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METHODOLOGIES AND APPLICATION

A random-fuzzy portfolio selection DEA model using value-at-risk and conditional value-at-risk

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Abstract

The complexity involved in portfolio selection has resulted in the development of a large number of methods to support ambiguous financial decision making. We consider portfolio selection problems where returns from investment securities are random variables with fuzzy information and propose a data envelopment analysis model for portfolio selection with downside risk criteria associated with value-at-risk (V@R) and conditional value-at-risk (CV@R). Both V@R and CV@R criteria are used to define possibility, necessity, and credibility measures, which are formulated as stochastic nonlinear programming programs with random-fuzzy variables. Our constructed stochastic nonlinear programs for analyzing portfolio selection are transformed into deterministic nonlinear programs. Moreover, we show an enumeration algorithm can solve the model without any mathematical programs. Finally, we demonstrate the applicability of the proposed framework and the efficacy of the procedures with a numerical example.

Keywords Value-at-risk · Conditional value-at-risk · Portfolio selection · Possibility measure · Necessity measure · Credibility measure · Random-fuzzy variable

1 Introduction

Financial decision making often involves semi-structured and unstructured situations characterized by fuzzy and random events. The purpose of this paper is four-fold. First, we develop a portfolio selection model where returns from investment securities are random variables with fuzzy information. The portfolio model is developed based on the value-at-risk (V@R) and conditional value-at-risk (CV@R). Second, both V@R and CV@R criteria are used to define possibility, necessity, and credibility measures, which are formulated as stochastic nonlinear programming programs with random-fuzzy variables. Third, the developed stochastic nonlinear programs are transformed into deterministic nonlinear programs. Fourth, four data envelopment analysis (DEA) portfolio models are proposed based on the developed framework.

In portfolio selection, a critical issue for the investors is to determine an optimal configuration of the investment portfolio. The mean–variance portfolio selection methodology proposed by Markowitz (1952) has been the basis for the development of modern portfolio theory, where the mean and variance (the first two moments) of the return from the invested securities represent the tradeoff between return and its risk, respectively. Markowitz (1959) has acknowledged the mean–variance portfolio selection methodology has several weaknesses, including variance capturing both the upside and downside movements of a security's return, equally treating extreme and non-extreme

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observations, and (iii) non-symmetrically distributed returns on investment securities. Due to these weaknesses, several researchers extended the mean–variance framework by considering alternative risk frameworks. The well-known alternatives are the mean-semivariance framework (Markowitz et al. 1993), and the framework associated with the mean and various forms of absolute deviation (Konno and Yamazaki 1991). Based on the data of Pakistan's Karachi Stock Exchange that is a highly volatile market, Ayub et al. (2015) concluded that downside risk is a better measure of risk than a variance. These studies assume the returns are random variables with specific probability distributions. Taking into account the complexity of securities investment environments in practice, we can easily recognize that predicting the return from historical data alone is difficult, and hence the prediction without consideration of many uncertain factors can be inaccurate. Indeed, the investors face considerable non-probabilistic factors consisting of socio-economic factors, people's psychological factors, and so forth (Chen et al. 2018). In this circumstance, we have difficulty in estimating the precise probability distribution of the returns from the historical data.

When selecting securities for investment, the investor tends to encounter and hence needs to deal with the uncertainty of future returns from actual securities. Regarding the complexity associated with the uncertainty, the theoretical literature documents several studies dealing with fuzziness and/or randomness. See Zadeh (1978) for the probability of a fuzzy event, Dubois and Prade (1980) for linguistic probabilities, Hirota (1981) for probabilistic sets, and Buckley (2004) and Buckley and Eslami (2004) for uncertain probabilities. Kwakernaak (1978, 1979) and Liu (2007) considered random-fuzzy variables to cope with uncertainty, where the random-fuzzy variable is a mapping on the universal set of random variables in the random-fuzzy environment.

Huang (2008) studied semivariance of a fuzzy variable and proposed two fuzzy mean-semivariance models, where semivariance is a downside risk measure that only considers the portfolio return below the expected value in the calculation. Believing that higher moments cannot be neglected in portfolio selection, Li et al. (2010) added skewness to Huang's (2008) mean-semivariance model and developed mean–variance-skewness models with fuzzy return variables. Hasuike et al. (2009) also suggested treating the return in credibility space, which is based on the uncertain theory of Liu (2007). Using semi-absolute deviation as a risk index, Vercher and Bermúdez (2015) proposed the possibility measure and the credibility measure. Zhongfeng et al. (2018) examined an uncertain mean-semi-absolute deviation model that considers changes in financial markets.

Gauging market risk and liquidity risk in terms of semivariance, Liu and Zhang (2013) constructed possibility and necessity measures based on the fuzzy set theory. Chen et al. (2017) also exploited Liu's (2007) uncertainty theory and proposed a semivariance portfolio model, where the security returns are estimated based on the degree of experts' belief. Tavana et al. (2019) presented a chance-constrained programming model, in which the security returns are random with rough information.

V@R, representing market risk, has been popular in financial risk management since the middle of the 1990s. V@R is computed as the maximal loss that does not exceed a given confidence level. Wang et al. (2011), for example, used a fuzzy V@R criterion to define downside risk. Despite its wide use, however, V@R is not coherent, which is not desirable from the axiomatic viewpoint (Artzner et al. 1999). Furthermore, V@R is unstable and difficult to deal with in the case when the losses are not normally distributed (Rockafellar and Uryasev 2002). Rockafellar and Uryasev (2002) considered a risk measure of CV@R that is defined by building on the V@R criterion. CV@R is a coherent measure in the sense of Artzner et al. (1999) and is obtained by minimizing a tractable auxiliary function without prespecifying the structure of V@R. Ma et al. (2015) presented V@R and CV@R measures in possibility and credibility spaces.

Although these studies consider a variety of portfolio models, they are not concerned about portfolio performance assessment explicitly. Morey and Morey (1999) proposed a mean–variance portfolio performance model using a data envelopment analysis (DEA) approach that was originally introduced by Charnes, Cooper, and Rhodes (1978) to address this limitation. Morey and Morey's (1999) model, which is a quadratic nonlinear DEA model, considers variance and mean to represent the single input and the single output, respectively. Adding a nonlinear constraint of skewness to Morey and Morey's (1999) DEA model, Joro and Na (2006) developed a mean–variance-skewness DEA model in which skewness is treated as a second output. Liu et al. (2015) investigated the traditional DEA model applied to portfolio analysis from the standpoint of sampling portfolio and concluded that linear programming DEA is an effective and practical approach in terms of the ability to approximate the portfolio efficiency. Recently, Chen et al. (2018) extended Liu et al.'s (2015) work into a fuzzy possibility environment and constructed three types of DEA portfolio performance measures, which are the variance possibility, the semivariance possibility, and the semi-absolute deviation possibility measures. Branda (2016) developed a mean-V@R DEA model based on directional distance function for the situation when there are positive and negative data.

The DEA portfolio performance studies have treated the return as a fuzzy variable or a random variable, not as a random-fuzzy variable. Note that while a random variable is characterized by mean and variance, the mean of a random-fuzzy variable is fuzzy, and hence we believe that incorporation of random-fuzziness will formulate the complexity of uncertainty better. Considering this perception, we develop a DEA portfolio performance framework, in which the security returns are random-fuzzy variables. In the developed framework, we focus only on the portfolio efficiency assessment based on V@R and CV@R criteria in the situation where securities returns are random-fuzzy variables. We believe that the framework will give much flexibility in carrying out portfolio performance analysis.

In summary, the classical portfolio selection methodology proposed by Markowitz (1952) utilizes the mean and variance of the return from the invested securities, under the assumption that the variance is the valid indicator of the investors. However, recent studies show that downside risk is a better indicator of risk than a variance. Given the complexity of securities, investment environments in practice, the practical portfolio analysis based on downside risks should consider several uncertain factors. Indeed, the investors face considerable non-probabilistic factors consisting of socio-economic factors, the people's psychological factors, and so forth. In these circumstances, decision-makers have difficulty in accurately estimating the downside risk of the returns from the historical data. To cope with the uncertain nature of the data, we propose a data envelopment analysis (DEA) model for portfolio selection with downside risk criteria associated with value-at-risk (V@R) and conditional value-at-risk (CV@R) under the assumption of random-fuzzy variables.

The remainder of the paper is organized as follows. Section 2 presents the basics of a random-fuzzy variable. Section 3 develops a portfolio performance framework that focuses on downside risks. Subsection 3.1 presents the mean–variance Markovitz portfolio selection method for quadratic programming problems. Subsections 3.2 gives possibility, necessity and credibility measures based on V@R, assuming that the returns are random-fuzzy variables. Section 4 introduces novel risk measures defined relative to CV@R under the assumption of random-fuzzy variables. Section 5 implements V@R and CV@R in the DEA portfolio performance framework under the assumption of random-fuzzy variables. Precisely, we propose a DEA portfolio performance framework based on possibility, necessity, and credibility measures capturing downside risk with the use of V@R and CV@R. Here, we also present an enumeration algorithm, which does not require solving any optimization problem. Section 6 presents the efficacy of our developed models using an illustrative example. Finally, we conclude the paper with concluding remarks and future extensions in Sect. 7.

2 Definitions of random fuzziness and basic results

To facilitate the understanding of our methodology, we present some preliminaries and definitions on fuzzy sets and random-fuzzy variables.

Definition 1 (Dubois and Prade 1980). Let $\tilde{A} = (\alpha, m, \beta)_{LR}$ be a fuzzy interval of LR-type, where α and β are the nonnegative left and right spreads, respectively, and m is the mean value of $\tilde{A}$. The membership function of $\tilde{A}$ is denoted as:

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\frac{m-x}{\alpha}\right), & x \le m, \\ 1, & x = m, \\ R\left(\frac{x-m}{\beta}\right), & x \ge m. \end{cases} \tag{1}$$

where L and R are the left and right functions, respectively.

In this study, we assume, $\tilde{A} = (\alpha, m, \beta)_{LR}$ is a triangular fuzzy number. If the mean m is also a fuzzy interval, then, $\tilde{A} = (\alpha, m_1, m_2, \beta)_{LR}$ is a trapezoidal fuzzy number where m_1 and m_2 are the mean values of $\tilde{A}$ (See Tavana et al. 2013, p. 34). An example of a membership function in Eq. (1) is given as $L(x) = R(x) = \begin{cases} 1-x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$.

Clearly, $\tilde{A} = (\alpha, m, \beta)_{LR} = (\alpha, m, \beta)$ and thus $\tilde{A}$ is a triangular fuzzy number. In this case, the triangular fuzzy numbers are represented by $(m-\alpha, m, m+\beta)$. LR fuzzy number is generalized form for triangular fuzzy number and trapezoidal fuzzy number. Then if we use $L(x) = R(x) = \begin{cases} 1-x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$., it is clear $\tilde{A} = (\alpha, m, \beta)_{LR}$ converts to fuzzy triangular number as $\tilde{A} = (\alpha, m, \beta)$.

Definition 2 (Zadeh 1978; Zimmermann 1996) Let $(\Theta, P(\Theta), Pos)$ be a possibility space with Θ being a nonempty set that includes all possible events, and $P(\Theta)$ is the power set of Θ. For each $A \subseteq P(\Theta)$, there is a nonnegative number $Pos(A)$, so-called a *possibility measure*, with the following properties:

(i) $Pos(\emptyset) = 0, \quad Pos(\Theta) = 1;$

(ii) $A \subseteq B \;\Rightarrow\; Pos(A) \le Pos(B)$ for any $A, B \in P(\Theta)$;

(iii) $Pos(\cup_k A_k) = Sup_k Pos(A_k)$.

Definition 3 (Zimmermann 1996; Dubois and Prade 1978, 1988) The *necessity measure* of A, denoted by $Nec(A)$, is defined on $(\Theta, P(\Theta), Pos)$ as $Nec(A) = 1 - Pos(A^c)$ where A^c is the complement set of A. For any sets A and B, the properties of $Nec(A)$ are presented as follows:

$(a)\quad Nec(\emptyset) = 0,\ Nec(\Theta) = 1;$

$(b)\quad Pos(A) \geq Nec(A);$

$(c)\quad A \subseteq B \text{ implies } Nec(A) \leq Nec(B);$

$(d)\quad Pos(A) < 1 \Rightarrow Nec(A) = 0;$

$(e)\quad Nec(A) > 0 \Rightarrow Pos(A) = 1$

Definition 4 (Liu and Liu 2002) Let $(\Theta, P(\Theta), Pos)$ be the possibility space. The *credibility measure* of a fuzzy event A, $Cr(A)$, is defined as $Cr(A) = 0.5(Pos(A) + Nec(A))$ with the following properties:

$(a)\quad Cr(\emptyset) = 0, \quad Cr(\Theta) = 1;$

$(b)\quad A \subseteq B \quad \Rightarrow \quad Pos(A) \leq Pos(B)$ for any $A, B \in P(\Theta)$;

$(c)\quad Cr(A) + Cr(A^c) = 1$ for any $A \in P(\Theta)$;

$(d)\quad Cr(\cup_i A_i) = Sup_i Cr(A_i)$ for any subset $A_i \in P(\Theta)$ with $Sup_i Cr(A_i) < 0.5$;

$(e)\quad Cr(A \cup B) \leq Cr(A) + Cr(B)$ for any $A, B \in P(\Theta)$;

$(f)\quad Pos(A) \geq Cr(A) \geq Nec(A)$ for any $A \in P(\Theta)$.

Definition 5 (Liu and Liu 2002) Let ξbe a fuzzy variable on the possibility space $(\Theta, P(\Theta), Pos)$. The possibility, necessity, and credibility of a fuzzy event $\{\xi \geq r\}$ are represented by:

$$Pos(\xi \geq r) = \underset{t \geq r}{Sup}\ \mu_\xi(t),$$

$$Nes(\xi \geq r) = 1 - \underset{t < r}{Sup}\ \mu_\xi(t),$$

$$Cr(\xi \geq r) = 0.5[Pos(\xi \geq r) + Nec(\xi \geq r)].$$

where $\mu_\xi : \Re \to [0, 1]$ is the membership function of ξ and r is a real number. Note here that $Cr(\xi \geq r) = 1 - Cr(\xi < r)$.

Definition 6 A random-fuzzy variable ξ is *normally distributed* if ξ is a normally distributed random variable whose density function $f(x)$ is defined by:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\hat{\mu})^2}{2\sigma^2}},\ -\infty < x < +\infty$$

where $\hat{\mu} = (\mu^L, \mu^m, \mu^R)$ is a fuzzy variable. To indicate that ξ is a normally distributed random-fuzzy variable, we write $\xi \sim N(\hat{\mu}, \sigma^2)$.

Definition 7 (Liu and Liu 2002) Let ξ be a random-fuzzy variable on the possibility space $(\Theta, P(\Theta), Pos)$. Then the expected value of ξ is denoted as:

$$E(\xi) = \int_0^{+\infty} Cr\{\theta \in \Theta | E[\xi(\theta)] \geq r\} dr - \int_{-\infty}^{0} Cr\{\theta \in \Theta | E[\xi(\theta)] \leq r\} dr$$

Notice that at least one of the above two integrals is finite. If ξ is a continuous random-fuzzy variable on a possibility space $(\Theta, P(\Theta), Pos)$, then the expected value of ξ is:

$$\mathrm{E}(\xi) = \int_0^{+\infty} Cr\left\{\int_{x\in\Theta} xf(x)dx \geq r\right\} dr - \int_{-\infty}^{0} Cr\left\{\int_{x\in\Theta} xf(x)dx \leq r\right\} dr$$

where $f(x)$ is a density function with fuzzy parameters.

Remark 1 (Tavana et al. 2013) Let $\hat{\mu} = (X^L, X^m, X^R)$ be a random-fuzzy variable where $\tilde{X}^m \sim N(\hat{\mu}, \sigma^2)$. Then the expected value of $\tilde{X}$ is as follows:

$$E[\xi] = X^m + \frac{1}{2}\left[X^L(T(0) - T(1)) + X^R(P(1) - P(0))\right]$$

where $T(x)$ and $p(x)$ are the continuously differentiable functions on [0, 1], and $\frac{\partial \mathrm{T}(x)}{\partial x} = L(x)$, and $\frac{\partial \mathrm{P}(x)}{\partial x} = R(x)$.

Lemma 1 (Khanjani Shiraz et al. 2014) *Let $\bar{\lambda}_1 = (\lambda_1^L, \lambda_1^m, \lambda_1^R)_{LR}$ and $\bar{\lambda}_2 = (\lambda_2^L, \lambda_2^m, \lambda_2^R)_{LR}$ be two LR-type fuzzy numbers with continuous membership functions. For a given confidence level $\alpha \in [0, 1]$. If $Pos(\bar{\lambda}_1 \geq \bar{\lambda}_2) \geq \alpha$, then we have: $\lambda_1^m + R^{-1}(\alpha)\lambda_1^R \geq \lambda_2^m - \lambda_2^L R^{-1}(\alpha)$*

Lemma 2 (Khanjani Shiraz et al. 2014) *Let $\bar{\lambda}_1 = (\lambda_1^L, \lambda_1^m, \lambda_1^R)_{LR}$ and $\bar{\lambda}_2 = (\lambda_2^L, \lambda_2^m, \lambda_2^R)_{LR}$ be two LR-type fuzzy numbers with continuous membership function. For a given confidence level $\alpha \in [0, 1]$,$Nec(\bar{\lambda}_1 \geq \bar{\lambda}_2) \geq \alpha$ implies $\lambda_1^m - L^{-1}(1-\alpha)\lambda_1^L \geq \lambda_2^m + \lambda_2^R L^{-1}(1-\alpha)$*

Lemma 3 (Tavana et al. 2012) *Let $\bar{\lambda}_1 = (\lambda_1^L, \lambda_1^m, \lambda_1^R)_{LR}$* and *$\bar{\lambda}_2 = (\lambda_2^L, \lambda_2^m, \lambda_2^R)_{LR}$ be two independent LR-type fuzzy numbers with continuous membership functions. For a given confidence level $\alpha \in [0, 1]$,*

(a) $\alpha \leq 0.5$, $Cr(\bar{\lambda}_1 \geq \bar{\lambda}_2) \geq \alpha$ if and only if $\lambda_1^m + \lambda_1^R R^{-1}(2\alpha) \geq \lambda_2^m - \lambda_2^L R^{-1}(2\alpha)$,

(b) $\alpha > 0.5$, $Cr(\bar{\lambda}_1 \geq \bar{\lambda}_2) \geq \alpha$ if and only if $\lambda_1^m - \lambda_1^L L^{-1}(2(1-\alpha)) \geq \lambda_2^m + \lambda_2^R L^{-1}(2(1-\alpha))$.

3 Portfolio selection model

3.1 Markowitz mean–variance model

The Markowitz's (1952) mean–variance framework is the foundation of modern portfolio theory. This subsection provides the basics of the mean–variance framework.

Suppose we have n securities. For any security $j = 1, \ldots, n$, let r_j be the return rate from the j-th security and let x_j be the investment proportion of funds invested in the j-th security. Suppose the portfolio selection problem is to minimize the variance of total portfolio return $\sum_{j=1}^{n} \tilde{r}_j x_j$ subject to the expected return constraint, where each $\tilde{r}_j$ is assumed to be a normally distributed random variable, that is: $\tilde{r}_j \sim N\left(r_j, \sigma_j^2\right)$. That is, the variance minimization problem takes the form:

$$\begin{aligned} &\min_{x} \quad Var\left[\sum_{j=1}^{n} \tilde{r}_j x_j\right] \\ &s.t. \\ &E\left\{\sum_{j=1}^{n} \tilde{r}_j x_j\right\} \geq \mu, \\ &\sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0,\ j = 1, \ldots, n. \end{aligned} \tag{2}$$

Markowitz (1952) formulated the portfolio return variance minimization problem of (2) as the following quadratic programming problem:

$$\begin{aligned} &\min \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} x_i \sigma_{ij} x_j} \\ &s.t. \\ &\sum_{i=1}^{n} x_j r_i \geq \mu, \\ &\sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0,\ j = 1, \ldots, n. \end{aligned}$$

3.2 Value-at-risk

In financial risk management, several risk measures are proposed for measuring downside risks. The V@R criterion has been widely used because it overcomes the undesirable feature associated with the Markovitz mean–variance method that includes both upside and downside risks.

For an arbitrary $\alpha \in (0,1)$, the value-at-risk ($V@R$) measure for a random-fuzzy variable X can be defined as the optimal solution of the following chance-constrained problem:

$$V@R_\alpha(X) = \min\{\varepsilon : \Pr(X \leq \varepsilon) \geq \alpha\},$$

of which objective is the loss associated with a decision variable X. Then the following relationship holds:

$$V@R_\alpha(X) \leq \varepsilon \Leftrightarrow Pr(X \leq \varepsilon) \geq \alpha$$

Based on these results, we construct the following generic portfolio model with random-fuzzy returns as follows:

$$\begin{aligned} &\min \quad V@R_\alpha\left[\sum_{j=1}^{n} x_j \tilde{r}_j\right] \\ &s.t. \\ &E\left[\sum_{i=1}^{n} x_j \tilde{r}_j\right] \geq \eta, \\ &\sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0,\ j = 1, \ldots, n. \end{aligned} \tag{3}$$

Now we implement the possibility, necessity, and credibility measures discussed in the previous subsection. Equation (3) can be written as follows:

$$\begin{aligned} &\min f \\ &s.t. \\ &\varphi\left[V@R_\alpha\left[\sum_{j=1}^{n} x_j \tilde{r}_j\right] \leq f\right] \geq \beta, \\ &E\left[\sum_{j=1}^{n} x_j \tilde{r}_j\right] \geq \eta, \\ &\sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0,\ j = 1, .., n. \end{aligned} \tag{4}$$

where[1] $\varphi = Pos,\ Nec,\ Cr$.

To provide the practical possibility, necessity, and credibility measures, we establish the following theorem:

Theorem 1 *Suppose that $\hat{a}_j$ are the LR (left–right) random-fuzzy numbers with normal distributions $N\left(\hat{\mu}, \sigma^2\right)$ where $\hat{\mu}_j = \left(a_j^L, a_j^m, a_j^R\right)$.* Then we have:

(a) $Pos\left[V@R_\alpha\left(\sum_{j=1}^{n} \hat{a}_j x_j\right) \leq \varepsilon\right] \geq \beta$ if and only if

[1] If $\varphi = Pos$ in Eq. (4), then a V@R-Possibility measure is described as

$$\min f \ \ s.t. \quad Pos\left[V@R_\alpha\left[\sum_{j=1}^{n} x_j \tilde{r}_j\right] \leq f\right] \geq \beta,\ \ E\left[\sum_{j=1}^{n} x_j \tilde{r}_j\right] \geq \eta,$$

$$\sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0,\ j = 1, \ldots, n.$$

$$\sum_{j=1}^{n} x_j\left(a_j^m - R^{-1}(\beta)a_j^L\right) + \Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_i\sigma_{il}x_l} \le \varepsilon$$

(b) $Nec\left[V@R_\alpha\left(\sum_{j=1}^{n}\hat{a}_j x_j\right) \le \varepsilon\right] \ge \beta$ if and only if

$$\sum_{j=1}^{n} x_j\left(a_j^m + L^{-1}(1-\beta)a_j^R\right) + \Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_i\sigma_{il}x_l} \le \varepsilon$$

(c) $Cr\left[V@R_\alpha\left(\sum_{j=1}^{n}\hat{a}_j x_j\right) \le \varepsilon\right] \ge \beta$ if and only if

$$\begin{cases} \sum_{j=1}^{n} x_j\left(a_j^m - R^{-1}(2\beta)a_j^L\right) + \Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_i\sigma_{il}x_l} \le \varepsilon, & \beta \le 0.5 \\ \sum_{j=1}^{n} x_j\left(a_j^m + L^{-1}(2(1-\beta))a_j^R\right) + \Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_i\sigma_{il}x_l} \le \varepsilon, & \beta > 0.5 \end{cases}$$

Proof We prove part (a). Note that:

$$Pos\left[V@R_\alpha\left(\sum_{j=1}^{n}\hat{a}_j x_j\right) \le \varepsilon\right] \ge \beta$$
$$\Leftrightarrow Pos\left[\Pr\left(\sum_{j=1}^{n}\hat{a}_j x_j \le \varepsilon\right) \ge \alpha\right] \ge \beta.$$

Let $\hat{h} = \sum_{j=1}^{n}\hat{a}_j x_j - \varepsilon$ where $\hat{a}_j \sim N\left(\hat{\mu}_j, \sigma_j^2\right)$. Obviously, $\hat{h}$ is also normally distributed with $\mu_{\hat{h}} = \sum_{j=1}^{n}\hat{\mu}_j x_j - \varepsilon$ and $\sigma_{\hat{h}}^2 = \sum_{i=1}^{n}\sum_{l=1}^{n} x_i\sigma_{il}x_l$. Accordingly, the standardized normal distribution can be used to transform the probability related to a deterministic form as follows:

$$Pos\left[Pr\left(\sum_{j=1}^{n}\hat{a}_j x_j \le \varepsilon\right) \ge \alpha\right] \ge \beta$$
$$\Leftrightarrow Pos\left[\sum_{j=1}^{n}\hat{\mu}_j x_j + \Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_i\sigma_{il}x_l} \le \varepsilon\right] \ge \beta$$

Lemma 1 yields

$$\sum_{j=1}^{n} x_j\left(a_j^m - R^{-1}(\beta)a_j^L\right) + \Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_i\sigma_{il}x_l} \le \varepsilon$$

The proof (a) is complete. The proof of parts (b) and (c) is similar. □

We have established the (a) and (b) parts of Theorem 1 that are based on possibility theory and the (c) part that is based on Liu's (2002) uncertainty theory. Using Eq. (4), part (a) of Theorem 1 and Remark 1, the possibility measure is constructed as follows:

$$\begin{aligned} &\min f \\ &\text{s.t.} \\ &\sum_{j=1}^{n} x_j\left(r_j^m - R^{-1}(\beta)r_j^L\right) + \Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_i\sigma_{il}x_l} \le f, \\ &\sum_{j=1}^{n} x_j\left(r_j + \frac{1}{2}\left[r_j^L(T(0) - T(1)) + r_j^R(\mathrm{P}(1) - \mathrm{P}(0))\right]\right) \ge \eta, \\ &\sum_{j=1}^{n} x_j = 1, \quad x_j \ge 0, \ j = 1, \ldots, n. \end{aligned}$$

which is obtained with the use of the (a) part of Theorem 1.

V@R-Possibility measure[2]

$$\begin{aligned} &\min \sum_{j=1}^{n} x_j\left(r_j^m - R^{-1}(\beta)r_j^L\right) + \Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_i\sigma_{il}x_l} \\ &\text{s.t.} \\ &\sum_{j=1}^{n} x_j = 1, \quad x_j \ge 0, \ j = 1, \ldots, n. \end{aligned} \tag{5}$$

Similarly, from Eq. (4) and the part (b) of Theorem 1, we obtain the following deterministic necessity measure.

V@R-Necessity measure

$$\begin{aligned} &\min \quad \sum_{j=1}^{n} x_j\left(r_j^m + L^{-1}(1-\beta)r_j^R\right) + \Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_i\sigma_{il}x_l} \\ &\text{s.t.} \\ &\sum_{j=1}^{n} x_j\left(r_j + \frac{1}{2}\left[r_j^L(T(0) - T(1)) + r_j^R(\mathrm{P}(1) - \mathrm{P}(0))\right]\right) \ge \eta, \\ &\sum_{j=1}^{n} x_j = 1, \quad x_j \ge 0, \ j = 1, \ldots, n. \end{aligned} \tag{6}$$

From Eq. (4) and the (c) part of Theorem 1, we obtain the following two deterministic credibility measures: one is for $\beta \le 0.5$, and the other is for $\beta > 0.5$:

***V@R-Credibility measure* for $\beta \le 0.5$**

$$\begin{aligned} &\min_{\beta \le 0.5} \sum_{j=1}^{n} x_j\left(r_j^m - R^{-1}(2\beta)r_j^L\right) + \Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_i\sigma_{il}x_l} \\ &\text{s.t.} \\ &\sum_{j=1}^{n} x_j\left(r_j + \frac{1}{2}\left[r_j^L(T(0) - T(1)) + r_j^R(\mathrm{P}(1) - \mathrm{P}(0))\right]\right) \ge \eta, \\ &\sum_{j=1}^{n} x_j = 1, \quad x_j \ge 0, \ j = 1, \ldots, n. \end{aligned}$$

[2] When we refer to a measure, we use V@R instead of $V@R_\alpha$ for simplicity. This applies to the case of CV@R.

***V@R-Credibility measure* for** $\beta > 0.5$

$$\max_{\beta>0.5} \sum_{j=1}^{n} x_j \left(r_j^m + L^{-1}(2(1-\beta)) r_j^R \right) + \Phi^{-1}(\alpha) \sqrt{\sum_{i=1}^{n} \sum_{l=1}^{n} x_i \sigma_{il} x_l}$$

s.t.

$$\sum_{j=1}^{n} x_j \left(r_j + \frac{1}{2} \left[r_j^L (T(0) - T(1)) + r_j^R (\mathrm{P}(1) - \mathrm{P}(0)) \right] \right) \geq \eta,$$

$$\sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0, \; j = 1, \ldots, n. \tag{7}$$

4 Portfolio selection with random-fuzzy returns in conditional value-at-risk approach

In this section, we propose a new conditional value-at-risk measure with random-fuzzy variables by extending the value-at-risk measures in the previous section. For continuous X, the conditional value-at-risk is the expected loss, conditional on the condition that the loss exceeds the value at risk at a given confidence level:

Definition 9 Let X be a continuous random variable representing a loss of the security investment. Given a parameter $0 < \alpha < 1$, the conditional value-at-risk of X is $CV@R_\alpha(Z) = E[X | X \geq V@R_\alpha(X)]$.

To implement $CV@R_\alpha$ in optimization problems, Rockafellar and Uryasev (2000, 2002) proposed the following minimization formula:

$$CV@R_\alpha(Z) = \min_{\eta \in \Re} \left\{ \eta + \frac{1}{1-\alpha} E[Z - \eta]^+ \right\}$$

where $[\cdot]^+ = \max\{\cdot, 0\}$ denotes the positive part and η is a real auxiliary variable, where $F(\eta) = P(Z \leq \eta)$, $\eta \in \Re$.

Lemma 4 (Chatterjee 2014, Acerbi's Integral Formula for $CV@R$). *The $CV@R$ of a random-fuzzy variable X, which represents a loss at a confidence level α, can be expressed as*:

$$CV@R_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 V@R_p(X) \, \mathrm{d}p.$$

Replacing $V@R_\alpha$ by $CV@R_\alpha$ in Eq. (4), we obtain the following pair of optimization problems for the conditional value-at-risk case.

$$\begin{array}{ll}
\min \quad CV@R_\alpha\left[\sum_{j=1}^{n} x_j \tilde{r}_j\right] & \min f \\
\text{s.t.} & \text{s.t.s.t.} \\
E\left[\sum_{j=1}^{n} x_j \tilde{r}_j\right] \geq \eta, & \varphi\left[CV@R_\alpha\left[\sum_{j=1}^{n} x_j \tilde{r}_j\right] \leq f\right] \geq \beta, \\
\sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0, \; j = 1, \ldots, n. & E\left[\sum_{j=1}^{n} x_j \tilde{r}_j\right] \geq \eta, E\left[\sum_{j=1}^{n} x_j \tilde{r}_j\right] \geq \eta, \\
 & \sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0, \; j = 1, \ldots, n
\end{array} \tag{8}$$

where $\varphi = Pos, Nec, Cr$. Now we are ready to establish Theorem 2 to obtain the deterministic optimization problems for the possibility, necessity, and credibility measures associated with the right-hand side of Eq. (8).

Theorem 2 *Suppose that $\hat{a}_j$ are the LR random-fuzzy numbers with normal distributions $N(\hat{\mu}, \sigma^2)$ where $\hat{\mu}_j = \left(a_j^L, a_j^m, a_j^R\right)$. Then we have*:(a) $Pos\left[CV@R_\alpha\left(\sum_{j=1}^{n} \hat{a}_j x_j\right) \leq \varepsilon\right] \geq \beta$ if and only if:

$$\sum_{j=1}^{n} x_j \left(a_j^m - R^{-1}(\beta) a_j^L \right) + \frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha} \sqrt{\sum_{i=1}^{n} \sum_{l=1}^{n} x_i \sigma_{il} x_l} \leq \varepsilon$$

(b) $Nec\left[CV@R_\alpha\left(\sum_{j=1}^{n} \hat{a}_j x_j\right) \leq \varepsilon\right] \geq \beta$ if and only if:

$$\sum_{j=1}^{n} x_j \left(a_j^m + L^{-1}(1-\beta) a_j^R \right) + \frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha} \sqrt{\sum_{i=1}^{n} \sum_{l=1}^{n} x_i \sigma_{il} x_l} \leq \varepsilon$$

(c) $Cr\left[CV@R_\alpha\left(\sum_{j=1}^{n} \hat{a}_j x_j\right) \leq \varepsilon\right] \geq \beta$ if and only if

$$\begin{cases} \sum_{j=1}^{n} x_j \left(a_j^m - R^{-1}(2\beta) a_j^L \right) + \frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha} \sqrt{\sum_{i=1}^{n} \sum_{l=1}^{n} x_i \sigma_{il} x_l}, & \beta \leq 0.5, \\ \sum_{j=1}^{n} x_j \left(a_j^m + L^{-1}(2(1-\beta)) a_j^R \right) + \frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha} \sqrt{\sum_{i=1}^{n} \sum_{l=1}^{n} x_i \sigma_{il} x_l}, & \beta > 0.5. \end{cases}$$

Proof Let $X \sim N(\hat{\mu}, \sigma^2)$ be a normal distribution with $\hat{\mu} = \sum_{j=1}^{n} x_j \hat{\mu}_j$ and $\sigma^2 = \sum_{i=1}^{n} \sum_{l=1}^{n} x_i \sigma_{il} x_l$. Then by using the definition of $CV@R_\alpha$, we obtain:

$$CV@R_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 V@R_p(X)\mathrm{d}p$$

$$X \sim N(\mu, \sigma^2), V@R_\alpha(X) = \mu + \Phi^{-1}(\alpha)\sigma$$

$$CV@R_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 \left[\mu + \Phi^{-1}(p)\sigma\right]\mathrm{d}p$$

$$= \mu + \frac{\sigma}{1-\alpha}\int_\alpha^1 \Phi^{-1}(p)\mathrm{d}p$$

$$\Phi(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{s} e^{-\frac{1}{2}v^2}\mathrm{d}v$$

$$\int_\alpha^1 \Phi^{-1}(p)\mathrm{d}p = \frac{1}{\sqrt{2\pi}}\int_{\Phi^{-1}(\alpha)}^{\Phi^{-1}(1)} ue^{-\frac{1}{2}u^2}\mathrm{d}u$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\Phi^{-1}(\alpha)\right)^2\right)$$

$$p = \Phi(u)$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{s} e^{-\frac{1}{2}v^2}\mathrm{d}v$$

Utilizing these results, we have that:

$$CV@R_\alpha(X) = \mu + \frac{\sigma}{1-\alpha}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\Phi^{-1}(\alpha)\right)^2\right) = \mu + \frac{\sigma}{1-\alpha}\varphi\left(\Phi^{-1}(\alpha)\right)$$

where φ is the standard normal distribution. By Lemma 1, we have:

$$Pos\left[CV@R_\alpha\left(\sum_{j=1}^{n}\hat{a}_j x_j\right) \le \varepsilon\right] \ge \beta \Leftrightarrow$$

$$Pos\left\{\hat{\mu} + \frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\sigma\right\} \ge \beta \Rightarrow \sum_{j=1}^{n} x_j\left(a_j^m - R^{-1}(\beta)a_j^L\right) + \frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_i\sigma_{il}x_l} \le \varepsilon$$

The proof (a) is complete. Parts (b) and (c) can be proved similarly. □

For the portfolio selection framework, the possibility, necessity, and credibility measures based on conditional value-at-risk are obtained as follows:

4.1 CV@R-Possibility measure

$$\begin{aligned} &\min \sum_{j=1}^{n} x_j\left(r_j^m - R^{-1}(\beta)r_j^L\right) + \frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_i\sigma_{il}x_l} \\ &\text{s.t.} \\ &\sum_{j=1}^{n} x_j\left(r_j^m + \frac{1}{2}\left[r_j^L(T(0)-T(1)) + r_j^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right) \ge \eta, \\ &\sum_{j=1}^{n} x_j = 1, \quad x_j \ge 0,\ j = 1,\ldots,n. \end{aligned} \tag{9}$$

Exploiting Eq. (9) and the (b) part of Theorem 2 yields:

CV@R-Necessity measure

$$\begin{aligned} &\min \sum_{j=1}^{n} x_j\left(r_j^m + L^{-1}(1-\beta)r_j^R\right) + \frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_i\sigma_{il}x_l} \\ &s.t. \\ &\sum_{j=1}^{n} x_j\left(r_j + \frac{1}{2}\left[r_j^L(T(0)-T(1)) + r_j^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right) \ge \eta, \\ &\sum_{j=1}^{n} x_j = 1, \quad x_j \ge 0,\ j = 1,..,n. \end{aligned} \tag{10}$$

From Eq. (8) and the (c) part of Theorem 2, we obtain the following two deterministic credibility measures:

***V@R-Credibility measure* for $\beta \le 0.5$**

$$\begin{aligned} &\min_{\beta \le 0.5} \sum_{j=1}^{n} x_j\left(r_j^m - R^{-1}(2\beta)r_j^L\right) + \frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_i\sigma_{il}x_l} \\ &\text{s.t.} \\ &\sum_{j=1}^{n} x_j\left(r_j + \frac{1}{2}\left[r_j^L(T(0)-T(1)) + r_j^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right) \ge \eta, \\ &\sum_{j=1}^{n} x_j = 1, \\ &x_j \ge 0,\ j = 1,\ldots,n. \end{aligned}$$

***CV@R-Credibility measure* for $\beta > 0.5$**

$$\begin{aligned} &\min_{\beta > 0.5} \sum_{j=1}^{n} x_j\left(r_j^m + L^{-1}(2(1-\beta))r_j^R\right) + \frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_i\sigma_{il}x_l} \\ &\text{s.t.} \\ &\sum_{j=1}^{n} x_j\left(r_j + \frac{1}{2}\left[r_j^L(T(0)-T(1)) + r_j^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right) \ge \eta, \\ &\sum_{j=1}^{n} x_j = 1, \quad x_j \ge 0,\ j = 1,\ldots,n. \end{aligned} \tag{11}$$

5 DEA based random-fuzzy portfolio estimation models

5.1 Basics of data envelopment analysis

In this subsection, we describe Data Envelopment Analysis (DEA) first developed by Charnes, Cooper, and Rhodes (CCR) (Charnes et al. 1978). DEA has been widely used for evaluating the performance and measuring the relative efficiency of a group of firms or Decision-Making Units (DMUs) that uses multiple inputs and multiple outputs.

Let us assume that there are n DMUs to be evaluated where every $\mathrm{DMU}_j, j = 1, 2, \ldots, n$, produces s outputs, $y_{rj} (r = 1, \ldots, s)$, using m inputs, x_{ij} $(i = 1, 2, \ldots, m)$. The following problem is posed to calculate the input-efficiency of a given DMU_o:

$$\begin{aligned}
&\min\ \theta\\
&\text{s.t.}\\
&\sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{io},\ i = 1, \ldots, m,\\
&\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro},\ r = 1, \ldots, s,\\
&\lambda_j \ge 0, \quad j = 1, \ldots, n.
\end{aligned} \tag{12}$$

DEA estimates the production possibility set using efficient DMUs. A DMU is said to be technically efficient if the optimal value of θ is equal to one; otherwise, DMU_p is technically inefficient.

5.2 Value-at-risk DEA measure

Next, we implement the developed framework into DEA for the value-at-risk case. Here, the value-at-risk associated with a portfolio is as input and the portfolio's return as output. The V@R-Possibility DEA measure takes the form:

V@R-Possibility DEA measure

$$\begin{aligned}
&\min\ \theta\\
&\text{s.t.}\\
&\sum_{j=1}^{N} \lambda_j \left(\sum_{i=1}^{n} x_{ij} \left(a_i^m - R^{-1}(\beta) a_i^L\right) + \Phi^{-1}(\alpha) \sqrt{\sum_{i=1}^{n} \sum_{l=1}^{n} x_{ij} \sigma_{il} x_{lj}} \right.\\
&\quad \le \theta \left(\sum_{i=1}^{n} x_{io} \left(a_i^m - R^{-1}(\beta) a_i^L\right) + \Phi^{-1}(\alpha) \sqrt{\sum_{i=1}^{n} \sum_{l=1}^{n} x_{io} \sigma_{il} x_{lo}} \right),\\
&\sum_{j=1}^{N} \lambda_j \left(\sum_{i=1}^{n} x_{ij} \left(r_i + \frac{1}{2} \left[r_i^L (T(0) - T(1)) + r_i^R (\mathrm{P}(1) - \mathrm{P}(0)) \right] \right) \right)\\
&\quad \ge \left(\sum_{i=1}^{n} x_{io} \left(r_i + \frac{1}{2} \left[r_i^L (T(0) - T(1)) + r_i^R (\mathrm{P}(1) - \mathrm{P}(0)) \right] \right) \right),\\
&\lambda_j \ge 0,\ j = 1, .., n.
\end{aligned} \tag{13}$$

The V@R-Necessity DEA measure is formulated as follows:

V@R-Necessity DEA measure

$$\begin{aligned}
&\min\ \theta\\
&\text{s.t.}\\
&\sum_{j=1}^{N} \lambda_j \left(\sum_{i=1}^{n} x_{ij} \left(a_i^m + L^{-1}(1-\beta) a_i^R\right) + \Phi^{-1}(\alpha) \sqrt{\sum_{i=1}^{n} \sum_{l=1}^{n} x_{io} \sigma_{il} x_{lo}} \right)\\
&\quad \le \theta \left(\sum_{j=1}^{n} x_{io} \left(a_i^m + L^{-1}(1-\beta) a_i^R\right) + \Phi^{-1}(\alpha) \sqrt{\sum_{i=1}^{n} \sum_{l=1}^{n} x_{io} \sigma_{il} x_{lo}} \right),\\
&\sum_{j=1}^{N} \lambda_j \left(\sum_{i=1}^{n} x_{ij} \left(r_i + \frac{1}{2} \left[r_i^L (T(0) - T(1)) + r_i^R (\mathrm{P}(1) - \mathrm{P}(0)) \right] \right) \right)\\
&\quad \ge \left(\sum_{i=1}^{n} x_{io} \left(r_i + \frac{1}{2} \left[r_i^L (T(0) - T(1)) + r_i^R (\mathrm{P}(1) - \mathrm{P}(0)) \right] \right) \right),\\
&\lambda_j \ge 0,\ j = 1, \ldots, n.
\end{aligned} \tag{14}$$

The two forms of V@R-Credibility DEA measure are given as follows:

V@R-Credibility DEA measure for $\beta \le 0.5$

$$\begin{aligned}
&\min_{\beta \le 0.5}\ \theta\\
&\text{s.t.}\\
&\sum_{j=1}^{N} \lambda_j \left(\sum_{i=1}^{n} x_{ij} \left(a_i^m - R^{-1}(2\beta) a_i^L\right) + \Phi^{-1}(\alpha) \sqrt{\sum_{i=1}^{n} \sum_{l=1}^{n} x_{ij} \sigma_{il} x_{lj}} \right)\\
&\quad \le \theta \left(\sum_{i=1}^{n} x_{io} \left(a_i^m - R^{-1}(2\beta) a_i^L\right) + \Phi^{-1}(\alpha) \sqrt{\sum_{i=1}^{n} \sum_{l=1}^{n} x_{io} \sigma_{il} x_{lo}} \right),\\
&\sum_{j=1}^{N} \lambda_j \left(\sum_{i=1}^{n} x_{ij} \left(r_i + \frac{1}{2} \left[r_i^L (T(0) - T(1)) + r_i^R (\mathrm{P}(1) - \mathrm{P}(0)) \right] \right) \right)\\
&\quad \ge \left(\sum_{i=1}^{n} x_{io} \left(r_i + \frac{1}{2} \left[r_i^L (T(0) - T(1)) + r_i^R (\mathrm{P}(1) - \mathrm{P}(0)) \right] \right) \right),\\
&\lambda_j \ge 0,\ j = 1, .., n.
\end{aligned} \tag{15}$$

V@R-Credibility DEA measure for $\beta > 0.5$

$$\begin{aligned}
&\min_{\beta > 0.5}\ \theta\\
&\text{s.t.}\\
&\sum_{j=1}^{N} \lambda_j \left(\sum_{i=1}^{n} x_{ij} \left(r_i^m + L^{-1}(2(1-\beta)) r_i^R\right) + \Phi^{-1}(\alpha) \sqrt{\sum_{i=1}^{n} \sum_{l=1}^{n} x_{ij} \sigma_{il} x_{lj}} \right)\\
&\quad \le \theta \left(\sum_{i=1}^{n} x_{io} \left(r_i^m + L^{-1}(2(1-\beta)) r_i^R\right) + \Phi^{-1}(\alpha) \sqrt{\sum_{i=1}^{n} \sum_{l=1}^{n} x_{io} \sigma_{il} x_{lo}} \right),\\
&\sum_{j=1}^{N} \lambda_j \left(\sum_{i=1}^{n} x_{ij} \left(r_i + \frac{1}{2} \left[r_i^L (T(0) - T(1)) + r_i^R (\mathrm{P}(1) - \mathrm{P}(0)) \right] \right) \right)\\
&\quad \ge \left(\sum_{i=1}^{n} x_{io} \left(r_i + \frac{1}{2} \left[r_i^L (T(0) - T(1)) + r_i^R (\mathrm{P}(1) - \mathrm{P}(0)) \right] \right) \right),\\
&\lambda_j \ge 0,\ j = 1, \ldots, n.
\end{aligned} \tag{16}$$

In this subsection, we developed not only V@R-Possibility measure (13) and V@R-Necessity measure (14), but also two-types of V@R-Credibility measure: one is for $\beta \leq 0.5$,, and the other is for $\beta > 0.5$, as shown in (15) and (16), respectively.

5.3 Conditional value-at-risk DEA measure

Similar to the value-at-risk models studied in the previous subsection, we use the conditional value-at-risk associated with the portfolio as the input, and the portfolio's return as the output. Thus, the risk-oriented DEA based random-fuzzy portfolio estimation model is formulated as in the following Eqs. (17)–(20).

CV@R-Possibility DEA measure:

$$\begin{aligned}
&\min\ \theta\\
&\text{s.t.}\\
&\sum_{j=1}^{N}\lambda_j\left(\sum_{i=1}^{n}x_{ij}\left(r_i^m - R^{-1}(\beta)r_i^L\right)+\frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n}x_{ij}\sigma_{il}x_{lj}}\right)\\
&\quad\leq\theta\left(\sum_{i=1}^{n}x_{io}\left(r_i^m - R^{-1}(\beta)r_i^L\right)+\frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n}x_{io}\sigma_{il}x_{lo}}\right),\\
&\sum_{j=1}^{N}\lambda_j\left(\sum_{i=1}^{n}x_{ij}\left(r_i+\frac{1}{2}\left[r_i^L(T(0)-T(1))+r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)\right)\\
&\quad\geq\left(\sum_{i=1}^{n}x_{io}\left(r_i^m+\frac{1}{2}\left[r_i^L(T(0)-T(1))+r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)\right),\\
&\lambda_j\geq 0,\ \ j=1,\ldots,n.
\end{aligned}\tag{17}$$

CV@R-Necessity DEA measure:

$$\begin{aligned}
&\min\ \theta\\
&\text{s.t.}\\
&\sum_{j=1}^{N}\lambda_j\left(\sum_{i=1}^{n}x_{ij}\left(r_i^m + L^{-1}(1-\beta)r_i^R\right)+\frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n}x_{ij}\sigma_{il}x_{lj}}\right)\\
&\quad\leq\theta\left(\sum_{i=1}^{n}x_{io}\left(r_i^m + L^{-1}(1-\beta)r_i^R\right)+\frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n}x_{io}\sigma_{il}x_{lo}}\right),\\
&\sum_{j=1}^{N}\lambda_j\left(\sum_{i=1}^{n}x_{ij}\left(r_i+\frac{1}{2}\left[r_i^L(T(0)-T(1))+r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)\right)\\
&\quad\geq\left(\sum_{i=1}^{n}x_{io}\left(r_i^m+\frac{1}{2}\left[r_i^L(T(0)-T(1))+r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)\right),\\
&\lambda_j\geq 0,\ \ j=1,\ldots,n.
\end{aligned}\tag{18}$$

CV@R-Credibility DEA measure for $\beta \leq 0.5$:

$$\begin{aligned}
&\min_{\beta\leq 0.5}\ \theta\\
&\text{s.t.}\\
&\sum_{j=1}^{N}\lambda_j\left(\sum_{i=1}^{n}x_{ij}\left(r_i^m - R^{-1}(2\beta)r_i^L\right)+\frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n}x_{ij}\sigma_{il}x_{lj}}\right)\\
&\quad\leq\theta\left(\sum_{i=1}^{n}x_{io}\left(r_i^m - R^{-1}(2\beta)r_i^L\right)+\frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n}x_{io}\sigma_{il}x_{lo}}\right),\\
&\sum_{j=1}^{N}\lambda_j\left(\sum_{i=1}^{n}x_{ij}\left(r_i+\frac{1}{2}\left[r_i^L(T(0)-T(1))+r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)\right)\\
&\quad\geq\left(\sum_{i=1}^{n}x_{io}\left(r_i^m+\frac{1}{2}\left[r_i^L(T(0)-T(1))+r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)\right),\\
&\lambda_j\geq 0,\ \ j=1,\ldots,n.
\end{aligned}\tag{19}$$

CV@R-Credibility DEA measure for $\beta > 0.5$

$$\begin{aligned}
&\min_{\beta > 0.5}\ \theta\\
&\text{s.t.}\\
&\sum_{j=1}^{N}\lambda_j\left(\sum_{i=1}^{n}x_{ij}\left(r_i^m + L^{-1}(2(1-\beta))r_i^R\right)+\frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n}x_{ij}\sigma_{il}x_{lj}}\right)\\
&\quad\leq\theta\left(\sum_{j=1}^{N}\lambda_j\left(\sum_{i=1}^{n}x_{io}\left(r_o^m + L^{-1}(2(1-\beta))r_o^R\right)+\frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n}x_{io}\sigma_{il}x_{lo}}\right)\right),\\
&\sum_{j=1}^{N}\lambda_j\left(\sum_{i=1}^{n}x_{ij}\left(r_i+\frac{1}{2}\left[r_i^L(T(0)-T(1))+r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)\right)\\
&\quad\geq\left(\sum_{i=1}^{n}x_{io}\left(r_i^m+\frac{1}{2}\left[r_i^L(T(0)-T(1))+r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)\right),\\
&\lambda_j\geq 0,\ \ j=1,\ldots,n.
\end{aligned}\tag{20}$$

5.4 Algorithm for DEA models

In this subsection, we present an enumeration algorithm for computing an efficient portfolio for computational ease. One of the most important advantages of our algorithm is that efficiency scores can be obtained without solving LP problems. While using mathematical programming, it is necessary to deal with high degrees of programming expertise, coding and debugging time depending on the environment being used, utilizing enumeration algorithms more often than not requires programming skills exceeding the one required from the mathematical programming (Kerstens and Van De Woestyne 2018).

Therefore, we present an enumeration approach to compute the efficiency of portfolios. After solving portfolio optimization models, we obtain the risk-ratio and the return-ratio for each portfolio defined as $RE_j^o = \frac{\text{Return for portfolio under evaluation}}{\text{Return for portfolio } j}$ and $RI_j^o = \frac{\text{Risk for portfolio } j}{\text{Risk for portfolio under evaluation}}$.

5.5 Enumeration algorithm

Step 1: Create input–output (return-risk) data $\left(RE_j^o, RI_j^o\right)$,
where

$$V@R:\begin{cases}
\left.\begin{aligned}
RE_j^o &= \frac{\sum_{i=1}^{n} x_{io}\left(r_i+\frac{1}{2}\left[r_i^L(T(0)-T(1))+r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)}{\sum_{i=1}^{n} x_{ij}\left(r_i+\frac{1}{2}\left[r_i^L(T(0)-T(1))+r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)},\\
RI_j^o &= \frac{\sum_{i=1}^{n} x_{ij}\left(a_i^m-R^{-1}(\beta)a_i^L\right)+\Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{ij}\sigma_{il}x_{lj}}}{\sum_{i=1}^{n} x_{io}(a_i^m-R^{-1}(\beta)a_i^L)+\Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{io}\sigma_{il}x_{lo}}},
\end{aligned}\right\}\ \text{Possibility}\\
\left.\begin{aligned}
RE_j^o &= \frac{\sum_{i=1}^{n} x_{io}\left(r_i+\frac{1}{2}\left[r_i^L(T(0)-T(1))+r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)}{\sum_{i=1}^{n} x_{ij}\left(r_i+\frac{1}{2}\left[r_i^L(T(0)-T(1))+r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)},\\
RI_j^o &= \frac{\sum_{i=1}^{n} x_{ij}\left(a_i^m+L^{-1}(1-\beta)a_i^R\right)+\Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{ij}\sigma_{il}x_{lj}}}{\sum_{j=1}^{n} x_{io}(a_i^m+L^{-1}(1-\beta)a_i^R)+\Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{io}\sigma_{il}x_{lo}}},
\end{aligned}\right\}\ \text{Necessity}\\
\left.\begin{aligned}
RE_j^o &= \frac{\sum_{i=1}^{n} x_{io}\left(r_i+\frac{1}{2}\left[r_i^L(T(0)-T(1))+r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)}{\sum_{i=1}^{n} x_{ij}\left(r_i+\frac{1}{2}\left[r_i^L(T(0)-T(1))+r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)},\\
RI_j^o &= \frac{\sum_{i=1}^{n} x_{ij}\left(a_i^m-R^{-1}(2\beta)a_i^L\right)+\Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{ij}\sigma_{il}x_{lj}}}{\sum_{i=1}^{n} x_{io}(a_i^m-R^{-1}(2\beta)a_i^L)+\Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{io}\sigma_{il}x_{lo}}},
\end{aligned}\right\}\ \text{Credibility }(\beta\leq 0.5)\\
\left.\begin{aligned}
RE_j^o &= \frac{\sum_{i=1}^{n} x_{io}\left(r_i+\frac{1}{2}\left[r_i^L(T(0)-T(1))+r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)}{\sum_{i=1}^{n} x_{ij}\left(r_i+\frac{1}{2}\left[r_i^L(T(0)-T(1))+r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)},\\
RI_j^o &= \frac{\sum_{i=1}^{n} x_{ij}\left(r_i^m+L^{-1}(2(1-\beta))r_i^R\right)+\Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{ij}\sigma_{il}x_{lj}}}{\sum_{i=1}^{n} x_{io}(r_i^m+L^{-1}(2(1-\beta))r_i^R)+\Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{io}\sigma_{il}x_{lo}}},
\end{aligned}\right\}\ \text{Credibility }(\beta>0.5)
\end{cases}$$

$$
CV@R:\begin{cases}
\left.\begin{aligned}
RE_j^o &= \frac{\sum_{i=1}^{n} x_{io}\left(r_i+\frac{1}{2}\left[r_i^L(T(0)-T(1))+r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)}{\sum_{i=1}^{n} x_{ij}\left(r_i+\frac{1}{2}\left[r_i^L(T(0)-T(1))+r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)},\\
RI_j^o &= \frac{\sum_{i=1}^{n} x_{ij}\left(a_i^m-R^{-1}(\beta)a_i^L\right)+\frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{ij}\sigma_{il}x_{lj}}}{\sum_{i=1}^{n} x_{io}(a_i^m-R^{-1}(\beta)a_i^L)+\frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{io}\sigma_{il}x_{lo}}},
\end{aligned}\right\}\ \text{Possibility}\\
\left.\begin{aligned}
RE_j^o &= \frac{\sum_{i=1}^{n} x_{io}\left(r_i+\frac{1}{2}\left[r_i^L(T(0)-T(1))+r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)}{\sum_{i=1}^{n} x_{ij}\left(r_i+\frac{1}{2}\left[r_i^L(T(0)-T(1))+r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)},\\
RI_j^o &= \frac{\sum_{i=1}^{n} x_{ij}\left(a_i^m+L^{-1}(1-\beta)a_i^R\right)+\frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{ij}\sigma_{il}x_{lj}}}{\sum_{j=1}^{n} x_{io}(a_i^m+L^{-1}(1-\beta)a_i^R)+\frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{io}\sigma_{il}x_{lo}}},
\end{aligned}\right\}\ \text{Necessity}\\
\left.\begin{aligned}
RE_j^o &= \frac{\sum_{i=1}^{n} x_{io}\left(r_i+\frac{1}{2}\left[r_i^L(T(0)-T(1))+r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)}{\sum_{i=1}^{n} x_{ij}\left(r_i+\frac{1}{2}\left[r_i^L(T(0)-T(1))+r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)},\\
RI_j^o &= \frac{\sum_{i=1}^{n} x_{ij}\left(a_i^m-R^{-1}(2\beta)a_i^L\right)+\frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{ij}\sigma_{il}x_{lj}}}{\sum_{i=1}^{n} x_{io}(a_i^m-R^{-1}(2\beta)a_i^L)+\frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{io}\sigma_{il}x_{lo}}},
\end{aligned}\right\}\ \text{Credibility } (\beta\le 0.5)\\
\left.\begin{aligned}
RE_j^o &= \frac{\sum_{i=1}^{n} x_{io}\left(r_i+\frac{1}{2}\left[r_i^L(T(0)-T(1))+r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)}{\sum_{i=1}^{n} x_{ij}\left(r_i+\frac{1}{2}\left[r_i^L(T(0)-T(1))+r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)},\\
RI_j^o &= \frac{\sum_{i=1}^{n} x_{ij}\left(r_i^m+L^{-1}(2(1-\beta))r_i^R\right)+\frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{ij}\sigma_{il}x_{lj}}}{\sum_{i=1}^{n} x_{io}(r_i^m+L^{-1}(2(1-\beta))r_i^R)+\frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{io}\sigma_{il}x_{lo}}},
\end{aligned}\right\}\ \text{Credibility } (\beta > 0.5)
\end{cases}
$$

Table 1 Random-fuzzy returns

Security	Return
1	$\tilde{r}_1 \sim (\hat{\mu}_1,\ 0.01)$ with $\hat{\mu}_1 = (0.1, 0.24, 0.2)$
2	$\tilde{r}_2 \sim N(\hat{\mu}_2,\ 0.02)$ with $\hat{\mu}_2 = (0.2, 0.5, 0.3)$
3	$\tilde{r}_3 \sim N(\hat{\mu}_3,\ 0.01)$ with $\hat{\mu}_3 = (0.2, 0.33, 0.2)$
4	$\tilde{r}_4 \sim N(\hat{\mu}_4,\ 0.04)$ with $\hat{\mu}_4 = (0.3, 0.4, 0.4)$
5	$\tilde{r}_5 \sim N(\hat{\mu}_5,\ 0.04)$ with $\hat{\mu}_5 = (0.1, 0.5, 0.7)$
6	$\tilde{r}_6 \sim N(\hat{\mu}_6,\ 0.05)$ with $\hat{\mu}_6 = (0.4, 0.6, 0.3)$
7	$\tilde{r}_7 \sim N(\hat{\mu}_7,\ 0.05)$ with $\hat{\mu}_7 = (0.2, 0.46, 0.6)$
8	$\tilde{r}_8 \sim N(\hat{\mu}_8,\ 0.05)$ with $\hat{\mu}_1 = (0.3, 0.5, 0.4)$
9	$\tilde{r}_9 \sim N(\hat{\mu}_9,\ 0.06)$ with $\hat{\mu}_1 = (0.2, 0.46, 0.34)$

Step 2: Based on the possibility, necessity, and credibility measures with V@R and CV@R. Compute $P_j^{\delta} = \left\{RE_j^o \times RI_j^o\right\}\ \forall j\ .\ \delta = V@R,\ CV@R$

Step 3: Compute the algorithmic portfolio efficiency (PE) measure as follows:

$$PE_o^{\varphi} = \min_{j=1,\ldots,n}\left\{P_j^{\delta}\right\}, \delta = V@R,\ CV@R,\ \varphi = Pos, Nec, Cr$$

Theorem 3 *The result from the algorithm is equal to results from all previous DEA models.*

Table 2 Efficiency scores and ranks for the V@R-Possibility DEA measure

DMU	Risk	Return	DEA score V@R-Possibility	DEA rank
1	0.8654	0.6454	0.8202	24
2	0.8342	0.6381	0.8413	23
3	0.8044	0.6302	0.8616	22
4	0.7870	0.6251	0.8735	21
5	0.7496	0.6124	0.8985	20
6	0.7097	0.5977	0.9262	19
7	0.6853	0.5878	0.9433	18
8	0.6701	0.5810	0.9536	17
9	0.6601	0.576	0.9597	16
10	0.6463	0.5685	0.9674	14
11	0.6237	0.5553	0.9792	11
12	0.6089	0.5459	0.9860	9
13	0.5985	0.5386	0.9897	7
14	0.5777	0.5227	0.9951	5
15	0.5551	0.5043	0.9991	2
16	0.5389	0.4900	1.0000	1
17	0.5266	0.4781	0.9985	3
18	0.5167	0.4679	0.9959	4
19	0.5085	0.4586	0.9919	6
20	0.5015	0.4501	0.9871	8
21	0.4954	0.4421	0.9815	10
22	0.4901	0.4344	0.9748	12
23	0.4856	0.4273	0.9678	13
24	0.4817	0.4208	0.9608	15

$\alpha = \beta = 0.95$

Proof of Theorem 3 We consider Eq. (12), whose dual is written as:

$$\begin{aligned}
&\max\ u\left(\sum_{i=1}^{n} x_{io}\left(r_i + \frac{1}{2}\left[r_i^L(T(0)-T(1)) + r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)\right)\\
&\text{s.t.}\\
&v\left(\sum_{i=1}^{n} x_{io}\left(a_i^m - R^{-1}(\beta)a_i^L\right) + \Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{io}\sigma_{il}x_{lo}}\right) = 1,\\
&u\left(\sum_{i=1}^{n} x_{ij}\left(r_i + \frac{1}{2}\left[r_i^L(T(0)-T(1)) + r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)\right)\\
&\quad - v\left(\sum_{i=1}^{n} x_{ij}\left(a_i^m - R^{-1}(\beta)a_i^L\right) + \Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{ij}\sigma_{il}x_{lj}}\right)\\
&\quad \leq 0\ (j = 1,\ldots,n),\\
&\quad u \geq 0,\quad v \geq 0.
\end{aligned} \tag{21}$$

From the first constraint of (20), we have:

$$v = \frac{1}{\sum_{i=1}^{n} x_{io}\left(a_i^m - R^{-1}(\beta)a_i^L\right) + \Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{io}\sigma_{il}x_{lo}}}.$$

By substituting v in the second constraint in (20), we obtain:

$$\begin{aligned}
&u\left(\sum_{i=1}^{n} x_{ij}\left(r_i + \frac{1}{2}\left[r_i^L(T(0)-T(1)) + r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)\right)\\
&\quad \leq \frac{\sum_{i=1}^{n} x_{ij}\left(a_i^m - R^{-1}(\beta)a_i^L\right) + \Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{ij}\sigma_{il}x_{lj}}}{\sum_{i=1}^{n} x_{io}\left(a_i^m - R^{-1}(\beta)a_i^L\right) + \Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{io}\sigma_{il}x_{lo}}},\\
&\quad (j = 1,\ldots,n)
\end{aligned}$$

which implies that (20) can be converted to Eq. (12). This indicates that (12) is written as:

$$\begin{aligned}
&\max\quad u\left(\sum_{i=1}^{n} x_{io}\left(r_i + \frac{1}{2}\left[r_i^L(T(0)-T(1)) + r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)\right)\\
&\text{s.t.}\\
&u\left(\sum_{i=1}^{n} x_{io}\left(r_i + \frac{1}{2}\begin{bmatrix} r_i^L(T(0)-T(1))\\ +r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\end{bmatrix}\right)\right)\\
&\quad \leq \left(\begin{array}{l}\dfrac{\sum_{i=1}^{n} x_{ij}\left(a_i^m - R^{-1}(\beta)a_i^L\right) + \Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{ij}\sigma_{il}x_{lj}}}{\sum_{i=1}^{n} x_{io}\left(a_i^m - R^{-1}(\beta)a_i^L\right) + \Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{io}\sigma_{il}x_{lo}}}\\ \times\dfrac{\sum_{i=1}^{n} x_{io}\left(r_i + \frac{1}{2}\left[r_i^L(T(0)-T(1)) + r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)}{\sum_{i=1}^{n} x_{ij}\left(r_i + \frac{1}{2}\left[r_i^L(T(0)-T(1)) + r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)}\end{array}\right),\\
&(j = 1,\ldots,n)\\
&u \geq 0.
\end{aligned} \tag{22}$$

Since u is a variable and the objective function is maximized on the variable, it is clear that the optimal value of the objective function is

$$\min_{j=1,2,\ldots,n}\left\{\begin{array}{l}\dfrac{\left(\sum_{i=1}^{n} x_{ij}\left(a_i^m - R^{-1}(\beta)a_i^L\right) + \Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{ij}\sigma_{il}x_{lj}}\right)}{\sum_{i=1}^{n} x_{io}\left(a_i^m - R^{-1}(\beta)a_i^L\right) + \Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{io}\sigma_{il}x_{lo}}}\\ \times\dfrac{\left(\sum_{i=1}^{n} x_{io}\left(r_i + \frac{1}{2}\left[r_i^L(T(0)-T(1)) + r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)\right)}{\left(\sum_{i=1}^{n} x_{ij}\left(r_i + \frac{1}{2}\left[r_i^L(T(0)-T(1)) + r_i^R(\mathrm{P}(1)-\mathrm{P}(0))\right]\right)\right)}\end{array}\right\}$$

We conclude that the above algorithm can provide the efficiency scores of the DEA model (12), where the input is a risk, and the output is a return, without solving mathematical optimization problems. The proof of this theorem shows the validity of the Algorithm. Using a similar argument, we can show that the other DEA models can be obtained via the enumeration algorithm. □

We should note that Eq. (21) and its dual form, Eq. (22) are developed based on the conditional value-at-risk (CV@R) are coherent risk measures that satisfy monotonicity, homogeneity, sub-additivity, and translation invariance.

Table 3 Optimal values for the decision variables for the V@R-Possibility measure

Portfolio	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
1	0.0000	0.0000	0.0000	0.0000	0.9490	0.0000	0.051	0.0000	0.0000
2	0.0000	0.0000	0.0000	0.0000	0.8675	0.0038	0.1287	0.0000	0.0000
3	0.0000	0.0000	0.0000	0.0000	0.7708	0.0580	0.1712	0.0000	0.0000
4	0.0000	0.0000	0.0000	0.0000	0.7077	0.0924	0.1999	0.0000	0.0000
5	0.0000	0.0728	0.0000	0.0000	0.6008	0.1051	0.2001	0.0212	0.0000
6	0.0000	0.1841	0.0000	0.0000	0.4904	0.1002	0.1829	0.0424	0.0000
7	0.0000	0.2448	0.0000	0.0000	0.4201	0.0946	0.1708	0.0540	0.0158
8	0.0000	0.2839	0.0000	0.0000	0.3731	0.0898	0.1629	0.0614	0.0290
9	0.0000	0.3122	0.0000	0.0000	0.3392	0.0858	0.1573	0.0668	0.0387
10	0.0000	0.3254	0.0000	0.0255	0.3052	0.0802	0.1493	0.0692	0.0451
11	0.0000	0.3158	0.0000	0.098	0.2653	0.0711	0.1359	0.0669	0.0470
12	0.0000	0.3083	0.0000	0.1500	0.2375	0.0639	0.1267	0.0652	0.0484
13	0.0000	0.3019	0.0000	0.1905	0.2165	0.0579	0.1199	0.0638	0.0495
14	0.0229	0.2838	0.0194	0.2140	0.1917	0.0503	0.1098	0.0599	0.0483
15	0.0572	0.2612	0.0445	0.2250	0.1689	0.0424	0.0997	0.0552	0.0459
16	0.0842	0.2430	0.0638	0.2337	0.1520	0.0357	0.0922	0.0514	0.0440
17	0.1069	0.2274	0.0796	0.2410	0.1386	0.0296	0.0864	0.0482	0.0424
18	0.1268	0.2133	0.0932	0.2474	0.1275	0.0239	0.0815	0.0454	0.0411
19	0.1449	0.2003	0.1054	0.2532	0.1178	0.0184	0.0774	0.0427	0.0398
20	0.1618	0.1880	0.1165	0.2587	0.1093	0.0131	0.0737	0.0403	0.0387
21	0.1778	0.1762	0.1268	0.2638	0.1016	0.0079	0.0704	0.0379	0.0376
22	0.1934	0.1646	0.1367	0.2688	0.0944	0.0026	0.0674	0.0356	0.0365
23	0.2080	0.1527	0.1458	0.273	0.0875	0.0000	0.0644	0.0333	0.0354
24	0.2218	0.1404	0.1543	0.2764	0.0806	0.0000	0.0614	0.0309	0.0342

$\alpha = \beta = 0.95$

5.6 Stability regions for the proposed models

In this section, we study the stability conditions of the models proposed in this study. Consider the following multiplier form of the input-oriented constant returns to scale Model (12).

$$\begin{aligned} & \max \;\; uY_o \\ & \text{s.t.} \\ & vX_o = 1, \\ & uY_j - vX_j \le 0 \; (j = 1, \ldots, n), \\ & u \ge 0, \quad v \ge 0. \end{aligned} \tag{23}$$

which can be written as:

$$\begin{aligned} & Z^* = \max \quad uY_o \\ & \text{s.t:} \\ & uY_j \le \frac{X_j}{X_o}, \quad (j = 1, \ldots, n), \\ & u \ge 0. \end{aligned} \tag{24}$$

Observing that:

$$uY_j \le \frac{X_j}{X_o}, \quad (j = 1, \ldots, n), \Rightarrow u \le \frac{X_j}{X_o} \times \frac{1}{Y_j}, \quad (j = 1, \ldots, n),$$

$$\Rightarrow uY_o \le \frac{X_j}{X_o} \times \frac{Y_o}{Y_j}, \quad (j = 1, \ldots, n),$$

and the objective function of (24) is $Z^* = \max \quad uY_o$, we have:

$$Z^* = \min_{j=1,2,\ldots,n} \left\{ \frac{X_j}{Y_j} \times \frac{Y_o}{X_o} \right\}. \tag{25}$$

Table 4 Efficiency scores and ranks for the V@R-Necessity DEA measure

DMU	Risk	Return	DEA score: V@R-Necessity	DEA rank
1	1.1178	0.5788	0.9330	17
2	1.0728	0.5683	0.9545	15
3	1.0412	0.5600	0.9691	13
4	1.0204	0.5538	0.9779	11
5	1.0056	0.5488	0.9834	9
6	0.9943	0.5447	0.9871	7
7	0.9853	0.541	0.9894	6
8	0.964	0.5313	0.9931	5
9	0.9175	0.5082	0.9981	2
10	0.8658	0.4805	1.0000	1
11	0.8341	0.4619	0.9978	3
12	0.8103	0.4467	0.9933	4
13	0.7909	0.4332	0.9869	8
14	0.7742	0.4207	0.9791	10
15	0.7595	0.4086	0.9694	12
16	0.746	0.3966	0.9579	14
17	0.7339	0.3850	0.9453	16
18	0.7229	0.3736	0.9312	18
19	0.7124	0.3617	0.9148	19
20	0.7030	0.3503	0.8979	20
21	0.6954	0.3403	0.8818	21
22	0.6881	0.3297	0.8634	22
23	0.6808	0.3183	0.8424	23
24	0.6738	0.3063	0.8191	24

(1) $\alpha = \beta = 0.95$. (2) The model expressions differ between the $\beta \leq 0.5$ case and the $\beta > 0.5$ case

Table 5 Efficiency scores and ranks for the V@R-Credibility DEA measure

DMU	Risk	Return	DEA score: V@R-Credibility	DEA rank
1	1.1087	0.5816	0.9295	18
2	1.0612	0.5706	0.9527	15
3	1.029	0.5620	0.9677	13
4	1.0079	0.5557	0.9769	11
5	0.993	0.5508	0.9828	9
6	0.9818	0.5467	0.9866	8
7	0.9731	0.5431	0.9889	6
8	0.9576	0.5360	0.9918	5
9	0.915	0.5149	0.9971	3
10	0.8620	0.4865	1.0000	1
11	0.8247	0.4646	0.9982	2
12	0.8002	0.4490	0.9942	4
13	0.7810	0.4356	0.9882	7
14	0.7648	0.4235	0.9811	10
15	0.7506	0.4118	0.9721	12
16	0.7376	0.4003	0.9616	14
17	0.7251	0.3883	0.9488	16
18	0.7141	0.3769	0.9352	17
19	0.7039	0.3654	0.9198	19
20	0.6943	0.3536	0.9024	20
21	0.6871	0.3441	0.8873	21
22	0.6801	0.3340	0.8702	22
23	0.6733	0.3233	0.8508	23
24	0.6665	0.3117	0.8286	24

(1) $\alpha = \beta = 0.95$. (2) The model expressions differ between the $\beta \leq 0.5$ case and the $\beta > 0.5$ case

Note that the optimal objective values of (24) and (25) are the same. Seiford and Zhu (1998) stated the following definitions on two kinds of stability regions.

Definition 10 (**Input stability region**): A region of allowable input increases is said to be an input stability region if and only if DMU_o stays efficient after the occurrence of such increases.

Definition 11 (**Output stability region**): A region of allowable output decreases is said to be an output stability region if and only if DMU_o stays efficient after the occurrence of such decreases.

If $Z^* = 1$, then DMU_o is efficient according to Models (12), (23), or (24). Therefore, there is at least one index $t \in \{1, 2, \ldots, n\}$ such that $\frac{X_t}{X_o} = \frac{Y_t}{Y_o}$ and $\frac{X_j}{X_o} \geq \frac{Y_j}{Y_o}, \forall j \neq t$.

Now we investigate the input stability by increasing the inputs of efficient DMU_o proportionately, i.e., $\hat{X_o} = \beta X_o, \hat{Y_o} = \lambda Y_o, \beta \geq 1, 0 < \lambda \leq 1$. We can easily check whether or not the perturbed input vector falls in the stability region (Definition 10). If there is at least one index $t \in \{1, 2, \ldots, n\}$ such that $\frac{X_t}{\beta X_o} = \frac{Y_t}{Y_o}$ and $\frac{X_j}{X_o} \geq \frac{Y_j}{Y_o}, \forall j \neq t$, then, the input perturbation of DMU_o is stable. Similarly, according to Definition 11, the output perturbation of DMU_o is given as $\hat{X_o} = \lambda X_o, \ \hat{Y_o} = \alpha Y_o, \ 1 < \alpha \leq 1$, which leads to an output-stable region, if there is at least one index $t \in \{1, 2, \ldots, n\}$ such that $\frac{X_t}{X_o} = \frac{Y_t}{\lambda Y_o}$ and $\frac{X_j}{X_o} \geq \frac{Y_j}{Y_o}, \forall j \neq t$.

Based on these results, we can investigate the stability issue for the V@R and CV@R measures, by substituting

Table 6 Efficiency scores and ranks for the CV@R-Possibility DEA measure

DMU	Risk	Return	DEA score CV@R-Possibility	DEA rank
1	0.8734	0.6423	0.8188	24
2	0.8362	0.6335	0.8435	23
3	0.8119	0.6271	0.8600	22
4	0.796	0.6223	0.8704	21
5	0.7373	0.6028	0.9103	20
6	0.7066	0.5915	0.9320	19
7	0.6867	0.5834	0.9459	18
8	0.6741	0.5778	0.9544	17
9	0.6655	0.5736	0.9597	16
10	0.6441	0.562	0.9715	14
11	0.6253	0.5509	0.9809	11
12	0.6124	0.5427	0.9867	9
13	0.6031	0.5363	0.9901	7
14	0.5777	0.5171	0.9966	5
15	0.5576	0.5006	0.9996	2
16	0.5429	0.4876	1.0000	1
17	0.5315	0.4767	0.9986	3
18	0.5222	0.4671	0.9959	4
19	0.5146	0.4584	0.9918	6
20	0.508	0.4504	0.9872	8
21	0.5024	0.4429	0.9815	10
22	0.4974	0.4358	0.9755	12
23	0.493	0.4288	0.9684	13
24	0.4893	0.4226	0.9616	15

$\alpha = \beta = 0.95$

$X_j = \sum_{i=1}^{n} x_{ij}\left(a_i^m - R^{-1}(\beta)a_i^L\right) + \Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n}\sum_{l=1}^{n} x_{ij}\sigma_{il}x_{lj}}$ and $Y_j = \sum_{i=1}^{n} x_{ij}\left(r_i + \frac{1}{2}\left[r_i^L(T(0) - T(1)) + r_i^R(\mathrm{P}(1) - \mathrm{P}(0))\right]\right)$ in Models (23), (24), and (25).

6 Numerical study

This section presents a numerical example to illustrate our proposed DEA framework, which deals with downside risks. In this example, we consider a portfolio selection problem with nine portfolios whose returns are random-fuzzy variables, as reported in Table 1.

We solved the portfolio selection problems for possibility, necessity and credibility measures for each of the value-at-risk and conditional value-at-risk criteria. We implemented the proposed method for $\alpha = 0.95$ and $\beta = 0.95$. We then simulate by using *Matlab* and *GAMS* to construct 24 sample portfolios. We calculate the efficiency of 24 sample portfolios using our proposed method.

Table 3 reports the computational result (risk, return, and DEA score) obtained by solving the problems (5) for the value-at-risk criterion. We confirmed this result by applying the algorithm developed in Sect. 5. DMU 16 (portfolio 16) with the highest efficiency score, 1.0000, is ranked first, considering the computational results presented in column 4 of Table 2. DMU 15, with a score of 0.9991, is ranked second. The third and fourth best performers are DMU 17 with a score of 0.9985 and DMU 18 with a score of 0.9959. DMU 14 with a score of 0.9951 and DMU 19 with a score of 0.9919 ranked eighth and ninth. The worst portfolio is DMU 1, with a score of 0.8202. Table 3 reports optimal input values from the V@R-Possibility DEA measure. The optimal investment strategy for DMU 16 is given as follows:

$$x_1 = 0.0842\,,\ x_2 = 0.2430\,,\ x_3 = 0.0638\,, x_4 = 0.2337\,,$$
$$x_5 = 0.1520\,,$$
$$x_6 = 0.0357\,,\ x_7 = 0.0922\,, x_8 = 0.0514\,,\ x_9 = 0.0440$$

Tables 4 and 5 report the computational results (risk, return, and DEA score) obtained by solving the V@R-Necessity DEA problem (6) and the V@R-Credibility DEA problem (7) for the value-at-risk criterion. DMU 10, with the highest efficiency score of 1.0000, is ranked first. The risk and return are (0.8658, 0.4805), and (0.8620, 0.4865) for V@R-Necessity and V@R-Credibility DEA measures, respectively. The worst portfolio is DMU 24, with scores of 0.8191 and 0.8286 for V@R-Necessity and V@R-Credibility, respectively.

The risk measures for the conditional value-at-risk Criterion are reported in Tables 6, 7, 8 and 9, which correspond to the CV@R-Possibility (16), CV@R-Necessity (17) and CV@R-Credibility DEA measures (19), respectively. Similar to the V@R-Possibility case, DMU 16 (portfolio 16), with the highest efficiency score, 1.0000, is ranked first, according to the CV@R-Possibility measure. The corresponding value of risk is 0.5429. Table 7 reports the optimal portfolio for the CV@R-Possibility measure (9) as follows:

$$x_1 = 0.0869\,,\ x_2 = 0.2432\,,\ x_3 = 0.0679\,, x_4 = 0.2345\,,$$
$$x_5 = 0.1457\,,$$
$$x_6 = 0.0375\,,\ x_7 = 0.0892\,, x_8 = 0.0512\,,\ x_9 = 0.0438$$

Table 7 Optimal values for the decision variables for the CV@R-Possibility measure

Portfolio	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
1	0.0000	0.0000	0.0000	0.0000	0.9139	0.0000	0.0861	0.0000	0.0000
2	0.0000	0.0000	0.0000	0.0000	0.8094	0.0440	0.1466	0.0000	0.0000
3	0.0000	0.0000	0.0000	0.0000	0.7310	0.0867	0.1822	0.0000	0.0000
4	0.0000	0.0000	0.0000	0.0000	0.6749	0.1148	0.2065	0.0039	0.0000
5	0.0000	0.1439	0.0000	0.0000	0.5273	0.1092	0.1852	0.0344	0.0000
6	0.0000	0.2235	0.0000	0.0000	0.4444	0.1032	0.1724	0.0496	0.0068
7	0.0000	0.2701	0.0000	0.0000	0.3884	0.0974	0.163	0.0585	0.0226
8	0.0000	0.3024	0.0000	0.0000	0.3497	0.0927	0.1567	0.0647	0.0338
9	0.0000	0.3267	0.0000	0.0000	0.3207	0.0888	0.1522	0.0694	0.0422
10	0.0000	0.3255	0.0000	0.0575	0.2814	0.0804	0.1404	0.0688	0.0461
11	0.0000	0.3167	0.0000	0.1186	0.2487	0.072	0.1295	0.0668	0.0477
12	0.0000	0.3097	0.0000	0.1643	0.2249	0.0653	0.1217	0.0652	0.0489
13	0.0000	0.3037	0.0000	0.2007	0.2064	0.0596	0.1158	0.0639	0.0499
14	0.0314	0.2806	0.0282	0.2167	0.1807	0.0512	0.1046	0.0590	0.0476
15	0.0621	0.2601	0.0504	0.2266	0.1608	0.0439	0.0959	0.0547	0.0455
16	0.0869	0.2432	0.0679	0.2345	0.1457	0.0375	0.0892	0.0512	0.0438
17	0.1079	0.2286	0.0825	0.2413	0.1335	0.0317	0.0839	0.0483	0.0423
18	0.1266	0.2154	0.0951	0.2473	0.1232	0.0263	0.0794	0.0456	0.0411
19	0.1435	0.2032	0.1065	0.2528	0.1142	0.0212	0.0756	0.0432	0.0399
20	0.1593	0.1917	0.1169	0.2579	0.1062	0.0162	0.0721	0.0408	0.0388
21	0.1743	0.1807	0.1266	0.2627	0.099	0.0113	0.069	0.0387	0.0378
22	0.1887	0.1699	0.1358	0.2673	0.0922	0.0065	0.0662	0.0365	0.0368
23	0.2028	0.1593	0.1447	0.2719	0.0859	0.0016	0.0635	0.0344	0.0358
24	0.2160	0.1483	0.1528	0.2754	0.0797	0.0000	0.0608	0.0323	0.0348

$\alpha = \beta = 0.95$

These optimal portfolio values for CV@R-Possibility are a little different from those for V@R-Possibility given in Table 3. That is, the suggested portfolio proportions on portfolios 5, 7, 8 and 9 for CV@R-Possibility are lower than those for V@R-Possibility. As a result, if the fund managers think that the conditional risk-at-value criterion is more relevant than the risk-at-value criterion and the possibility method is relevant, then they should invest less on these portfolios.

We further test the performance of portfolios with necessity and credibility methods under the conditional value-at-risk criterion. The results implied by the necessity and credibility methods under the conditional value-at-risk criterion are somewhat similar to those with the value-at-risk criterion. For example, DMU 10 is efficient, and DMU 9, with a score of 0.9989, is ranked the second. But the results from the two criteria are not always the same. For example, the worst-performing portfolio is DMU 24, with (risk, return) = (0.6913,0.3176) for the CV@R-Necessity measure and (risk, return) = (0.6834,0.3224) for the CV@R-Credibility DEA measures.

Figure 1 shows how the simulated risk and return estimates differ between the possibilistic V@R case and the possibilistic CV@R case. For the V@R case, the risk estimates are calculated by the objective functions of Eqs. (5), (6) and (7), and the return estimates are obtained by the first constraints of the corresponding equations. Similarly, for the CV@R case, we use objective functions and the first constraints of Eqs. (9), (10) and (11). Figure 1 can identify the V@R-based possibility frontier and the CV@R-based possibility frontier. Similarly, Fig. 2 plots the values showing the difference between the V@R and CV@R cases for the necessity method. Figure 3 is those for the credibilistic method. For each figure, a V@R-based frontier is located to the left of the corresponding CV@R-based frontier. A comparison of the three figures indicates: (i) that the V@R (CV@R)-based possibilistic frontiers are located to the left of the V@R (CV@R)-based necessity frontiers and (ii) that the V@R (CV@R)-based credibilistic frontiers are located between the corresponding possibilistic and necessity frontiers as is expected.

Table 8 Efficiency scores and ranks for the CV@R-Necessity DEA measure

DMU	Risk	Return	DEA score: CV@R-Necessity	DEA rank
1	1.1192	0.5767	0.9373	17
2	1.0765	0.5668	0.9577	15
3	1.0473	0.559	0.9709	12
4	1.0281	0.5533	0.9789	11
5	1.0143	0.5487	0.9840	9
6	1.0038	0.5448	0.9872	8
7	0.9954	0.5414	0.9893	6
8	0.9682	0.5291	0.9940	4
9	0.9183	0.5043	0.9989	2
10	0.8711	0.4789	1.0000	1
11	0.8394	0.4604	0.9977	3
12	0.8168	0.4460	0.9932	5
13	0.7987	0.4335	0.9873	7
14	0.7835	0.4220	0.9797	10
15	0.7700	0.4109	0.9707	13
16	0.7577	0.4000	0.9603	14
17	0.7460	0.3888	0.9480	16
18	0.7356	0.3779	0.9345	18
19	0.7261	0.3673	0.9201	19
20	0.717	0.3562	0.9036	20
21	0.7098	0.3466	0.8882	21
22	0.7034	0.3375	0.8728	22
23	0.6973	0.3278	0.8551	23
24	0.6913	0.3176	0.8357	24

$\alpha = \beta = 0.95$

Table 9 Efficiency scores and ranks in CV@R-Credibility DEA measure

DMU	Risk	Return	DEA score CV@R-Credibility	DEA rank
1	1.1092	0.5793	0.9338	24
2	1.0649	0.569	0.9554	23
3	1.0346	0.561	0.9695	22
4	1.0151	0.5552	0.9779	21
5	1.0013	0.5506	0.9832	20
6	0.9909	0.5467	0.9865	19
7	0.9827	0.5434	0.9887	18
8	0.9611	0.5335	0.9925	17
9	0.9153	0.5108	0.9978	16
10	0.8663	0.4845	1.0000	13
11	0.8296	0.4629	0.9977	11
12	0.8061	0.448	0.9937	9
13	0.7884	0.4357	0.9881	7
14	0.7735	0.4245	0.9813	4
15	0.7605	0.4139	0.9731	2
16	0.7488	0.4034	0.9633	1
17	0.7377	0.3928	0.9521	3
18	0.7272	0.3819	0.9390	5
19	0.7178	0.3712	0.9247	6
20	0.7085	0.3599	0.9083	8
21	0.701	0.3499	0.8925	10
22	0.6949	0.3412	0.8779	12
23	0.689	0.332	0.8616	14
24	0.6834	0.3224	0.8435	15

(1) $\alpha = \beta = 0.95$. (2) The model expressions differ between the $\beta \leq 0.5$ case and the $\beta > 0.5$ case

7 Conclusion

The complex nature of portfolio selection has resulted in the development of many approaches to support decision-making. We consider portfolio selection problems where returns from investment securities are random variables with fuzzy information and propose a DEA approach for portfolio selection with downside risk criteria associated with value-at-risk (V@R) and conditional value-at-risk (CV@R). In this paper, we have considered a portfolio selection problem where the returns are random variables with fuzzy information and developed two kinds of risk criteria based on value-at-risk (V@V) and conditional value-at-risk (CV@R). Both V@R and CV@R criteria are used to define possibility, necessity, and credibility measures, which are formulated as stochastic nonlinear programming programs with random-fuzzy variables. We then transformed these programs into deterministic non-linear convex optimization problems.

Moreover, we presented eight kinds of DEA programs to implement an efficiency assessment for the portfolio selection problem. To improve the practicality, we presented algorithmic procedures for the DEA programs. These algorithms do not require solving any optimization problems. Finally, we presented a numerical example dealing with portfolio selection. We should note that CV@R is more widely used than V@R because: (i) CV@R can quantify fat tails beyond that of V@R; and (ii) the CV@R is a coherent risk measure, according to the modern portfolio theory.

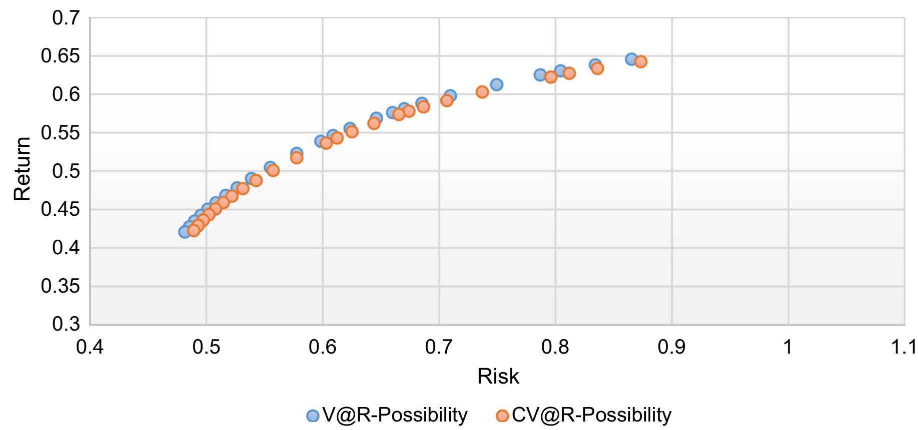

Fig. 1 Estimates of risk and return based upon V@R-Possibility and CV@R-Possibility criteria

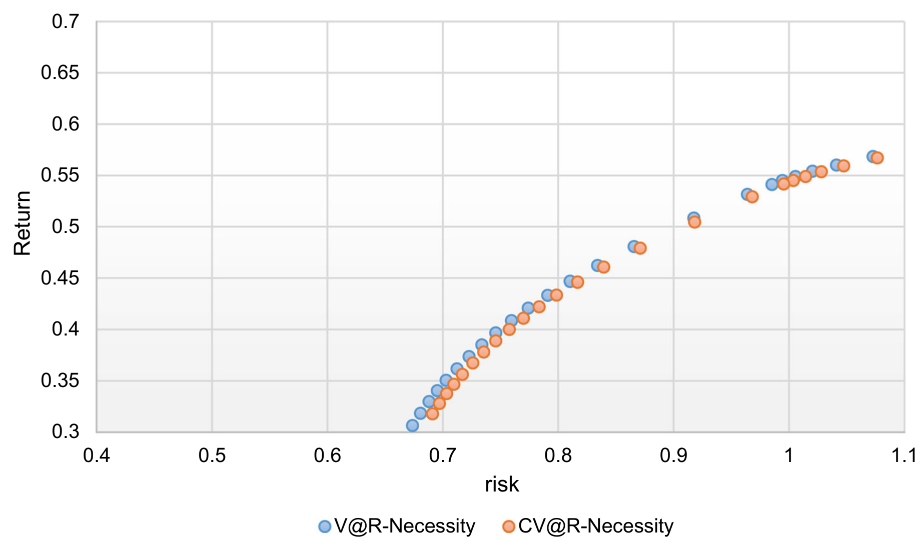

Fig. 2 Estimates of risk and return based on V@R-Necessity and CV@R-Necessity criteria

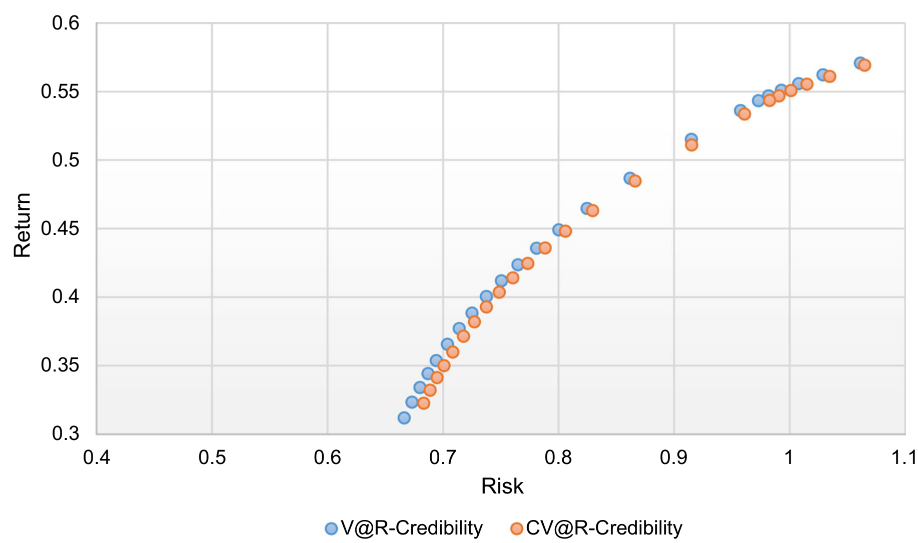

Fig. 3 Estimates of risk and return based on V@R-Credibility and CV@R-Credibility criteria

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Compliance with ethical standards

Conflict of interest The authors declare no conflict of interest

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