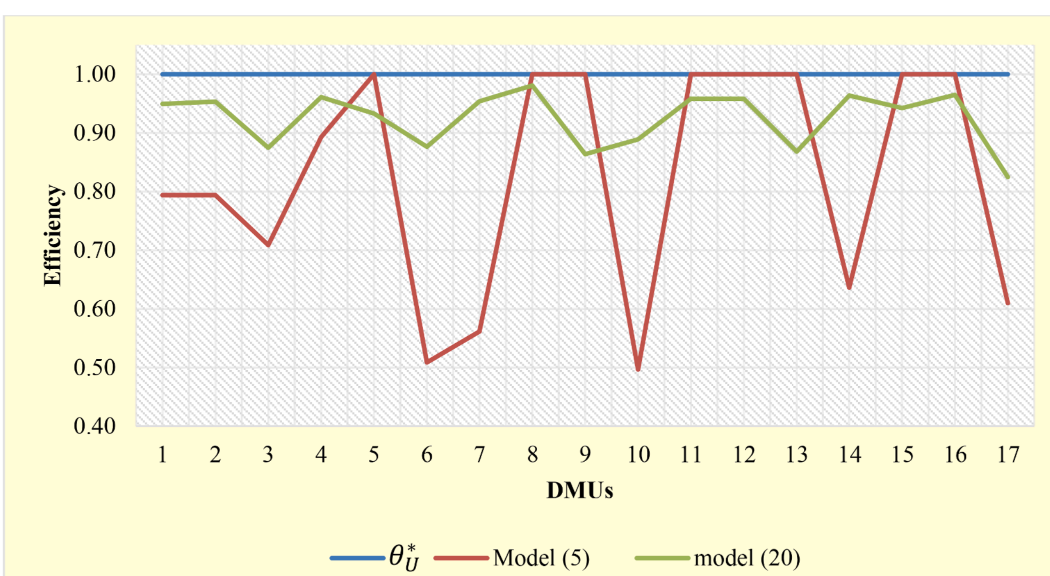

Fig. 6. Efficiency scores obtained from the optimistic Models θ_U^*, (5), and Model (20).

inputs and (un)desirable outputs in the application.

Table 4 shows that the efficiency scores of Model (5) are equal to or smaller than those obtained from the optimistic CCR model (θ_U^*). The efficiency scores for Model (6) are smaller than those for the pessimistic CCR model (θ_L^*) as expected (see Fig. 5). Table 4 presents the results from the models used by Khalili-Damghani et al. (2015), i.e., θ_L^* and θ_U^*, and Models (5), (6), and (20) presented in this study.

All 17 DMUs are classified as E^+ by θ_L^* and θ_U^*, while using the proposed interval of Models (5) and (6), 8 DMUs are classified as $E^+$and 9 DMUs are classified as E^-. So Models (5) and (6) produce a finer classification than the model proposed by Khalili-Damghani et al. (2015).

We rank the power plants with Model (20). We solved Model (20) for each γ with a step-length of 0.5 in $\gamma \in [0,6]$ to calculate the value of robust cross-efficiency scores for the DMUs presented in Table 4. Because there are six uncertain parameters involved in this assessment, we add constraint $\gamma = \gamma^x + \gamma^g + \gamma^b$ to Model (20) to control the number of imprecise inputs, desirable and undesirable outputs. $\gamma = 0$ represents no data perturbation, and $\gamma = 6$ represents maximum data perturbation. Fig. 6 displays the efficiency scores of the DMUs under both optimistic and pessimistic scenarios along with the mean efficiency scores for each γ, corresponding to Models (5), (6), and (20).

Table 4 shows that the proposed robust cross-efficiency Model (20) ranks DMU 8 as the top DMU, and hence we gain higher discriminatory power and higher-ranking stability over the interval models. Moreover, model (20) linearly ranks the DMUs.

6. Conclusion and future research directions

The RO approach provides a framework to study problems in operations research involving data uncertainty. DEA conventionally handles uncertainty differently. Another challenging problem in DEA is to solve the problem of the non-unique optimal weights. In this paper, we developed two DEA adaptations (an interval approach and a robust approach) to rank DMUs characterized by uncertain data and undesirable outputs. In addition, we presented an example from the literature and a real-world application to compare our method with an interval method. This example exhibited the ability of our approach to improving discernibility among DMUs.

Using the interval approach, we proposed an interval DEA model and used the RO approach to propose a robust cross-efficiency model to handle undesirability and uncertainty problems. The robust cross-efficiency model is proposed by combining the three important concepts in DEA, including interval models, cross-efficiency evaluation, and the RO approach. Initially, using the proposed interval model, we categorized the DMUs into three groups. However, we are not able to rank DMUs because in the interval approach, the efficiency value is not certain and lies in an interval. To solve this problem, we used the RO approach in the presence of uncertain data and undesirable output simultaneously. We then integrated the RO approach with cross-efficiency evaluation and solved the non-unique optimal weights, and presented a full ranking preference for of the DMUs. It should be noted that one of the most important advantages of the proposed robust model is the ability to control the number of uncertain data by adjusting the level of conservatism against the data uncertainty. A case study and a comparative example are presented to show the applicability of the proposed robust model. Developing robust DEA models in the presence of non-discretionary factors and uncertain data and applying the RO approach to handle negative data in DEA models can be considered for future works.

CRediT authorship contribution statement

Madjid Tavana: Conceptualization, Methodology, Writing - original draft, Writing - review & editing, Visualization, Supervision. **Mehdi Toloo:** Conceptualization, Methodology, Formal analysis, Writing - review & editing. **Nazila Aghayi:** Formal analysis, Investigation, Data curation, Validation. **Aliasghar Arabmaldar:** Writing - review & editing, Investigation, Resources.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

The following model provides a suitable value for the epsilon in Model (3):

$$
\begin{aligned}
&\varepsilon_d^* = \max \varepsilon_d \\
&s.t. \\
&\sum_{i=1}^{m} v_{id} x_{id} = 1 \\
&\sum_{r=1}^{s} u_{rd} y_{rj} + \sum_{t=1}^{k} w_{td} \bar{b}_{tj} - \sum_{i=1}^{m} v_{id} x_{ij} \leq 0, j = 1, \dots, n \\
&\varepsilon_d - \sum_{r=1}^{s} u_{rd} y_{rd} \leq 0 \\
&\varepsilon_d - \sum_{t=1}^{k} w_{td} \bar{b}_{td} \leq 0 \\
&v_{id} \geq 0, i = 1, \dots, m \\
&u_{rd} \geq 0, r = 1, \dots, s \\
&w_{td} \geq 0, t = 1, \dots, k
\end{aligned}
\tag{A1}
$$

Theorem A.1.. *Model (A.1) has a positive and bounded optimal objective value.*

Proof.. Initially, we will show that Model (A.1) is feasible. It is easy to verify that $(v_d, u_d, w_d, \varepsilon_d) = (1/x_{cd} e_c, 0, 0, 0)$ is a feasible solution for Model (A.1) where $x_{cd} = \max_i x_{id}$ and the vector e_c has 1 in the c^{th} position and 0 elsewhere. Now, we have to show that the objective value of Model (A.1) is positive. Suppose that $\varepsilon_d^* = 0$ and consider the following dual problem of Model (A.1):

$$\theta^* = \min\theta$$

$$s.t.$$

$$\sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{id}, i = 1, ..., m,$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} \ge \mu y_{rd}, r = 1, ..., s,$$

$$\sum_{j=1}^{n} \lambda_j \bar{b}_{tj} \ge \pi \bar{b}_{td}, t = 1, ..., k,$$

$$\mu + \pi = 1, \tag{A2}$$

$$\lambda_j \ge 0, j = 1, ..., n$$

$$\mu \ge 0$$

$$\pi \ge 0$$

where θ, λ, μ and π are the dual variables associated with the first four constraint set of Model (A.1), respectively. Reference to the strong duality theorem in linear programming, we obtain $\theta^* = \varepsilon_d^* = 0$. Moreover, from the first constraint set of Model (A.2), we have $\lambda_j = 0, \forall j = 1, ..., n$, which taking the second and third constraint sets into consideration leads to $\mu = \pi = 0$. However, this solution contradicts the last constraint $\mu + \pi = 1$, and therefore an optimal objective value of Model (A.1) is positive. To show the boundedness of Model (A.1), consider the feasible solution $\lambda_d = \frac{1}{2}$, $\lambda_{j \neq d} = 0, j = 1, ..., n$ and $\theta = \mu = \pi = \frac{1}{2}$ of the dual Model (A.2). Given the provided feasible solution for Model (A.2), which is a minimization problem, we have $0 < \theta^* \le \frac{1}{2}$ and accordingly, the model is feasible and bounded. Regarding the strong duality theorem, Model (A.1) has bounded optimal objective value, i.e., $0 < \varepsilon^* \le \frac{1}{2}$ which completes the proof. □

Theorem A.2.. *Model* (3) *is feasible if and only if* $\varepsilon \le \varepsilon_d^*$.

Proof.. Let $\left(\bar{v}_d, \bar{u}_d, \bar{w}_d, \varepsilon_d^*\right)$ be an optimal solution of Model (A.1). Furthermore, let $\bar{\alpha} = \frac{\sum_{r=1}^{s} \bar{u}_{rd} y_{rd}}{\varepsilon_d^*}$ and $\bar{\beta} = \frac{\sum_{t=1}^{k} \bar{w}_{td} \bar{b}_{td}}{\varepsilon_d^*}$. It is clear that $\left(\bar{\alpha}, \bar{\beta}, \bar{v}_d, \bar{u}_d, \bar{w}_d\right)$ is a feasible solution for Model (3) for all $\varepsilon \le \varepsilon_d^*$. What is left is to prove that Model (3) is infeasible for $\varepsilon > \varepsilon_d^*$. Suppose, contrary to our claim, $(\alpha, \beta, v_d, u_d, w_d)$ is a feasible solution for $\varepsilon > \varepsilon_d^*$. Hence, $(v_d, u_d, w_d, \varepsilon)$ is a feasible solution for Model (A.1) with a larger objective value than $\left(\bar{v}_d, \bar{u}_d, \bar{w}_d, \varepsilon_d^*\right)$, which contradicts the optimality condition. As a result, Model (3) is infeasible for $\varepsilon > \varepsilon_d^*$, which completes the proof. □

Appendix B

We formulate the following model for identifying a suitable value for ε_d^{*U} in Model (5):

$$\varepsilon_d^{*U} = \text{Max } \varepsilon_d^U$$

$$s.t.$$

$$\sum\nolimits_{i=1}^{m} v_{id} x_{id}^L = 1$$

$$\sum\nolimits_{i=1}^{m} v_{id} x_{ij}^U - \sum\nolimits_{r=1}^{s} u_{rd} y_{rj}^L - \sum\nolimits_{t=1}^{k} w_{td} \bar{b}_{tj}^L \ge 0, j = 1, ..., n, j \neq d$$

$$\sum\nolimits_{i=1}^{m} v_{id} x_{id}^L - \sum\nolimits_{r=1}^{s} u_{rd} y_{rd}^U - \sum\nolimits_{t=1}^{k} w_{td} \bar{b}_{td}^U \ge 0 \tag{B1}$$

$$\sum\nolimits_{r=1}^{s} u_{rd} y_{rd}^U - \varepsilon_d^U \ge 0$$

$$\sum\nolimits_{t=1}^{k} w_{td} \bar{b}_{td}^U - \varepsilon_d^U \ge 0$$

$$v_{id} \ge 0, i = 1, ..., m$$

$$u_{rd} \ge 0, r = 1, ..., s$$

$$w_{td} \ge 0, t = 1, ..., k$$

In Model (5), the existence of weight constraints $\sum_{r=1}^{s} u_{rd} y_{rd}^{U} \geq \alpha \varepsilon_d^{*U}$ and $\sum_{t=1}^{k} w_{td} \overset{-U}{b}_{td} \geq \beta \varepsilon_d^{*U}$ for DMU_d in the upper-bound results in the corresponding weights assigned to desirable and undesirable outputs not being all equal to zero, and also reduces the differences between the weighted desirable and undesirable outputs. We have always $\varepsilon_d^{*U} > 0$ and to obtain a set of $\{\varepsilon_1^{*U}, ..., \varepsilon_n^{*U}\}$, Model (B.1) should be solved $n$times, once for each unit. Analogously, an epsilon model can be extended for Model (6).

Appendix C

We suggest the following model to find the epsilon in Model (14):

$$\varepsilon_d^{*R} = \text{Max } \varepsilon_d^R$$

$$s.t.$$

$$\sum_{i=1}^{m} v_{id} x_{id}^{L} + \alpha_d^x = 1$$

$$\sum_{i=1}^{m} v_{id} x_{ij}^{U} - \sum_{r=1}^{s} u_{rd} y_{rj}^{L} - \sum_{t=1}^{k} w_{td} \overset{-L}{b}_{tj} - \alpha_j^x - \alpha_j^g - \alpha_j^b \geq 0, j = 1, ..., n, j \neq d$$

$$\sum_{i=1}^{m} v_{id} x_{id}^{L} - \sum_{r=1}^{s} u_{rd} y_{rd}^{U} - \sum_{t=1}^{k} w_{td} \overset{-U}{b}_{td} + \alpha_d^x + \alpha_d^g + \alpha_d^b \geq 0 \tag{C1}$$

$$\sum_{r=1}^{s} u_{rd} y_{rd}^{U} - \alpha_d^g - \varepsilon_d^R \geq 0$$

$$\sum_{t=1}^{k} w_{td} \overset{-U}{b}_{td} - \alpha_d^b - \varepsilon_d^R \geq 0$$

$$v_{id} \geq 0, i = 1, ..., m$$

$$u_{rd} \geq 0, r = 1, ..., s$$

$$w_{td} \geq 0, t = 1, ..., k$$

Appendix D

Theorem D.1.. *Model (20) is always feasible.*

Proof:. It is sufficient to introduce a feasible solution to the model. Let's assume that $\widehat{z}_j^x = \widehat{z}_j^g = \widehat{z}_j^b = 0 (j = 1, ..., n)$. In this case, $\gamma_j^x = \gamma_j^g = \gamma_j^b = 0 (j = 1, ..., n)$, and $R_{ij} = p_{rj} = q_{tj} = 0 (\forall i, r, t, j = 1, \cdots, n)$. Furthermore, let's assume that $\omega_{id} = \mu_{rd} = \varphi_{td} = 0 (\forall i, r, t)$, for constraints related to $j = 1, ..., n, j \neq d$ and suppose that $v_{id} = \frac{1}{\sum_{i=1}^{m} x_{id}^{L}}$ and $u_{rd} = \frac{\alpha}{\sum_{r=1}^{s} y_{rd}^{U}}, w_{td} = \frac{\beta}{\sum_{t=1}^{k} \overset{-U}{b}_{td}} (\forall i, \forall r, \forall t)$ for the DMU under evaluation, i.e., d. Substituting these assumptions into Constraint (26a) yields:

$$\sum_{i=1}^{m} \frac{1}{\sum_{i=1}^{m} x_{id}^{L}} x_{id}^{L} + 0 = 1 \tag{D1}$$

Which always holds. In such a setting, the Constraints (18d) and (18e) are satisfied as well. Besides, let's assume that $s_j = \xi_j = z_j = 0 (\forall j)$. With this assumption, Constraints (19d), (19e), and (20b) are established. Moreover, the provided solution satisfies the Constraints (19b) and (20a). From Constraint (19a), we have:

$$\sum_{r=1}^{s} u_{rd} y_{rd}^{U} + \sum_{t=1}^{k} w_{td} \overset{-U}{b}_{td} = E_{dd}^{R} \tag{D3}$$

Therefore, for Constraint (27c), we have:

$$E_{dd}^{R} \left(\sum_{i=1}^{m} v_{id} x_{id}^{L} + 0 \right) - \sum_{r=1}^{s} u_{rd} y_{rd}^{U} - 0 - \sum_{t=1}^{k} w_{td} \overset{-U}{b}_{td} + 0 - \xi_d = 0 \tag{D4}$$

Substituting (D.3) in (D.4), Constraint (19c) is satisfied. Similarly, Constraint (20b) holds, and we have $\xi_d = 1 - E_{dd}^{R}$. At this point, since $\gamma_j^x = \gamma_j^g = \gamma_j^b = 0 (j = 1, ..., n)$, we can rewrite Constraints (18f), (18g) and (18h) as follows:

$$x_{ij}^{U} - x_{ij}^{L} = 0, \forall i, j = 1, ..., n,$$

$$y_{rj}^{U} - y_{rj}^{L} = 0, \forall r, j = 1, ..., n,$$

$$\overset{-U}{b}_{tj} - \overset{-L}{b}_{tj} = 0, \forall t, j = 1, ..., n$$

Therefore, Constraints (18f), (18g) and (18h), also hold, and this completes the proof. □

Theorem D.2.. *The objective function value of the robust Model* (20) *is smaller than or equal to the objective function value of the optimistic Model* (12).

Proof:. Since the objective functions for both Models (12) and (20) are of the form $\max\sum_{j=1}^{n}\xi_j$, consider the third constraint in Model (12) as well as constraint (20a). Then, summing up the third constraint in Model (12), we have:

$$\sum_{\substack{j=1\\ j\neq d}}^{n}\xi_j=\sum_{i=1}^{m}v_{id}\Big(\sum_{\substack{j=1\\ j\neq d}}^{n}x_{ij}^{U}\Big)-\sum_{r=1}^{s}u_{rd}\Big(\sum_{\substack{j=1\\ j\neq d}}^{n}y_{rj}^{L}\Big)-\sum_{t=1}^{k}w_{td}\Big(\sum_{\substack{j=1\\ j\neq d}}^{n}\overset{-L}{b_{tj}}\Big) \tag{D5}$$

Moreover, summing up Constraint (20*a*), we have:

$$\sum_{\substack{j=1\\ j\neq d}}^{n}\xi_j=\sum_{i=1}^{m}v_{id}\Big(\sum_{\substack{j=1\\ j\neq d}}^{n}x_{ij}^{U}\Big)-\sum_{r=1}^{s}u_{rd}\Big(\sum_{\substack{j=1\\ j\neq d}}^{n}y_{rj}^{L}\Big)-\sum_{t=1}^{k}w_{td}\Big(\sum_{\substack{j=1\\ j\neq d}}^{n}\overset{-L}{b_{tj}}\Big)-\Big(\sum_{\substack{j=1\\ j\neq d}}^{n}\hat{z}_j^{x}\gamma_j^{x}+\sum_{\substack{j=1\\ j\neq d}}^{n}\sum_{i=1}^{m}R_{ij}+\sum_{\substack{j=1\\ j\neq d}}^{n}\hat{z}_j^{g}\gamma_j^{g}+\sum_{\substack{j=1\\ j\neq d}}^{n}\sum_{r=1}^{s}p_{rj}+\sum_{\substack{j=1\\ j\neq d}}^{n}\hat{z}_j^{b}\gamma_j^{b}+\sum_{\substack{j=1\\ j\neq d}}^{n}\sum_{t=1}^{k}q_{tj}\Big) \tag{D6}$$

Since $R_{ij},p_{rj},q_{tj},\hat{z}_j^{x},\hat{z}_j^{g},\hat{z}_j^{b},\gamma_j^{x},\gamma_j^{g}$ and γ_j^{b} $\forall i,\forall r,\forall j\neq d$ all have positive values; clearly, Equation (D.6) is smaller than Equation (D.5). □

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