



A hybrid DEA-MOLP model for public school assessment and closure decision in the City of Philadelphia



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ABSTRACT

Data envelopment analysis (DEA) is generally used to evaluate past performance and multi objective linear programming (MOLP) is often used to plan for future performance goals. In this study, we establish an equivalence relationship between MOLP problems and combined-oriented DEA models using a direction distance function designed to account for desirable and undesirable inputs and outputs together with uncontrollable variables. This equivalence model can be effectively used to support interactive processes and performance measures designed to establish future performance goals while taking into account the preferences of decision makers (DMs). In particular, it allows DMs to consider different efficiency improvement strategies when subject to budgetary restrictions. The applicability of the proposed method and the efficacy of the procedures and algorithms are demonstrated using a case study where the performance of high schools in the City of Philadelphia is evaluated.

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1. Introduction

Data Envelopment Analysis (DEA), initially introduced by Charnes et al. [86], is a well-known non-parametric methodology for computing the relative efficiency of a set of homogeneous units, named Decision Making Units (DMU). The non-parametric property implies that this methodology does not rely on assumptions requiring the data to follow from any specific production function. DEA uses the data observed and some preliminary assumptions to determine a production possibility set which contains those operating points that are deemed feasible. Then, DEA formulates and solves a linear programming (LP) problem that produces an

efficiency score and a target operating point for each DMU. The target operating point lies on the efficient frontier and is computed in such a way that it generally uses the same or less inputs to produce identical or more output. The efficiency score is a measure of the relative improvements in inputs and outputs that can be defined between the DMU and its assigned target.

That is, DEA projects each DMU in turn on the efficient frontier by solving a LP problem guaranteeing that the projection (target unit) dominates the observed DMU. A DMU is efficient if it is not possible to find another feasible point that dominates it. However, if the LP model finds a feasible point that consumes less input and/or produces more output, then the DMU is inefficient. The projected feasible point, defined as a target unit for the inefficient unit, informs the Decision Maker (DM) of the amount (%) by which an inefficient DMU should decrease its inputs and/or increase its outputs to become efficient.

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1.1. Conventional DEA models

Conventional DEA models do not generally consider the preference structure or value judgments of the decision makers (DMs). [3]; p. 14) define value judgments as “logical constructs, incorporated within an efficiency assessment study, reflecting the DMs’ preferences in the process of assessing efficiency”. In recent years, several methods have been proposed to take the DMs’ preference information into consideration when generating efficiency scores and target levels. Generally, there are two main approaches, referred to as efficiency scores models and target setting models, designed to incorporate the specific preference structure of the DMs in DEA models. The efficiency scores models use the preference information to provide more meaningful efficiency scores while target setting models apply this information to derive more effective targets.

Among the efficiency scores models, value efficiency analysis [35,47,48], through the selection of a most preferred solution (MPS), is an appealing way of incorporating the preference information of DMs in order to compute value efficiency scores that is also compatible with the use of weight restrictions [36]. The first weight restriction DEA model was proposed by Ref. [73]. [7,16,19,60,71,72,75] have proposed various weight restriction models for performance measurement in DEA.

Among the target setting models, the most common approach of taking the preference information of DMs into account is to use multi objective linear programming (MOLP). The first interactive target setting model was introduced by Ref. [33]. He combined DEA with MOLP, used the DM to allocate a set of input levels as resources and selected the most preferred set of output levels from a set of viable points on the efficient frontier. [4,6,11,38,40,43,44,46,59,67,74,80] have proposed various target setting DEA models in the performance evaluation literature.

1.2. Output-oriented dual DEA models and their extensions

[80] investigated equivalence models and interactive tradeoff analysis procedures in MOLP, such that DEA-oriented performance assessment and target setting are integrated so that the DMs’ preferences are used interactively. They established three equivalence models between the output-oriented dual DEA model and the minimax reference point formulations, namely the super-ideal point model, the ideal point model and the shortest distance model. All these models can be used to support efficiency analysis in the same way as the conventional DEA model while also supporting tradeoff analysis for setting target values by individuals or groups.

In a similar vein [76], developed an equivalence model between DEA and MOLP, and showed how a DEA problem can be solved interactively without any prior judgement using a MOLP formulation [39] established a new equivalence model between the output-oriented dual DEA model and the min-ordering formulation in MOLP. These authors applied the Zionts-Wallenius [85] method to reflect the preferences of DMs in the process of efficiency assessment.

The output-oriented dual DEA models focus mainly on output increase. This means that the DM can only incorporate his preferences on output values to reach the MPS as the best target unit but has no control over the input values. To address this deficiency [20,40] introduced an improved equivalence model linking the combined-oriented DEA model with the min-ordering formulation and the minimax reference point formulation, respectively. These models manage to consider both a decrease in total input

consumption and an increase in total output production simultaneously.

It should be emphasized that the DEA model proposed in all these approaches is a radial model projecting all DMUs on to the efficient frontier by solving n LP problems. As a result [53], suggested a super-ideal model, which is identical to a target model that not only considers the decrease in total input consumption and the increase in total output production but also projects all inputs and outputs on to the efficient frontier by solving only one LP problem.

1.3. Hybrid minimax reference point-DEA models and their extensions

[81] analyzed the performance and target setting behavior of 14 branches from a major retail bank in the Greater Manchester county of England using a hybrid minimax reference point-DEA approach that incorporated the value judgments of both branch managers and head-office directors. The authors applied the model to search for the MPS along the efficient frontier for each bank branch [83] used the DEA-interactive minimax reference point approach to study data envelopes and efficient frontiers for different multiple input and multiple output DEA models [82] investigated the equivalence relationships between DEA and multi objective optimization models. They proved that minimax reference point models are equivalent to input-oriented dual CCR models under certain conditions, leading to the development of an interactive minimax reference point approach for hybrid efficiency and trade-off analyses, with the DMs’ preferences taken into account interactively.

However, these hybrid DEA-MOLP models lack a proper method designed to find target units for each inefficient DMU in the presence of undesirable inputs and outputs. Moreover, none of above-mentioned approaches considered finding the MPS for the DMs of each inefficient unit in the presence of uncontrollable factors. Recently [22], provided a new link between the output-oriented DEA model and the weighted minimax reference point formulation in the presence of undesirable outputs. Then, they used the satisfying trade-off method (STOM) proposed by Ref. [57] to help the DM search for the target unit associated to each inefficient unit. However, their approach is not designed to find MPS if there are both undesirable inputs and outputs in the production process.

1.4. Current paper contribution

In this paper, an extended equivalence model is defined between the combined-oriented DEA model and the super-ideal point model of the minimax reference point formulation such that:

- the increase in the total desirable outputs and undesirable inputs and the decrease in the total undesirable outputs and desirable inputs are simultaneously considered;
- uncontrollable variables, defined as those factors that influence the performance of the DMU but are out of the control of the management, are taken into account by the DM.

The main advantage of our DEA-MOLP method over other hybrid DEA models is the ability of the DM to select the variables, as well as their relative intensities, in order to move the DMU closer to the efficient frontier. In this regard, it should be noted that the efficient frontier is determined by many different types of inputs and outputs, such as desirable, undesirable and uncontrollable. As a result, the DM can find the strategic trade-offs among the different factors that are necessary to move the DMU closer to the efficient

frontier. This is particularly relevant if there is a budgetary restriction that constrains the DM from improving all the inefficient factors from a given DMU.

The remainder of this paper is organized as follows. Section 2 provides a brief discussion of the classical DEA model, a summary of the minimax reference point formulation of the MOLP problem and introduces an interactive MOLP method known as STOM, which incorporates the preferences of the DMs in the efficiency assessment process. We also review previous DEA studies applied to undesirable factors and uncontrollable variables and the relation between DEA models and MOLP problems. In Section 3, we establish a new link between the combined-oriented BCC model and the super-ideal point model of the minimax reference point formulation in the presence of undesirable inputs and outputs as well as uncontrollable variables. Section 4 presents a case study where the performance of high schools in the City of Philadelphia is evaluated. Section 5 concludes and suggests future research directions.

2. Background

2.1. Traditional DEA models

The data domain for a traditional DEA study is the set A of n data points, a^1, a^2, \dots, a^n , one for each DMU. Each data point is composed by two different types of elements, those pertaining to the m inputs, $x_j, x_j \neq 0$ and those corresponding to the s outputs, $y_j, y_j \neq 0$. Therefore, the data can be organized as follows:

$$A = [a^1, a^2, \dots, a^n], a^j = (x_j, y_j)^t \tag{1}$$

A is the $(m + s)$ by n matrix whose columns are data points per DMU. Based on the assumption of variable returns to scale (VRS), an input-oriented DEA model defined to obtain the efficiency score of DMU_p can be formulated as follows:

$$\begin{aligned} \min \quad & \alpha_p \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq \alpha_p x_{ip} \quad i = 1, 2, \dots, m, \\ & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{rp} \quad r = 1, 2, \dots, s, \\ & \sum_{j=1}^n \lambda_j = 1, \\ & \lambda_j \geq 0 \quad j = 1, 2, \dots, n. \end{aligned} \tag{2}$$

In this model, an efficiency score, α_p , is produced for DMU_p by minimizing all the inputs radially (proportionally). The objective value of (2) lies within $0 < \alpha_p \leq 1$. If $\alpha_p < 1$, then DMU_p is not efficient and the parameter α_p indicates the decrease in inputs required from DMU_p to become efficient.

In a similar way, given the VRS assumption, an output-oriented DEA model defined to obtain the efficiency score of DMU_p can be formulated as follows:

$$\begin{aligned} \max \quad & \beta_p \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{ip} \quad i = 1, 2, \dots, m, \\ & \sum_{j=1}^n \lambda_j y_{rj} \geq \beta_p y_{rp} \quad r = 1, 2, \dots, s, \\ & \sum_{j=1}^n \lambda_j = 1, \\ & \lambda_j \geq 0 \quad j = 1, 2, \dots, n. \end{aligned} \tag{3}$$

In this model, an efficiency score, $\frac{1}{\beta_p}$, is produced for DMU_p by maximizing all the outputs radially (proportionally). The objective value of (3) lies within $\beta_p \geq 1$. If $\beta_p > 1$, then DMU_p is not efficient and the parameter β_p indicates the extent by which DMU_p has to increase its outputs to become efficient.

A related approach designed to approximate the production technology is based on the directional distance function. The set of all technologically possible input–output combinations is defined as follows:

$$T = \{(x, y); x \text{ can produce } y\} \tag{4}$$

The directional technology distance function measures the distance from a particular observation (x, y) to the efficient frontier of the technology. Let $d = (d_x, d_y)$ be a directional vector. Then, the directional distance function defined on the technology T is given by:

$$\vec{D}_T(x, y, d_x, d_y) = \sup\{\psi : (x - \psi d_x, y + \psi d_y) \in T\} \tag{5}$$

This distance function seeks to simultaneously expand output and contract input. Based on the assumption of VRS for the technology set T , a combined-oriented DEA model designed to evaluate DMU_p can be formulated as follows:

$$\begin{aligned} \max \quad & \psi_p \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{ip} - \psi_p d_{ip} \quad i = 1, 2, \dots, m, \\ & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{rp} + \psi_p d_{rp} \quad r = 1, 2, \dots, s, \\ & \sum_{j=1}^n \lambda_j = 1, \\ & \lambda_j \geq 0 \quad j = 1, 2, \dots, n. \end{aligned} \tag{6}$$

In this model, an efficiency score, $1 - \psi_p$, is produced for DMU_p by decreasing all the inputs and increasing all the outputs. If $\psi_p \neq 0$, then DMU_p is inefficient and the parameter $\psi_p d_x$ indicates the extent by which DMU_p has to decrease its inputs to become efficient. Similarly, the parameter $\psi_p d_y$ indicates the extent by which DMU_p has to increase its outputs in order to become efficient.

DEA also provides a reference set (reference target) for improving an inefficient DMU. The reference set of an inefficient DMU involves a subset of efficient units that facilitate a benchmarking process. In other words, the target units inform the DM of the amount (percentage) by which an inefficient DMU should decrease its inputs and/or increase its outputs to become efficient.

Consider the input-oriented DEA Model (2). In order to find the reference set associated with the inefficient DMU_p , the following LP problem must be solved assuming α_p^* is the optimal solution of Model (2):

$$\begin{aligned} \max \quad & \sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+ \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} + s_i^- = \alpha_p^* x_{ip} \quad i = 1, 2, \dots, m, \\ & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{rp} \quad r = 1, 2, \dots, s, \\ & \sum_{j=1}^n \lambda_j = 1, \\ & \lambda_j \geq 0 \quad j = 1, 2, \dots, n, \\ & s_i^- \geq 0 \quad i = 1, 2, \dots, m, \\ & s_r^+ \geq 0 \quad r = 1, 2, \dots, s. \end{aligned} \tag{7}$$

Suppose that $\lambda_j^*(j = 1, 2, \dots, n)$, $s_i^{-*}(i = 1, 2, \dots, m)$ and $s_r^{+*}(r = 1, 2, \dots, s)$ are the optimal solutions of Model (7). The reference set of the inefficient DMU_p is defined as follows:

$$E_p = \left\{ j \mid \lambda_j^* > 0, j = 1, 2, \dots, n \right\} \quad (8)$$

In this case, the following point on the efficient frontier can be regarded as a target unit for the inefficient DMU_p :

$$\left(\hat{x}_p, \hat{y}_p \right) = \left(\alpha_p^* x_p - s^{-*}, y_p + s^{+*} \right) = \left(\sum_{j \in E_p} \lambda_j^* x_j, \sum_{j \in E_p} \lambda_j^* y_j \right) \quad (9)$$

Definition 1. Let α_p^* be the optimal solution of Model (2). Let $s_i^{-*}(i = 1, 2, \dots, m)$ and $s_r^{+*}(r = 1, 2, \dots, s)$ be the optimal solution of Model (7). The DMU_p is called.

- strong efficient for the input-oriented BCC model, if and only if $\alpha_p^* = 1$ and $\sum_{r=1}^s s_r^{+*} + \sum_{i=1}^m s_i^{-*} = 0$;
- weak efficient for the input-oriented BCC model, if and only if $\alpha_p^* = 1$ and $\sum_{r=1}^s s_r^{+*} + \sum_{i=1}^m s_i^{-*} \neq 0$;
- inefficient for the input-oriented BCC model, if and only if $\alpha_p^* < 1$.

Consider now the output-oriented DEA Model (3). In order to find the reference set associated with the inefficient DMU_p , the following LP problem must be solved assuming β_p^* is the optimal value of Model (3):

$$\begin{aligned} \max \quad & \sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+ \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} + s_i^- = x_{ip} \quad i = 1, 2, \dots, m, \\ & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = \beta_p^* y_{rp} \quad r = 1, 2, \dots, s, \\ & \sum_{j=1}^n \lambda_j = 1, \\ & \lambda_j \geq 0 \quad j = 1, 2, \dots, n, \\ & s_i^- \geq 0 \quad i = 1, 2, \dots, m, \\ & s_r^+ \geq 0 \quad r = 1, 2, \dots, s. \end{aligned} \quad (10)$$

The following point on the efficient frontier can be regarded as a target unit for the inefficient DMU_p :

$$\left(\hat{x}_p, \hat{y}_p \right) = \left(x_p - s^{-*}, \beta_p^* y_p + s^{+*} \right) = \left(\sum_{j \in E_p} \lambda_j^* x_j, \sum_{j \in E_p} \lambda_j^* y_j \right) \quad (11)$$

Where $\lambda_j^*(j = 1, 2, \dots, n)$, $s_i^{-*}(i = 1, 2, \dots, m)$ and $s_r^{+*}(r = 1, 2, \dots, s)$ are the optimal solutions of Model (10).

Definition 2. Let β_p^* be the optimal solution of Model (3). Let $s_i^{-*}(i = 1, 2, \dots, m)$ and $s_r^{+*}(r = 1, 2, \dots, s)$ be the optimal solutions of Model (10). The DMU_p is called:

- strong efficient for the output-oriented BCC model, if and only if $\beta_p^* = 1$ and $\sum_{r=1}^s s_r^{+*} + \sum_{i=1}^m s_i^{-*} = 0$;
- weak efficient for the output-oriented BCC model, if and only if $\beta_p^* = 1$ and $\sum_{r=1}^s s_r^{+*} + \sum_{i=1}^m s_i^{-*} \neq 0$;

- inefficient for the output-oriented BCC model, if and only if $\beta_p^* > 1$.

Finally, consider the combined-oriented DEA Model (6). In order to find the reference set associated with the inefficient DMU_p , the following LP problem must be solved assuming ψ_p^* as the optimal solution of Model (6):

$$\begin{aligned} \max \quad & \sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+ \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} + s_i^- = x_{ip} - \psi_p^* d_{ix} \quad i = 1, 2, \dots, m, \\ & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{rp} + \psi_p^* d_{ry} \quad r = 1, 2, \dots, s, \\ & \sum_{j=1}^n \lambda_j = 1, \\ & \lambda_j \geq 0 \quad j = 1, 2, \dots, n, \\ & s_i^- \geq 0 \quad i = 1, 2, \dots, m, \\ & s_r^+ \geq 0 \quad r = 1, 2, \dots, s. \end{aligned} \quad (12)$$

In this case, the following point on the efficient frontier can be regarded as the target unit for the inefficient DMU_p .

$$\begin{aligned} \left(\hat{x}_p, \hat{y}_p \right) &= \left(x_p - \psi_p^* d_x - s^{-*}, y_p + \psi_p^* d_y + s^{+*} \right) \\ &= \left(\sum_{j \in E_p} \lambda_j^* x_j, \sum_{j \in E_p} \lambda_j^* y_j \right) \end{aligned} \quad (13)$$

Where $\lambda_j^*(j = 1, 2, \dots, n)$, $s_i^{-*}(i = 1, 2, \dots, m)$ and $s_r^{+*}(r = 1, 2, \dots, s)$ are the optimal solutions of Model (12).

Definition 3. Let ψ_p^* be the optimal value of Model (6). Let $s_i^{-*}(i = 1, 2, \dots, m)$ and $s_r^{+*}(r = 1, 2, \dots, s)$ be the optimal solutions of Model (12). The DMU_p is called.

- strong efficient for the combined-oriented BCC model, if and only if $\psi_p^* = 0$ and $\sum_{r=1}^s s_r^{+*} + \sum_{i=1}^m s_i^{-*} = 0$;
- weak efficient for the combined-oriented BCC model, if and only if $\psi_p^* = 0$ and $\sum_{r=1}^s s_r^{+*} + \sum_{i=1}^m s_i^{-*} \neq 0$;
- inefficient for the combined-oriented BCC model, if and only if $\psi_p^* \neq 0$.

2.2. MOLP preliminaries and STOM method

A multiple objective linear programming (MOLP) problem consists of optimizing several linear objective functions subject to a set of linear constraints.

A MOLP problem with $q(\geq 2)$ conflicting objective functions $f_h : A \rightarrow R(h = 1, 2, \dots, q)$, can be formulated as follows in a general form:

$$\begin{aligned} \max f(\lambda) &= [f_1(\lambda), \dots, f_q(\lambda)] \\ \text{s.t.} \quad & \lambda \in A \end{aligned} \quad (14)$$

In this model, the decision variables $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)^t$ belong to the non-empty feasible space Λ .

Generally, there are no solutions for MOLP problems that can simultaneously optimize all the objective functions. As a result, the primary goal in MOLP is to find the efficient solutions and to help

the DM select the MPS. It is worth noting that a MPS is an efficient solution that optimizes the objective function of the DM.

Definition 4. A feasible solution $\hat{\lambda} \in A$ is called an efficient (non-dominated) solution if there is no feasible solution $\lambda \in \Lambda$ such that $f(\lambda) \geq f(\hat{\lambda})$ and $f(\lambda) \neq f(\hat{\lambda})$.

Based on Definition 4, a solution represented by a point in the decision variable space is an efficient optimal solution if it is not possible to move the point within the feasible space to increase the value of an objective function without deteriorating at least one of the other objectives.

Definition 5. A feasible solution $\hat{\lambda} \in A$ is called a weak efficient solution if there is no feasible solution $\lambda \in \Lambda$ such that $f(\lambda) > f(\hat{\lambda})$.

This means that a solution represented by a point in the decision variable space is a weak efficient solution if it is not possible to move the point within the feasible region to increase the value of all the objective functions.

Definition 6. The point $f^* = (f_1^*, \dots, f_q^*)$, given by $f_h^* = \max_{\lambda \in A} f_h(\lambda)$, ($h = 1, 2, \dots, q$), is called the ideal point of the MOLP problem (14).

The weighted minimax formulation can be used to generate any efficient solutions of the MOLP problem (14) [66,78,79]. Considering λ as an efficient solution of (14) and f_h^* as the maximum feasible value of objective h , there exist a weighting vector \vec{w} satisfying $w_1 = 1$ and $w_h > 0$ for $h = 2, 3, \dots, q$, and a reference point f^{ref} , such that λ can be generated by solving the following weighted minimax problem:

$$\begin{aligned} \min_{\lambda} \max_{1 \leq h \leq q} \{w_h |(f_h^{ref} - f_h(\lambda))|\} \\ \text{s.t.} \\ \lambda \in A \end{aligned} \tag{15}$$

It is worth noting that if the ideal point $f^* = [f_1^*, \dots, f_q^*]$ is used as the reference point $f^{ref} = [f_1^{ref}, \dots, f_q^{ref}]$, then the weighted minimax reference point formulation will be called the ideal point model.

On the other hand, if $f^* \leq f^{ref}$, then the weighted minimax formulation (15) can be equivalently converted into the following form, called the super-ideal point model, by assuming $\phi = \max_{1 \leq h \leq q} \{w_h |(f_h^{ref} - f_h(\lambda))|\}$:

$$\begin{aligned} \min \phi \\ \text{s.t.} \\ w_h (f_h^{ref} - f_h(\lambda)) \leq \phi, h = 1, 2, \dots, q \\ \lambda \in A \end{aligned} \tag{16}$$

The minimax formulation using a particular reference point can be applied to design an interactive procedure that helps the DM search for the MPS on the efficient frontier by systematically changing the weighting parameters $w_h, h = 1, 2, \dots, q$.

Interactive methods are resolution methods for multi objective problems where the information exchange between the DM and the analyst is carried out in a continuous way during the resolution process. These methods progressively incorporate the information provided by the analyst so as to lead the DM to his MPS. Based on the information required, interactive methods are classified into five groups: comparison methods, tradeoffs or local weights methods, level specification methods, classification methods and non-trading-off methods.

One of these classifying methods, namely, the satisfying trade-off method (STOM) proposed by Ref. [57]; is used in this paper. The first step of the method consists of finding the reference point of the achievement function that does not change within the whole solution process. After that, the reference levels of the objectives are provided by the DM at any iteration and incorporated in the weights of the achievement function.

Let (λ^{k-1}, f^{k-1}) be the solution of iteration $k-1$. Given this solution, the DM should classify the objective functions into three categories, namely:

- i) The class of objective functions that should be improved further.
- ii) The class of objective functions that have to be maintained.
- iii) The class of objective functions that must be relaxed.

Suppose that $\{f_h, h \in I_I^k\}$ represent the objective functions that should be improved further and $\{\Delta f_h, h \in I_I^k\}$ the amounts to be improved. Similarly, let $\{f_h, h \in I_M^k\}$ denote the objective functions that are to be maintained and $\{f_h, h \in I_R^k\}$ the objective functions that must be relaxed by the corresponding amounts $\{\Delta f_h, h \in I_R^k\}$. Thus, the new reference point is determined as follows:

$$y_h^k = f_h^{k-1} + \Delta f_h^k \quad \forall h \in I_I^k \tag{17}$$

$$y_h^k = f_h^{k-1} \quad \forall h \in I_M^k$$

$$y_h^k = f_h^{k-1} - \Delta f_h^k \quad \forall h \in I_R^k$$

Now, according to this new reference point, the new values of w_h^k are computed as follows:

$$w_h^k = \frac{1}{f_h^{ref} - y_h^k} \quad h = 1, 2, \dots, q \tag{18}$$

After that, the following weighted minimax MOLP problem is solved:

$$\begin{aligned} \min \phi \\ \text{s.t.} \\ w_h^k (f_h^{ref} - f_h(\lambda)) \leq \phi, h = 1, 2, \dots, q \\ \lambda \in A \end{aligned} \tag{19}$$

This process is repeated in an interactive way until a satisfactory solution is obtained.

2.3. Previous DEA studies applied to undesirable factors and uncontrollable variables

The production process in DEA uses a set of inputs to produce a set of outputs. In traditional DEA models, each producer uses varying levels of inputs and produces varying levels of outputs. According to the efficiency criterion of DEA, either producing more output with the same input or producing the same output with less input is more efficient. However, both desirable (good) and undesirable (bad) input and output factors may be present in performance measurement problems. In this case, the DMUs with more desirable outputs and undesirable inputs and less undesirable outputs and desirable inputs are considered efficient. As reported in Table 1, six main different approaches have been developed to deal with and incorporate undesirable factors in the DEA framework.

Table 1
Classification of DEA models for dealing with undesirable factors.

• Ignoring undesirable factors in DEA models	• [58] • [41] • [50, 51]
• Treating undesirable outputs as inputs and treating undesirable inputs as outputs	• [18]
• Applying a nonlinear monotonic decreasing transformation to the undesirable factors	• [34]
• Using a linear monotonic decreasing transformation to deal with undesirable factors	• [63]
• Treating undesirable factors in the non-linear DEA model	• [25]
• Aggregating undesirable factors in a ratio form with the desirable factors	• [84]

In the current paper, we use the method developed by Ref. [63] to deal with undesirable factors. The reasons for this particular choice are described below. First, it is a very simple method, easy to understand and calculate. Second, it preserves the properties of the production process. Third, unlike the non-linear transformation approach, this method does not eliminate the ratio or interval scale of the original data [23]. Fourth, since this method uses the strong disposability assumption of undesirable outputs to build the production possibility set, it can actually provide the shadow price of undesirable outputs. Fifth, this method is particularly convenient when incorporating undesirable inputs and outputs indirectly into DEA, a property that will prove useful when applying our method to the assessment of public schools.

Suppose that the DEA data domain is expressed as follows:

$$A = [a^1, a^2, \dots, a^n], \quad a^j = (x_j^g, x_j^b, y_j)^t \quad (20)$$

where x_j^g and x_j^b represent the desirable and undesirable inputs, respectively [63] used a linear monotonic decreasing transformation to deal with undesirable inputs in the input-oriented BCC model framework. These authors proposed a method that first multiplies each undesirable input x_j^b by -1 and then finds a proper translation vector u to change all negative undesirable inputs into positive values. The data domain of (20) becomes:

$$a^j = (x_j^g, \bar{x}_j^b, y_j)^t, \quad \bar{x}_j^b = -x_j^b + u > 0 \quad (21)$$

Based on (21) and Model (2), we have

$$\begin{aligned} \min \quad & \alpha_p \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij}^g \leq \alpha_p x_{ip}^g \quad i = 1, 2, \dots, m_1, \\ & \sum_{j=1}^n \lambda_j \bar{x}_{ij}^b \leq \alpha_p \bar{x}_{ip}^b \quad i = 1, 2, \dots, m, \\ & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{rp} \quad r = 1, 2, \dots, s, \\ & \sum_{j=1}^n \lambda_j = 1, \\ & \lambda_j \geq 0 \quad j = 1, 2, \dots, n. \end{aligned} \quad (22)$$

It is worth noting that Model (22) contracts desirable inputs and expands undesirable inputs.

Now suppose that the DEA data domain is expressed as follows:

$$A = [a^1, a^2, \dots, a^n], \quad a^j = (x_j, y_j^g, y_j^b)^t \quad (23)$$

where y_j^g and y_j^b represent the desirable and undesirable outputs, respectively. In the output-oriented BCC model framework [63], first multiply each undesirable output y_j^b by -1 and then find a proper translation vector u' to change all negative undesirable outputs into positive values. The data domain of (23) becomes:

$$a^j = (x_j, y_j^g, \bar{y}_j^b)^t, \quad \bar{y}_j^b = -y_j^b + u' > 0 \quad (24)$$

In this case, based on (24), Model (3) becomes:

$$\begin{aligned} \max \quad & \beta_p \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{ip} \quad i = 1, 2, \dots, m, \\ & \sum_{j=1}^n \lambda_j y_{rj}^g \geq \beta_p y_{rp}^g \quad r = 1, 2, \dots, s_1, \\ & \sum_{j=1}^n \lambda_j \bar{y}_{rj}^b \geq \beta_p \bar{y}_{rp}^b \quad r = 1, 2, \dots, s_2, \\ & \sum_{j=1}^n \lambda_j = 1, \\ & \lambda_j \geq 0 \quad j = 1, 2, \dots, n. \end{aligned} \quad (25)$$

Note that Model (25) expands desirable outputs and contracts undesirable outputs.

Another important issue in performance measurement problems is how to treat uncontrollable variables, which often reflect the impact of the operating environment. Generally speaking, uncontrollable variables refer to those factors that influence the performance of DMUs and are, at the same time, out of the control of the management. Usually, the management can decide on some controllable factors internal to production activities, while the impact of the operating environment is out of the control of the management.

Traditional studies that have constructed research models using only controllable factors implicitly assume that all the inefficiencies of DMUs are caused by bad management. Since the impact of uncontrollable variables is not filtered out, the performance of those DMUs in an adverse operating environment will be underestimated. Previous works that incorporate uncontrollable variables into DEA can be broadly classified as separation models [10,15,26,32], one-stage models [9,52], two-stage models [27,52,61], three-stage models [29] and four-stage models [28].

Assume that the DEA data domain in the presence of uncontrollable inputs is expressed as follows:

$$a^j = (x_j, x_j^n, y_j)^t \quad (26)$$

where x_j and x_j^n represent the controllable and uncontrollable inputs, respectively. Following [9] and based on Equation (26), Model (2) becomes:

$$\begin{aligned}
 \min \quad & \alpha_p \\
 \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq \alpha_p x_{ip} \quad i = 1, 2, \dots, m_1, \\
 & \sum_{j=1}^n \lambda_j x_{ej}^n \leq x_{ep}^n \quad e = 1, 2, \dots, m_3, \\
 & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{rp} \quad r = 1, 2, \dots, s, \\
 & \sum_{j=1}^n \lambda_j = 1, \\
 & \lambda_j \geq 0 \quad j = 1, 2, \dots, n.
 \end{aligned} \tag{27}$$

It should be noted that the variable α_p in Model (27) is not applied to the uncontrollable input constraints because these values are fixed exogenously and it is not possible for management to change them.

Similarly, by expressing the DEA data domain as $a^j = (x_j, y_j, y_j^n)^t$, where y_j and y_j^n represent the controllable and uncontrollable outputs, respectively, Model (3) can be reformulated as follows:

$$\begin{aligned}
 \max \quad & \beta_p \\
 \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{ip} \quad i = 1, 2, \dots, m, \\
 & \sum_{j=1}^n \lambda_j y_{rj} \geq \beta_p y_{rp} \quad r = 1, 2, \dots, s_1, \\
 & \sum_{j=1}^n \lambda_j y_{lj}^n \geq y_{lp}^n \quad l = 1, 2, \dots, s_3, \\
 & \sum_{j=1}^n \lambda_j = 1, \\
 & \lambda_j \geq 0 \quad j = 1, 2, \dots, n.
 \end{aligned} \tag{28}$$

2.4. Previous studies applied to the relation between DEA and MOLP

In this section, we review some existing approaches of DEA and MOLP problems such that the DEA-oriented performance assessment and target setting can be integrated in a way that the DMs' preferences are used interactively.

[80] proved that, under certain conditions, the output-oriented DEA Model (3) can be transformed into the super-ideal point Model (16). Then, they concluded that the output-oriented DEA Model (3) is actually constructed to locate a specific efficient solution for the observed DMU_p , termed as a DEA efficient solution, on the efficient frontier of the following generic MOLP formulation:

$$\begin{aligned}
 \max f(\lambda) = & \left[\sum_{j=1}^n \lambda_j y_{1j}, \dots, \sum_{j=1}^n \lambda_j y_{rj}, \dots, \sum_{j=1}^n \lambda_j y_{sj} \right] \\
 \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{ip} \quad i = 1, 2, \dots, m, \\
 & \sum_{j=1}^n \lambda_j = 1, \\
 & \lambda_j \geq 0 \quad j = 1, 2, \dots, n.
 \end{aligned} \tag{29}$$

Hence, they applied the interactive gradient projection approach in MOLP to support integrated DEA-oriented performance assessment and target setting.

[76] showed that the equivalence model described above can be solved satisfactorily using five well-known interactive methods in MOLP and compared the results derived from these MOLP methods

when applied to the efficiency analysis of several retail banks in the UK.

As described in the introduction section [81], and [83] investigated several hybrid DEA-minimax reference point approaches designed to incorporate the value judgments and preferences of DMs. In particular, the latter authors proposed a new technical efficiency score, which can be intuitively interpreted as the degree of minimum effort that is necessary for a DMU to expand its outputs for achieving feasible efficiency, and a new preferred efficiency score, which can be intuitively interpreted as the degree of minimum effort that is necessary for a DMU to balance its outputs for achieving the MPS (or target DMU) along its efficient frontier.

It is worth noting that these models consider only the output-oriented DEA model, which is a radial model that focuses on output increase. To overcome this limitation [82], proved that, under certain conditions, the input-oriented DEA Model (2) is equivalent to the following minimax reference point model:

$$\begin{aligned}
 \min \quad & \alpha \\
 \text{s.t.} \quad & w_i (f_i(\lambda) - f_i^{ref}) \leq \alpha, \quad i = 1, 2, \dots, m \\
 & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{rp} \quad r = 1, 2, \dots, s, \\
 & \sum_{j=1}^n \lambda_j = 1, \\
 & \lambda_j \geq 0 \quad j = 1, 2, \dots, n.
 \end{aligned} \tag{30}$$

Therefore, the efficiency score of the inefficient DMU_p together with its target unit can be generated by solving the following formulation, which allows for the use of interactive methods in MOLP to solve the DEA problem:

$$\begin{aligned}
 \min f(\lambda) = & \left[\sum_{j=1}^n \lambda_j x_{1j}, \dots, \sum_{j=1}^n \lambda_j x_{ij}, \dots, \sum_{j=1}^n \lambda_j x_{mj} \right] \\
 \text{s.t.} \quad & \text{Constraints of (30)}
 \end{aligned} \tag{31}$$

This type of equivalence model leads to the development of a minimax reference point approach for supporting integrated performance analysis and resource planning that takes into account the preferences of DMs in an interactive fashion.

It should be noted that none of these equivalence approaches provides the MPS when both the input and output values of DMUs can be modified simultaneously. To address this shortage [20], established an equivalence model between the general combined-oriented DEA Model (6) and the following minimax reference point model:

$$\begin{aligned}
 \min \quad & \psi \\
 \text{s.t.} \quad & w_i (f_i^{ref} - f_i(\lambda)) \leq \psi, \quad i = 1, 2, \dots, m \\
 & w_r (f_r^{ref} - f_r(\lambda)) \leq \psi \quad r = 1, 2, \dots, s, \\
 & \sum_{j=1}^n \lambda_j = 1, \\
 & \lambda_j \geq 0 \quad j = 1, 2, \dots, n.
 \end{aligned} \tag{32}$$

Then, they used the Zions–Wallenius method to incorporate the preferences of the DM in the process of assessing efficiency. They illustrated the applicability of the proposed equivalence model through a real-life case study involving 20 bank branches.

Another limitation of the equivalence models developed by Refs. [76,80,81,83] is that they are not applicable in the presence of undesirable outputs. To overcome this shortcoming [22], defined an

equivalence model between the DEA Model (25) and the corresponding reference point model, leading to the following MOLP problem for the observed inefficient DMU_p :

$$\begin{aligned} \max \quad & f(\lambda) = \left[\sum_{j=1}^n \lambda_j y_{1j}^g, \dots, \sum_{j=1}^n \lambda_j y_{s_1 j}^g, \sum_{j=1}^n \lambda_j \bar{y}_{1j}^b, \dots, \sum_{j=1}^n \lambda_j \bar{y}_{s_2 j}^b \right] \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{ip} \quad i = 1, 2, \dots, m, \\ & \sum_{j=1}^n \lambda_j = 1, \\ & \lambda_j \geq 0 \quad j = 1, 2, \dots, n. \end{aligned} \tag{33}$$

Similarly to the equivalence approaches described throughout this section, this formulation can be used to perform an interactive search for the DEA efficient solution on the frontier.

Remark 3.1. The main drawback of the above-described approaches relating the DEA and MOLP problems is that the target unit found for each inefficient DMU_p is based on the formulation $(\bar{x}_p, \bar{y}_p) = [\sum_{j \in E_p} \lambda_j^* x_j, \sum_{j \in E_p} \lambda_j^* y_j]$, which assumes that $\lambda_j^* (j \in E_p)$ is the optimal solution of models (2), (3) or (6). Since these models are radial and the slack variables are not optimized, the computed target could be a weak efficient point using the VRS technology. In the next section, we extend the existing approaches such that the computed target unit is obtained based on formulations (9), (11) or (13), leading to a strong efficient point using the VRS technology.

3. An improved relation between DEA and MOLP

In this section, we develop an interactive approach based on the minimax reference point model that supports integrated performance analysis and target setting. In particular, a generalized combined-oriented DEA model is used for non-parametric performance analysis, and the equivalence between the DEA and minimax models is exploited to enable equivalent performance analysis in a MOLP process.

Assume that the DEA data domain is expressed as follows:

$$A = [a^1, a^2, \dots, a^n], a^i = (x_j^g, x_j^b, x_j^n, y_j^g, y_j^b, y_j^n)^t \tag{34}$$

where $x_j^g, x_j^b, x_j^n, y_j^g, y_j^b,$ and y_j^n represent the desirable (controllable) inputs, undesirable (controllable) inputs, uncontrollable inputs, desirable (controllable) outputs, undesirable (controllable) outputs and uncontrollable outputs, respectively.

We use a linear monotonic decreasing transformation to deal with the undesirable inputs and outputs. That is, we multiply each undesirable input x_j^b and each undesirable output y_j^b by -1 and then find proper translation vectors, u and u' , respectively, to change all negative undesirable inputs and outputs into positive values. The data domain of (35) becomes

$$\begin{aligned} a^j &= (x_j^g, \bar{x}_j^b, x_j^n, y_j^g, \bar{y}_j^b, y_j^n)^t, \\ \bar{x}_j^b &= -x_j^b + u > 0, \\ \bar{y}_j^b &= -y_j^b + u' > 0 \end{aligned} \tag{35}$$

The following model, based on Equation (35) and termed general combined-oriented DEA model, is proposed for efficiency analysis:

$$\begin{aligned} \max \quad & \psi_p \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij}^g \leq x_{ip}^g - \psi_p d_{ix}^g \quad i = 1, 2, \dots, m_1, \\ & \sum_{j=1}^n \lambda_j \bar{x}_{kj}^b \leq \bar{x}_{kp}^b - \psi_p \bar{d}_{kx}^b \quad k = 1, 2, \dots, m_2, \\ & \sum_{j=1}^n \lambda_j x_{ej}^n \leq x_{ep}^n \quad e = 1, 2, \dots, m_3, \\ & \sum_{j=1}^n \lambda_j y_{rj}^g \geq y_{rp}^g + \psi_p d_{ry}^g \quad r = 1, 2, \dots, s_1, \\ & \sum_{j=1}^n \lambda_j \bar{y}_{tj}^b \geq \bar{y}_{tp}^b + \psi_p \bar{d}_{ty}^b \quad t = 1, 2, \dots, s_2, \\ & \sum_{j=1}^n \lambda_j y_{lj}^n \geq y_{lp}^n \quad l = 1, 2, \dots, s_3, \\ & \sum_{j=1}^n \lambda_j = 1, \\ & \lambda_j \geq 0 \quad j = 1, 2, \dots, n. \end{aligned} \tag{36}$$

Note that Model (36) simultaneously expands desirable outputs and undesirable inputs and contracts desirable inputs and undesirable outputs.

If $\psi_p \neq 0$, then DMU_p is inefficient. In order to find the reference set of the inefficient DMU_p based on the generalized combined-oriented DEA Model (36), the following LP problem is solved assuming ψ_p^* is the optimal solution of Model (36):

$$\begin{aligned} \max \quad & \sum_{i=1}^{m_1} s_i^{g-} + \sum_{k=1}^{m_2} s_k^{b-} + \sum_{e=1}^{m_3} s_e^{n-} + \sum_{r=1}^{s_1} s_r^{g+} + \sum_{t=1}^{s_2} s_t^{b+} + \sum_{l=1}^{s_3} s_l^{n+} \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij}^g + s_i^{g-} = x_{ip}^g - \psi_p^* d_{ix}^g \quad i = 1, 2, \dots, m_1, \\ & \sum_{j=1}^n \lambda_j \bar{x}_{kj}^b + s_k^{b-} = \bar{x}_{kp}^b - \psi_p^* \bar{d}_{kx}^b \quad k = 1, 2, \dots, m_2, \\ & \sum_{j=1}^n \lambda_j x_{ej}^n + s_e^{n-} = x_{ep}^n \quad e = 1, 2, \dots, m_3, \\ & \sum_{j=1}^n \lambda_j y_{rj}^g - s_r^{g+} = y_{rp}^g + \psi_p^* d_{ry}^g \quad r = 1, 2, \dots, s_1, \\ & \sum_{j=1}^n \lambda_j \bar{y}_{tj}^b - s_t^{b+} = \bar{y}_{tp}^b + \psi_p^* \bar{d}_{ty}^b \quad t = 1, 2, \dots, s_2, \\ & \sum_{j=1}^n \lambda_j y_{lj}^n - s_l^{n+} = y_{lp}^n \quad l = 1, 2, \dots, s_3, \\ & \sum_{j=1}^n \lambda_j = 1, \\ & s_i^{g-} \geq 0 \quad i = 1, 2, \dots, s_1, \\ & s_k^{b-} \geq 0 \quad k = 1, 2, \dots, s_2, \\ & s_e^{n-} \geq 0 \quad e = 1, 2, \dots, s_3, \\ & s_r^{g+} \geq 0 \quad r = 1, 2, \dots, m_1, \\ & s_t^{b+} \geq 0 \quad t = 1, 2, \dots, m_2, \\ & s_l^{n+} \geq 0 \quad l = 1, 2, \dots, m_3, \\ & \lambda_j \geq 0 \quad j = 1, 2, \dots, n. \end{aligned} \tag{37}$$

Let $(\psi_p^*, \lambda^*, s^{g-*}, s^{b-*}, s^{n-*}, s^{g+*}, s^{b+*}, s^{n+*})$ be the optimal solutions of Model (37). Denote the reference set of the inefficient DMU_p by $E_p = \{j | \lambda_j^* > 0, j = 1, 2, \dots, n\}$. The following point on the efficient frontier can be regarded as a target unit of the inefficient DMU_p :

$$(\widehat{x}_p, \widehat{y}_p) = \left(\sum_{j \in E_p} \lambda_j^* x_j^g, \sum_{j \in E_p} \lambda_j^* x_j^b, \sum_{j \in E_p} \lambda_j^* x_j^e, \sum_{j \in E_p} \lambda_j^* y_j^g, \sum_{j \in E_p} \lambda_j^* y_j^b, \sum_{j \in E_p} \lambda_j^* y_j^e \right) \tag{38}$$

Now consider the following MOLP problem:

$$\begin{aligned} \max \quad & h(\lambda) = [f_1(\lambda), \dots, f_{m_1}(\lambda), \bar{f}_1(\lambda), \dots, \bar{f}_{m_2}(\lambda), g_1(\lambda), \dots, g_{s_1}(\lambda), \\ & \bar{g}_1(\lambda), \dots, \bar{g}_{s_2}(\lambda)] \\ \text{s.t.} \quad & \lambda \in A \end{aligned} \tag{40}$$

As discussed in Section 2, the following super-ideal point model can be used to generate any efficient solution of MOLP (40), as follows:

$$\begin{aligned} \min \quad & \sigma \\ \text{s.t.} \quad & w_i \left((f_i^{ref} - f_i(\lambda)) \right) \leq \sigma \quad i = 1, \dots, m_1, \\ & \bar{w}_k \left((\bar{f}_k^{ref} - \bar{f}_k(\lambda)) \right) \leq \sigma \quad k = 1, \dots, m_2, \\ & v_r \left((g_r^{ref} - g_r(\lambda)) \right) \leq \sigma \quad r = 1, \dots, s_1, \\ & \bar{v}_t \left((\bar{g}_t^{ref} - \bar{g}_t(\lambda)) \right) \leq \sigma \quad t = 1, \dots, s_2, \\ & \lambda \in A \end{aligned} \tag{41}$$

We are now in a position to prove that formulation (36) is equivalent to formulation (41) under certain conditions.

In formulation (36), the i -th composite desirable input and the k -th composite undesirable input can be represented by $f_i(\lambda)$ and $\bar{f}_k(\lambda)$, respectively, as follows:

$$f_i(\lambda) = x_{ip}^g - \sum_{j=1}^n \lambda_j x_{ij}^g, \quad i = 1, 2, \dots, m_1 \tag{42}$$

$$\bar{f}_k(\lambda) = \bar{x}_{kp}^b - \sum_{j=1}^n \lambda_j \bar{x}_{kj}^b, \quad k = 1, 2, \dots, m_2 \tag{43}$$

Similarly, the r -th composite desirable output and the t -th composite undesirable output can be represented by $g_r(\lambda)$ and $\bar{g}_t(\lambda)$, respectively, as follows:

$$g_r(\lambda) = \sum_{j=1}^n \lambda_j y_{rj}^g - y_{rp}^g, \quad r = 1, 2, \dots, s_1, \tag{44}$$

$$\bar{g}_t(\lambda) = \sum_{j=1}^n \lambda_j \bar{y}_{tj}^b - \bar{y}_{tp}^b, \quad t = 1, 2, \dots, s_2, \tag{45}$$

Assume that

$$A_p = \left\{ \lambda \mid \sum_{j=1}^n \lambda_j x_{ej}^n \leq x_{ep}^n, (e = 1, 2, \dots, m_3), \sum_{j=1}^n \lambda_j y_{lj}^n \geq y_{lp}^n, (l = 1, 2, \dots, s_3), \sum_{j=1}^n \lambda_j = 1, \lambda_j \geq 0 (j = 1, 2, \dots, n) \right\} \tag{46}$$

The maximum feasible value of the i -th composite desirable

input for the observed DMU_p is denoted by $\widehat{f}_{ip} = f_i(\lambda^*)$, where λ^* can be found by solving the following problem:

$$\widehat{f}_{ip} = \max_{\lambda \in A_p} f_i(\lambda) = x_{ip}^g - \sum_{j=1}^n \lambda_j x_{ij}^g, \quad i = 1, 2, \dots, m_1 \tag{47}$$

The maximum feasible value of the k -th composite undesirable input for the observed DMU_p is denoted by $\widehat{\bar{f}}_{kp} = \bar{f}_k(\lambda^*)$, where λ^* can be found by solving the following problem:

$$\widehat{\bar{f}}_{kp} = \max_{\lambda \in A_p} \bar{f}_k(\lambda) = \bar{x}_{kp}^b - \sum_{j=1}^n \lambda_j \bar{x}_{kj}^b, \quad k = 1, 2, \dots, m_2, \tag{48}$$

The maximum feasible value of the r -th composite desirable output for the observed DMU_p is denoted by $\widehat{g}_{rp} = g_r(\lambda^*)$, where λ^* can be found by solving the following problem:

$$\widehat{g}_{rp} = \max_{\lambda \in A_p} g_r(\lambda) = \sum_{j=1}^n \lambda_j y_{rj}^g - y_{rp}^g, \quad r = 1, 2, \dots, s_1 \tag{49}$$

Finally, the maximum feasible value of the t -th composite undesirable output for the observed DMU_p is denoted by $\widehat{\bar{g}}_{tp} = \bar{g}_t(\lambda^*)$, where λ^* can be found by solving the following problem:

$$\widehat{\bar{g}}_{tp} = \max_{\lambda \in A_p} \bar{g}_t(\lambda) = \sum_{j=1}^n \lambda_j \bar{y}_{tj}^b - \bar{y}_{tp}^b, \quad t = 1, 2, \dots, s_2 \tag{50}$$

Assume that the feasible region Λ in formulation (41) is set to be the same as the region defined in formulation (46), i.e., assume $\Lambda = \Lambda_p$. The equivalence relationship between the general combined-oriented DEA Model (36) and the minimax formulation (41) can be established by the following theorem.

Theorem 1. Assume $d_{ix}^g > 0$ ($i = 1, 2, \dots, m_1$), $\bar{d}_{kx}^b > 0$ ($k = 1, 2, \dots, m_2$), $d_{ry}^g > 0$ ($r = 1, 2, \dots, s_1$) and $\bar{d}_{ty}^b > 0$ ($t = 1, 2, \dots, s_2$). The general combined-oriented DEA Model (36) can be equivalently transformed into the super-ideal point Model (41) using Equations (47)-(50) and the following definitions:

$$w_i = \frac{1}{d_{ix}^g}, \quad i = 1, 2, \dots, m_1 \tag{51}$$

$$\bar{w}_k = \frac{1}{\bar{d}_{kx}^b}, \quad k = 1, 2, \dots, m_2 \tag{52}$$

$$v_r = \frac{1}{d_{ry}^g}, \quad r = 1, 2, \dots, s_1 \tag{53}$$

$$\bar{v}_t = \frac{1}{\bar{d}_{ty}^b}, \quad t = 1, 2, \dots, s_2 \tag{54}$$

$$f_i^{ref} = \frac{T^{max}}{w_i} = d_{ix}^g T^{max}, \quad i = 1, 2, \dots, m_1 \tag{55}$$

$$\bar{f}_k^{ref} = \frac{T^{max}}{\bar{w}_k} = \bar{d}_{kx}^b T^{max}, \quad k = 1, 2, \dots, m_2 \tag{56}$$

$$g_r^{ref} = \frac{T^{max}}{v_r} = d_{ry}^g T^{max}, \quad r = 1, 2, \dots, s_1 \tag{57}$$

$$\bar{g}_t^{ref} = \frac{T^{max}}{\bar{v}_t} = \bar{d}_{ty}^b T^{max}, \quad t = 1, 2, \dots, s_2 \tag{58}$$

$$\sigma = T^{max} - \psi_p \tag{59}$$

$$T^{max} = \max_{\substack{1 \leq i \leq m_1 \\ 1 \leq k \leq m_2 \\ 1 \leq r \leq s_1 \\ 1 \leq t \leq s_2}} \left\{ w_i \hat{f}_{ip}, \bar{w}_k \hat{f}_{kp}, v_r \hat{g}_{rp}, \bar{v}_t \hat{g}_{tp} \right\} \tag{60}$$

Proof. Please, see the Appendix section.

Theorem 1 states that if the reference point in Model (41) is set by $h^{ref} = [f_1^{ref}, \dots, f_{m_1}^{ref}, \bar{f}_1^{ref}, \dots, \bar{f}_{m_2}^{ref}, g_1^{ref}, \dots, g_{s_1}^{ref}, \bar{g}_1^{ref}, \dots, \bar{g}_{s_2}^{ref}]$ and w_i, \bar{w}_k, v_r and \bar{v}_t are given by Equations (51)–(54), respectively, then the general combined-oriented DEA model will be identical to the following super-ideal point model:

$$\begin{aligned} & \min \quad \sigma \\ & s.t. \\ & w_i \left(f_i^{ref} - x_{ip}^g + \sum_{j=1}^n \lambda_j x_{ij}^g \right) \leq \sigma \quad i = 1, \dots, m_1, \\ & \bar{w}_k \left(\bar{f}_k^{ref} - \bar{x}_{kp}^b + \sum_{j=1}^n \lambda_j \bar{x}_{kj}^b \right) \leq \sigma \quad k = 1, \dots, m_2, \\ & v_r \left(g_r^{ref} - \sum_{j=1}^n \lambda_j y_{rj}^g + y_{rp}^g \right) \leq \sigma \quad r = 1, \dots, s_1, \\ & \bar{v}_t \left(\bar{g}_t^{ref} - \sum_{j=1}^n \lambda_j \bar{y}_{tj}^b + \bar{y}_{tp}^b \right) \leq \sigma \quad t = 1, \dots, s_2, \\ & \lambda \in \mathcal{A}_p. \end{aligned} \tag{61}$$

Thus, if the conditions described in Theorem 1 are satisfied, the super-ideal point Model (61) can be applied to obtain the same efficiency score and efficient composite inputs and outputs for the DMU_p under evaluation as those derived from the general combined-oriented DEA Model (36).

The MOLP problem on which the super-ideal point Model (61) is constructed is the following one:

$$\begin{aligned} \max \quad & h_1(\lambda) = \left[x_{1p}^g - \sum_{j=1}^n \lambda_j x_{1j}^g, \dots, x_{m_1 p}^g - \sum_{j=1}^n \lambda_j x_{m_1 j}^g, \bar{x}_{1p}^b - \sum_{j=1}^n \lambda_j \bar{x}_{1j}^b, \dots, \bar{x}_{m_2 p}^b - \sum_{j=1}^n \lambda_j \bar{x}_{m_2 j}^b \right] \\ \max \quad & h_2(\lambda) = \left[\sum_{j=1}^n \lambda_j y_{1j}^g - y_{1p}^g, \dots, \sum_{j=1}^n \lambda_j y_{s_1 j}^g - y_{s_1 p}^g, \sum_{j=1}^n \lambda_j \bar{y}_{1j}^b - \bar{y}_{1p}^b, \dots, \sum_{j=1}^n \lambda_j \bar{y}_{s_2 j}^b - \bar{y}_{s_2 p}^b \right] \\ s.t. \quad & \sum_{j=1}^n \lambda_j x_{ej}^n \leq x_{ep}^n, \quad e = 1, 2, \dots, m_3, \\ & \sum_{j=1}^n \lambda_j y_{lj}^n \geq y_{lp}^n, \quad l = 1, 2, \dots, s_3, \\ & \sum_{j=1}^n \lambda_j = 1, \\ & \lambda_j \geq 0, \quad j = 1, 2, \dots, n. \end{aligned}$$

In fact, Theorem 1 shows that the minimax formulation where a particular super-ideal point is defined as the reference one can be used to generate DEA scores and the corresponding composite inputs and outputs in the same way as the general combined-oriented DEA model. Moreover, this minimax formulation can also be used to develop the interactive STOM procedure that helps the DM to search for the MPS on the efficient frontier by systematically changing the weighting parameters.

The steps of the generalized interactive method used to identify the most preferred performance target of the DM for the inefficient DMU_p are summarized as follows:

Step 1: Solve the general combined-oriented DEA Model (36) for all DMUs to identify the efficient and inefficient DMUs.

Step 2: Solve Model (37) for each inefficient DMU_p and obtain the target unit of DMU_p based on formulation (38). If the DM accepts the target unit obtained as the MPS, then Stop. Else, go to Step 3.

Step 3: Find $f_i^{ref} (i = 1, \dots, m_1), \bar{f}_k^{ref} (k = 1, \dots, m_2), g_r^{ref} (r = 1, \dots, s_1)$ and $\bar{g}_t^{ref} (t = 1, \dots, s_2)$ using formulations (55), (56), (57) and (58), respectively. Let $z = 1$.

Step 4: Find $f_i^z(\lambda^*), \bar{f}_k^z(\lambda^*), g_r^z(\lambda^*)$ and $\bar{g}_t^z(\lambda^*)$ based on formulations (42), (43), (44) and (45), respectively, where λ^* is the optimal solution of Model (37).

Step 5: Classify each of the main components (desirable inputs, undesirable inputs, desirable outputs and undesirable outputs) of the target unit into three categories: the class of values that should be improved further, the class of values that have to be maintained and the class of values that must be relaxed. Let:

- $\{\hat{x}_{ip}^g, i \in I_i^{g,z}\}$: The desirable inputs that should be improved further, where $\{\Delta f_i^{g,z}, i \in I_i^{g,z}\}$ are the amounts to be improved.
- $\{\hat{x}_{ip}^g, i \in I_M^{g,z}\}$: The desirable inputs that have to be maintained.
- $\{\hat{x}_{ip}^g, i \in I_R^{g,z}\}$: The desirable inputs that should be relaxed, where $\{\Delta f_i^{g,z}, i \in I_R^{g,z}\}$ are the amounts to be relaxed.
- $\{\hat{x}_{kp}^b, k \in I_i^{b,z}\}$: The undesirable inputs that should be improved further, where $\{\Delta f_k^{b,z}, k \in I_i^{b,z}\}$ are the amounts to be improved.
- $\{\hat{x}_{kp}^b, k \in I_M^{b,z}\}$: The undesirable inputs that have to be maintained.
- $\{\hat{x}_{kp}^b, k \in I_R^{b,z}\}$: The undesirable inputs that should be relaxed, where $\{\Delta f_k^{b,z}, i \in I_R^{b,z}\}$ are the amounts to be relaxed.
- $\{\hat{y}_{rp}^g, r \in O_r^{g,z}\}$: The desirable outputs that should be improved further, where $\{\Delta g_r^{g,z}, r \in O_r^{g,z}\}$ are the amounts to be improved.

$\{\hat{y}_{tp}^g, r \in O_M^{g,z}\}$: The desirable outputs that have to be maintained.

$\{\hat{y}_{tp}^g, r \in O_R^{g,z}\}$: The desirable inputs that should be relaxed, where $\{\Delta g_r^{g,z}, r \in O_R^{g,z}\}$ are the amounts to be relaxed.

$\{\hat{y}_{tp}^b, t \in O_I^{b,z}\}$: The undesirable outputs that should be improved further, where $\{\Delta g_t^{b,z}, t \in O_I^{b,z}\}$ are the amounts to be improved.

$\{\hat{y}_{tp}^b, t \in O_M^{b,z}\}$: The undesirable outputs that have to be maintained.

$\{\hat{y}_{tp}^b, t \in O_R^{b,z}\}$: The desirable inputs that should be relaxed, where $\{\Delta g_t^{b,z}, t \in O_R^{b,z}\}$ are the amounts to be relaxed.

According to this classification, denote by:

$\{f_i^z, i \in FI_I^{g,z}\}$: The objective functions that are improved further, where $\{\Delta f_i^z, i \in FI_I^{g,z}\}$ are the amounts improved.

$\{f_i^z, i \in FI_M^{g,z}\}$: The objective functions that are maintained.

$\{f_i^z, i \in FI_R^{g,z}\}$: The objective functions that are relaxed, where $\{\Delta f_i^z, i \in FI_R^{g,z}\}$ are the amounts relaxed.

$\{f_k^z, k \in FI_I^{b,z}\}$: The objective functions that are improved further, where $\{\Delta f_k^z, k \in FI_I^{b,z}\}$ are the amounts improved.

$\{f_k^z, k \in FI_M^{b,z}\}$: The objective functions that are maintained.

$\{f_k^z, k \in FI_R^{b,z}\}$: The objective functions that are relaxed, where $\{\Delta f_k^z, k \in FI_R^{b,z}\}$ are the amounts relaxed.

$\{g_r^z, r \in FO_I^{g,z}\}$: The objective functions that are improved further, where $\{\Delta g_r^z, r \in FO_I^{g,z}\}$ are the amounts improved.

$\{g_r^z, r \in FO_M^{g,z}\}$: The objective functions that are maintained.

$\{g_r^z, r \in FO_R^{g,z}\}$: The objective functions that are relaxed, where $\{\Delta g_r^z, r \in FO_R^{g,z}\}$ are the amounts relaxed.

$\{g_t^z, t \in FO_I^{b,z}\}$: The objective functions that are improved further, where $\{\Delta g_t^z, t \in FO_I^{b,z}\}$ are the amounts improved.

$\{g_t^z, t \in FO_M^{b,z}\}$: The objective functions that are maintained.

$\{g_t^z, t \in FO_R^{b,z}\}$: The objective functions that are relaxed, where $\{\Delta g_t^z, t \in FO_R^{b,z}\}$ are the amounts relaxed.

Based on these categories, the new point of reference is defined as follows:

$$\begin{aligned}
 q_i^z &= f_i^z(\lambda^*) - \Delta f_i^z \quad \forall i \in FI_I^{g,z} \\
 q_i^z &= f_i^z(\lambda^*) \quad \forall i \in FI_M^{g,z} \\
 q_i^z &= f_i^z(\lambda^*) + \Delta f_i^z \quad \forall i \in FI_R^{g,z} \\
 \bar{q}_k^z &= \bar{f}_k^z(\lambda^*) + \Delta \bar{f}_k^z \quad \forall k \in FI_I^{b,z} \\
 \bar{q}_k^z &= \bar{f}_k^z(\lambda^*) \quad \forall k \in FI_M^{b,z} \\
 \bar{q}_k^z &= \bar{f}_k^z(\lambda^*) - \Delta \bar{f}_k^z \quad \forall k \in FI_R^{b,z} \\
 \pi_r^z &= g_r^z(\lambda^*) + \Delta g_r^z \quad \forall r \in FO_I^{g,z} \\
 \pi_r^z &= g_r^z(\lambda^*) \quad \forall r \in FO_M^{g,z} \\
 \pi_r^z &= g_r^z(\lambda^*) - \Delta g_r^z \quad \forall r \in FO_R^{g,z} \\
 \bar{\pi}_t^z &= \bar{g}_t^z(\lambda^*) - \Delta \bar{g}_t^z \quad \forall t \in FO_I^{b,z} \\
 \bar{\pi}_t^z &= \bar{g}_t^z(\lambda^*) \quad \forall t \in FO_M^{b,z} \\
 \bar{\pi}_t^z &= \bar{g}_t^z(\lambda^*) + \Delta \bar{g}_t^z \quad \forall t \in FO_R^{b,z}
 \end{aligned}
 \tag{63}$$

Step 6: Given the new point of reference obtained in Step 5, the new weights are computed as follows:

$$\begin{aligned}
 w_i^z &= \frac{1}{f_i^{ref} - q_i^z} \quad i = 1, 2, \dots, m_1 \\
 \bar{w}_k^z &= \frac{1}{\bar{f}_k^{ref} - \bar{q}_k^z} \quad k = 1, 2, \dots, m_2 \\
 v_r^z &= \frac{1}{g_r^{ref} - \pi_r^z} \quad r = 1, 2, \dots, s_1 \\
 \bar{v}_t^z &= \frac{1}{\bar{g}_t^{ref} - \bar{\pi}_t^z} \quad t = 1, 2, \dots, s_2
 \end{aligned}
 \tag{64}$$

Step 7: Solve the super-ideal point Model (61) with the new weights computed in Step 6 in order to find the optimal value σ^* .

Step 8: Let $\psi_p^* = T^{\max} - \sigma^*$ and solve Model (37).

Step 9: Find the new target unit of DMU_p based on formulation (38). If the DM accepts the target unit obtained as the MPS, then Stop. Else, let $z = z + 1$ and go to Step 4.

Fig. 1 Summarizes the main steps composing the generalized interactive methods just described.

4. Measuring the performance of high schools in the District of Philadelphia

The generalized interactive DEA-MOLP method introduced in the current paper has a substantial advantage over other DEA approaches and, in particular, over the standard parametric economic approach. Namely, our method allows DMs to adjust the different controllable inputs and outputs of a DMU when selecting the MPS while subject to budgetary restrictions. That is, it allows DMs to study how the interaction and potential trade-offs among the different input and output variables affect the efficiency of a DMU and its relative distance to the frontier. This feature, which is essential when focusing on real-life applications, is illustrated using the current example, together with the step-by-step implementation process of our interactive method.

DEA has been used to measure school efficiency for more than twenty years [68], but improvements that account for all the potential factors affecting efficiency are still required [65]. In particular, the literature lacks a general framework to represent the capacity adjustment process of the DMUs when subject to budgetary constraints. That is, the identification of the input and output variables that can be improved constitutes only part of the solution. Accounting for the capacity of the DM to suggest potential improvements and verify the relative increase in efficiency when different subsets of inputs and outputs are improved to one degree or another, completes the solution process.

[17] emphasized the necessity of a formal framework to analyze the strategic budgetary decisions and trade-offs faced by educational leaders, who must retrieve and use data to determine the efficient combinations of resources for public schools given their effect on student achievements [30]. The flexibility of our DEA-MOLP method allows for a direct implementation of the diagnosis and design approach to effective school management [64] while accounting for budgetary constraints.

Our case study is motivated by the persistent decline in district school enrollment across the School District of Philadelphia and the contrasting sustained increase in charter school enrollment. Fig. 2 illustrates both these trends. Due to this decline in enrollment, the different districts decided to close at least 23 school buildings [69]. It is generally agreed that policymakers use school efficiency as a critical performance indicator [14], with school competition having a direct positive effect on performance [37]. Thus, we will analyze the relative efficiency of the schools in the District of

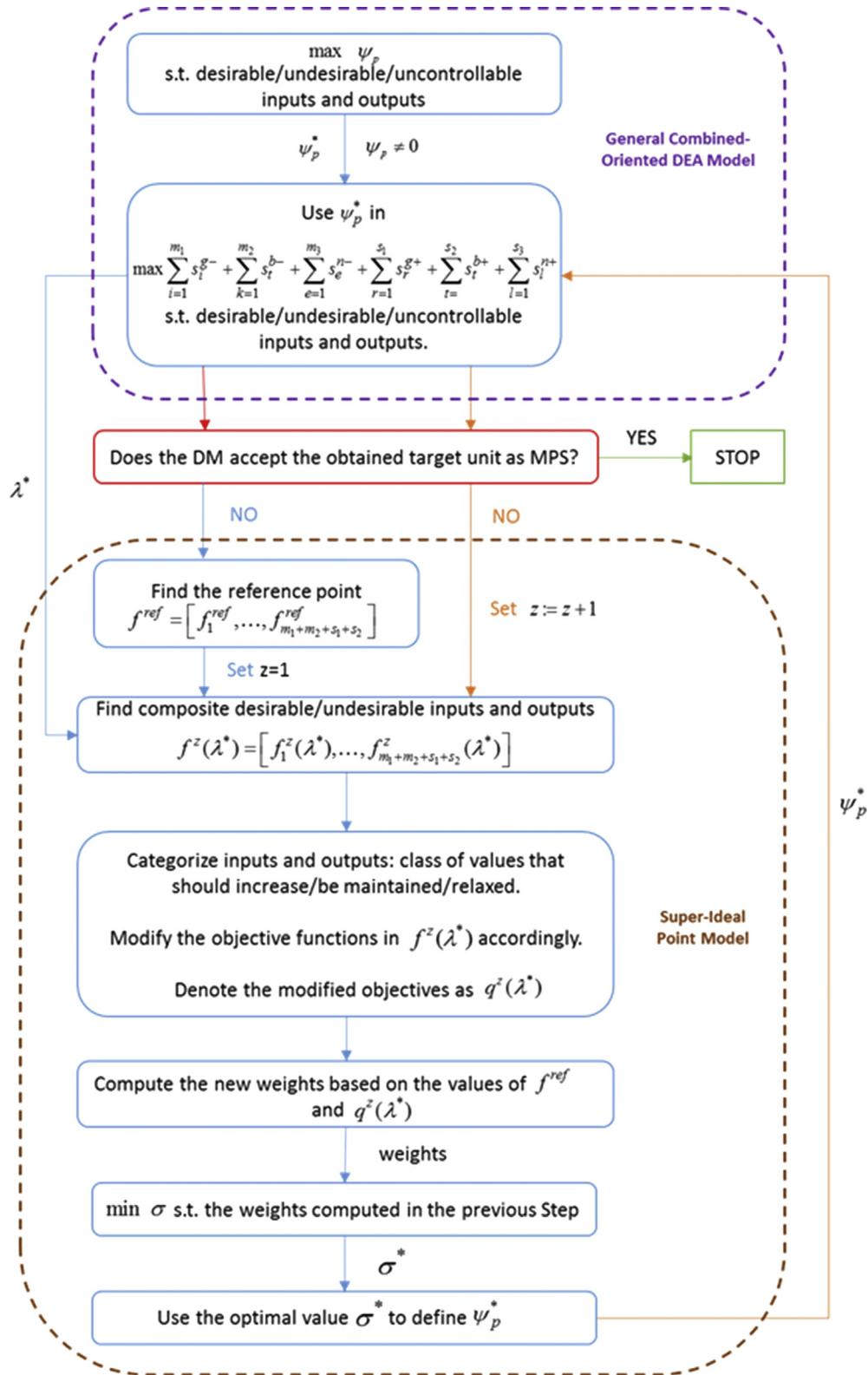


Fig. 1. The generalized interactive DEA-MOLP method.

Philadelphia using data retrieved from the Mayor’s Office of Education.

[12] follows an econometric approach to study the factors determining school closures using a panel of Illinois schools from 1991 to 2005. As in our case study, the budgetary constraints to

which district leaders and administrators are subject make school closures a potentially attractive choice, which aims at improving both fiscal conditions and student outcomes [12,56] highlights the complexity of the problem and emphasizes that, besides low enrollments and high per-student expenditures, several other factors

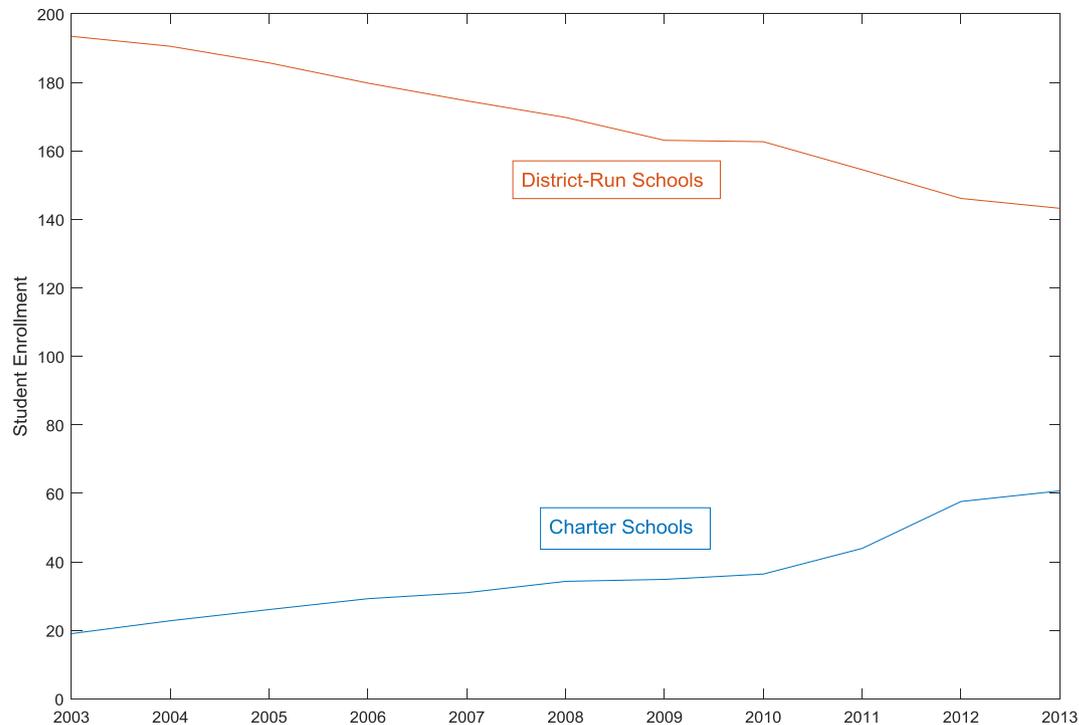


Fig. 2. Student enrollment trends: District-Run versus Charter schools. Data source: The Pew Charitable Trusts (2013).

Table 2
Statistics of the dataset.

Data	Kind	Min	Max	Average	S.D.
Student/Faculty Ratio	Desirable-controllable input	8	25	14.02	2.749473
Attendance	Undesirable-controllable input	0.833488	0.9962	0.912313	0.047823
Instructional Spending Per Student	Desirable-uncontrollable input	5702	16661	8795.91	2176.21
Total SAT Score	Desirable-controllable output	1281	1898	1542.18	106.439
Reported Incidents Per 100	Undesirable-controllable output	0.021	19.67	10.59365	6.45627
Percent to College	Desirable-uncontrollable output	61	100	87.39	6.932381

are significant in determining the closure of a school. She identified the following significant factors common to the schools that were closed:

- Schools that were poorer and spent less on students → One of the direct consequences is an increase in the student/faculty ratio. It should be noted that empirical findings regarding class size are mixed, though much of the research describes a positive impact of smaller classes on student achievement [70].
- Schools that had lower attendance rates
- Schools where students were not performing well in math and reading → Total SAT score is a fundamental output variable illustrating school efficiency.
- Schools located in poorer and more rural residential areas. In particular [12], highlights the proportions of black and low income students as one of the most significant predictors of school closure → The number of reported incidents can be assumed to constitute an adequate proxy for this environmental feature.

In addition to these factors, which can be controlled to a certain extent by the DMs, external uncontrollable factors must be included in the analysis. For example, district and school policies determine the instructional spending per student, which, together with the socioeconomic status of students, affect their performance and the subsequent educational choices [1]. It should be noted that

when measuring the efficiency of Australian schools [13], emphasize the importance of controlling for the socioeconomic environment to prevent biased efficiency estimates arising from inappropriate comparisons.

Following the above literature, we consider in our analysis the variables described in Table 2 for a total of 100 schools in the District of Philadelphia during the year 2012.

The values of these variables for the 100 schools being analyzed are presented in Figs. 3 and 4. In particular, Figs. 3 and 4 represent the input and output factors whose values must be increased and decreased by the DMUs, respectively, in order to increase their efficiency. Thus, higher values represent higher efficiency in Fig. 3, while lower values imply a higher efficiency in Fig. 4. It should be noted that we have represented the uncontrollable factors on the z-axis of each figure. That is, the dispersion generated across different schools by the “Percent to college” and “Instructional spending” variables cannot be controlled by the DM. Note, in particular, the substantial dispersion to which DMUs are subject in terms of the “Percent to college” variable.

We provide below a detailed description of the different steps that must be followed when implementing our DEA-MOLP interactive method to study the efficiency of these 100 schools.

Step 1: We solve the general combined-oriented DEA Model (36) for all the DMUs in order to identify the efficient and inefficient ones. The translation vectors considered are ($u = 1.9962$,

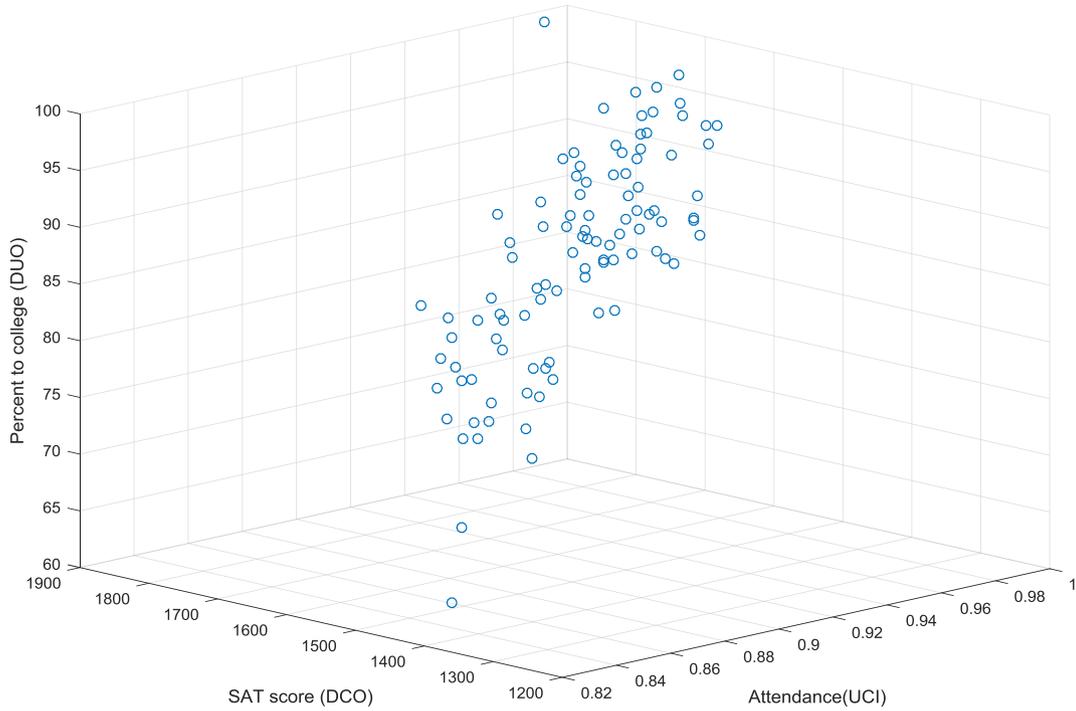


Fig. 3. Input and output factors positively related to efficiency.

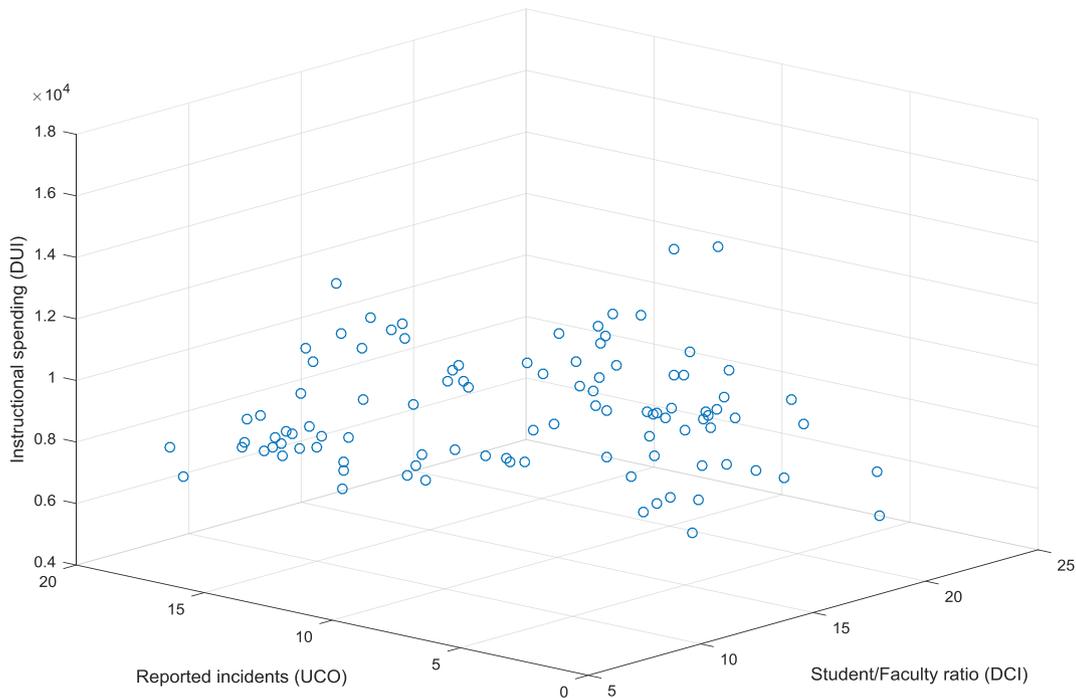


Fig. 4. Input and output factors negatively related to efficiency.

$u' = 20.670$), while the components of the directional vectors are given by $(d_{1x}^g = 17, \bar{d}_{1x}^b = 0.162712, d_{1y}^g = 617, \bar{d}_{1y}^b = 19.649)$. The DMUs with $\psi_p^* = 0$ are efficient and those with $\psi_p^* > 0$ are inefficient. In the latter case, the value of $E_p^* = 1 - \psi_p^*$ gives the efficiency scores of the inefficient units under consideration. The results are presented in Table 3.

As shown in Table 3, units 21, 22, 23, 35, 37, 39, 40, 41, 46, 48, 50, 51, 55, 66, 78, 91 and 98 were efficient schools. In order to produce a complete ranking of the DMUs, we have to implement any one of the DEA ranking methods available in the literature. Among the DEA ranking methods available, a simple and often used approach is super-efficiency. The basic idea of the super-efficiency approach introduced by Ref. [5] is that the DMU under evaluation should be eliminated from the DMUs that determine the reference set.

Applying this specific modification to Model (36) we get the super-efficiency formulation:

$$\begin{aligned}
 & \max \psi_p \\
 & s.t. \\
 & \sum_{j=1, j \neq p}^n \lambda_j x_{ij}^g \leq x_{ip}^g - \psi_p d_{ix}^g \quad i = 1, 2, \dots, m_1, \\
 & \sum_{j=1, j \neq p}^n \lambda_j \bar{x}_{kj}^b \leq \bar{x}_{kp}^b - \psi_p \bar{d}_{kx}^b \quad k = 1, 2, \dots, m_2, \\
 & \sum_{j=1, j \neq p}^n \lambda_j x_{ej}^n \leq x_{ep}^n \quad e = 1, 2, \dots, m_3, \\
 & \sum_{j=1, j \neq p}^n \lambda_j y_{rj}^g \geq y_{rp}^g + \psi_p d_{ry}^g \quad r = 1, 2, \dots, s_1, \\
 & \sum_{j=1, j \neq p}^n \lambda_j \bar{y}_{tj}^b \geq \bar{y}_{tp}^b + \psi_p \bar{d}_{ty}^b \quad t = 1, 2, \dots, s_2, \\
 & \sum_{j=1, j \neq p}^n \lambda_j y_{lj}^n \geq y_{lp}^n \quad l = 1, 2, \dots, s_3, \\
 & \sum_{j=1, j \neq p}^n \lambda_j = 1, \\
 & \lambda_j \geq 0 \quad j = 1, 2, \dots, n, j \neq p.
 \end{aligned} \tag{65}$$

as well as the corresponding super-efficiency scores of the efficient schools. These scores are presented in Table 3, together with the resulting ranking of all the schools.

Step 2: We solve Model (37) in order to obtain a target unit for each inefficient school. We illustrate the implementation of our approach by considering the first inefficient unit, i.e. DMU 1. After solving the version of Model (37) that corresponds to DMU 1, the following optimal solution is found:

$$\begin{aligned}
 \lambda_{40}^* &= 0.68640674, \\
 \lambda_{41}^* &= 0.22574904, \\
 \lambda_{50}^* &= 0.08784422, \\
 \lambda_j^* &= 0 \quad (j \neq 40, 41, 50)
 \end{aligned} \tag{66}$$

This means that the composite unit of DMU 1 on the efficient frontier can be represented as a linear combination composed of 0.68640674 of unit 40, 0.22574904 of unit 41, and 0.08784422 of unit 50. In this case, based on formulation (38), the following virtual unit was used to evaluate the performance of DMU 1:

$$(\hat{x}_1, \hat{y}_1) = (10.77425096, 0.98506419, 10147.00000000, 1717.66395046, 1.07937294, 94.36365386) \tag{67}$$

This virtual unit could be regarded as a target unit for DMU 1. In this regard, it should be noted that the values of the inputs and outputs of DMU 1 are given by:

$$(x_1, y_1) = (16.00000000, 0.91892200, 10147.00000000, 1528.00000000, 10.12500000, 87.00000000)$$

These values imply that for DMU 1 to become efficient, the value of the first (desirable) input should be reduced from 16 to 10.77425096, the value of the second (undesirable) input should be increased from 0.918922 to 0.98506419, the value of the first (desirable) output should be increased from 1528 to 1717.66395046, and the value of second (undesirable) output should be decreased from 10.125 to 1.07937294.

However, we will assume that the DM does not accept the DEA composite unit proposed as the most preferred solution for DMU 1, noting that it is difficult to decrease the “Reported Incidents per 100

Table 3
Efficiency scores, supper efficiency (SE) scores and ranking.

DMU	1 - ψ_p	SE	Rank	DMU	1 - ψ_p	SE	Rank
1	0.692603		90	51	1	-0.11685	12
2	0.734776		80	52	0.804714		70
3	0.953598		24	53	0.769419		72
4	0.717318		88	54	0.691842		91
5	0.858523		50	55	1	-0.83842	17
6	0.820769		61	56	0.821011		60
7	0.646314		95	57	0.995218		18
8	0.877996		45	58	0.719667		87
9	0.912773		34	59	0.993252		19
10	0.838933		53	60	0.818276		63
11	0.4406		100	61	0.918252		31
12	0.816123		65	62	0.602177		99
13	0.905799		36	63	0.807504		68
14	0.818838		62	64	0.637461		96
15	0.827652		57	65	0.845489		51
16	0.835826		56	66	1	-0.1396	13
17	0.927848		30	67	0.864378		48
18	0.914585		33	68	0.705434		89
19	0.888202		41	69	0.736093		78
20	0.95428		23	70	0.739232		77
21	1	-0.16482	14	71	0.885068		42
22	1	-0.03596	7	72	0.961909		22
23	1	-0.70588	16	73	0.724966		85
24	0.975595		21	74	0.725977		83
25	0.896643		37	75	0.735525		79
26	0.825173		59	76	0.947826		25
27	0.806296		69	77	0.883744		43
28	0.893593		39	78	1	-0.04151	8
29	0.811803		66	79	0.747509		75
30	0.881324		44	80	0.669259		93
31	0.871294		47	81	0.935338		28
32	0.827003		58	82	0.918066		32
33	0.928214		29	83	0.678513		92
34	0.843177		52	84	0.733488		81
35	1	-0.02226	4	85	0.89338		40
36	0.606295		98	86	0.750813		74
37	1	-0.0022	1	87	0.992834		20
38	0.94507		26	88	0.733403		82
39	1	-0.06289	9	89	0.838012		54
40	1	-0.11286	11	90	0.809635		67
41	1	-0.07473	10	91	1	-0.02811	5
42	0.859274		49	92	0.895261		38
43	0.625366		97	93	0.817665		64
44	0.837131		55	94	0.719948		86
45	0.725549		84	95	0.74096		76
46	1	-0.01211	3	96	0.912519		35
47	0.775299		71	97	0.754439		73
48	1	-0.27224	15	98	1	-0.0074	2
49	0.937258		27	99	0.66721		94
50	1	-0.03257	6	100	0.87521		46

students” by an amount of 9.04562706. Therefore, the approach proposed in this study is needed to search for the MPS along the frontier for DMU 1.

Step 3: Based on Equations (55)-(58), we have computed $f_1^{ref} = 10.19448947$, $\bar{f}_1^{ref} = 0.09757446$, $g_1^{ref} = 370$ and $\bar{g}_1^{ref} = 11.78303079$.

Step 4: Based on Equations (42)-(45), we have computed $f_1^1(\lambda^*) = 5.22574904$, $\bar{f}_1^1(\lambda^*) = 0.06614219$, $g_1^1(\lambda^*) = 189.66395046$ and $\bar{g}_1^1(\lambda^*) = 9.04562706$, where λ^* is the optimal solution obtained in (66).

Step 5: As indicated above, the DM considers that is difficult to decrease the “Reported Incidents per 100 students” from 10.125 to 1.07937294. However, he agrees to decrease its value from 10.125 to 7.00. At the same time, in order to compensate for this partial improvement on the “reported incidents” factor, he agrees to

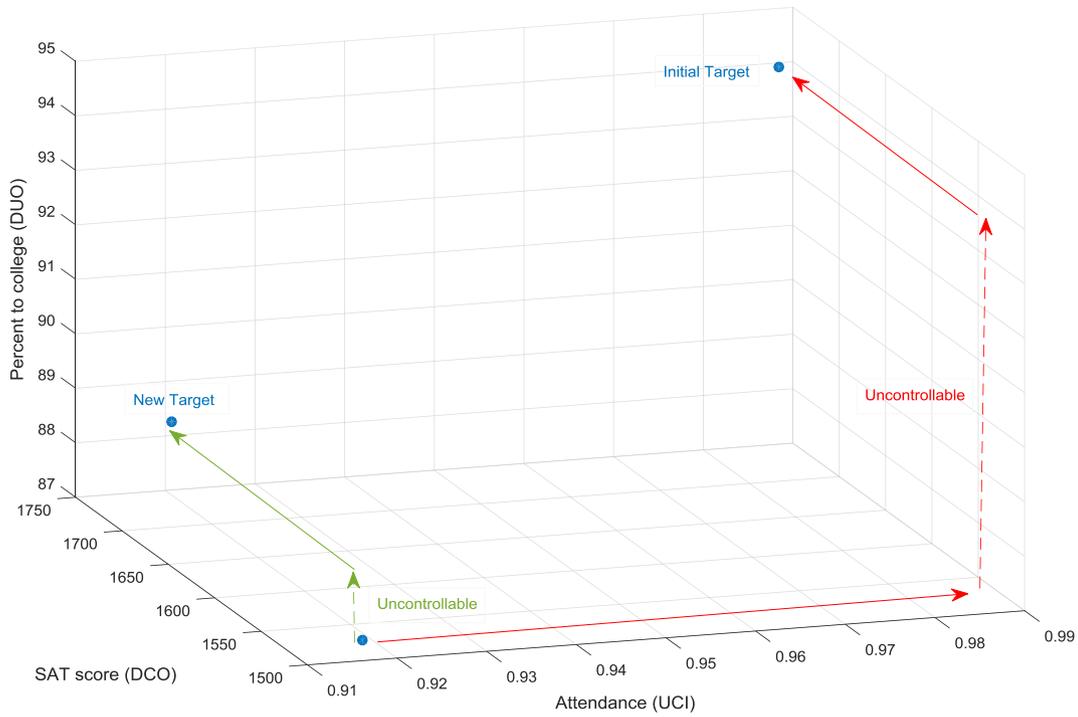


Fig. 5. Differences in efficiency improvement required by the initial and the new target: factors positively related to efficiency.

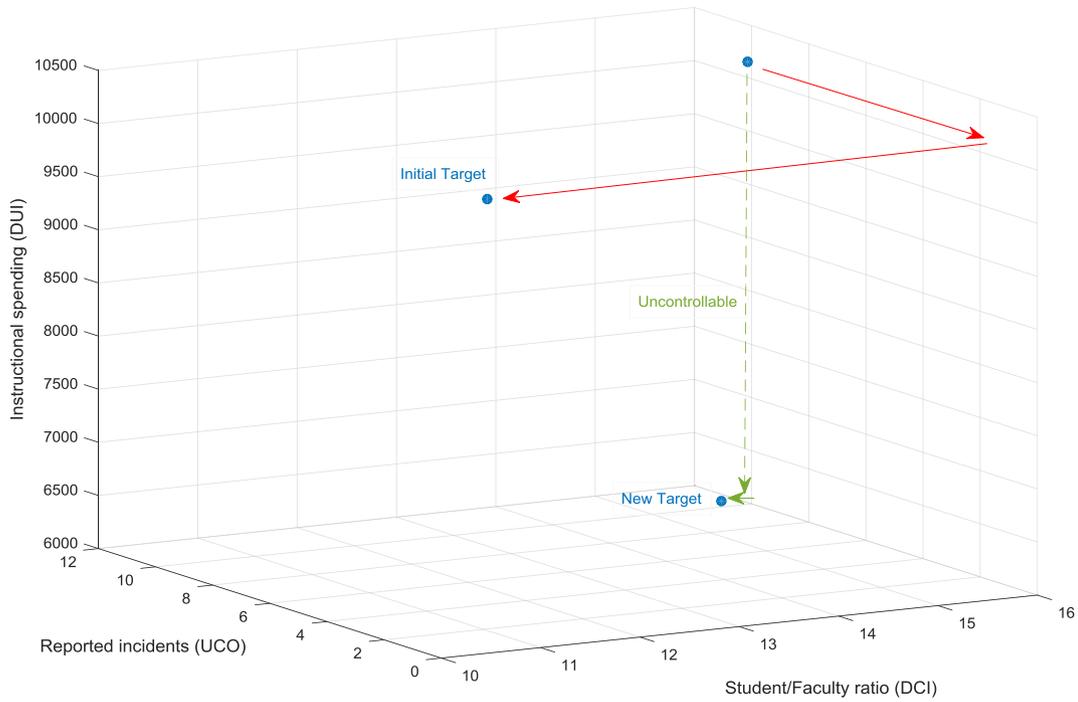


Fig. 6. Differences in efficiency improvement required by the initial and the new target: factors negatively related to efficiency.

decrease the “Student/Faculty Ratio” from 16.00 to 14.00.

Step 6: Given the previous efficiency trade-off defined by the DM, the new weights are given by:

$$\begin{aligned}
 w_1^1 &= 0.14349796, \\
 \bar{w}_1^1 &= 8.08759960, \\
 \nu_1^1 &= 0.0027027, \\
 \bar{\nu}_1^1 &= -1.23824248
 \end{aligned}
 \tag{68}$$

Step 7: We solve the super-ideal point Model (61) with the new weights given in (68), which leads to the optimal value $\sigma^* = 0.63731312$.

Step 8: Given $T^{\max} = 0.59967585$, we obtain $\psi_1^* = T^{\max} - \sigma^* = -0.03763727$. We solve Model (37), which leads to the following optimal solution:

$$\begin{aligned}\lambda_{48}^* &= 0.57353354, \\ \lambda_{55}^* &= 0.42646646, \\ \lambda_j^* &= 0 \quad (j \neq 48, 55)\end{aligned}\quad (69)$$

Step 9: We use Equation (38) to define the new target unit for DMU 1, which is given by:

$$\begin{aligned}(\hat{x}_1, \hat{y}_1) &= (15.70586582, \quad 0.91279796, \quad 6049.14369522, \\ &\quad 1676.04251849, \quad 10.02858681, \quad 89.24992975)\end{aligned}\quad (70)$$

If the DM is satisfied with the target level of inputs and outputs derived from the trade-offs defined between the factors, then the interactive process is terminated and the MPS is determined.

Figs. 5 and 6 represent the differences in efficiency improvement required from DMU 1 by the initial target (following from the initial combined-oriented DEA model) and the new target (generated after implementing the super-ideal point model) located on the efficient frontier. In particular, Fig. 6 illustrates how, after implementing the trade-off suggested by the DM between both factors, the corresponding target on the efficient frontier is mainly defined by uncontrollable improvements. Similarly, Fig. 5 shows that the trade-off implemented leaves the DMU requiring an improvement in its SAT score, while being attendance-efficient.

4.1. Applications to public sector decision making

The case study presented in Section 4 contributes to the increasing interest of the literature in the potential applications of DEA to analyze efficiency in the public sector [24]. In this regard, the European Commission has already warned about the significant variability in the efficiency of public spending on education and R&D across European countries [55]. These authors emphasize the existing potential for improvement in the efficiency of public spending while noting the difficulties faced to find suitable data to measure and evaluate efficiency.

It has however been suggested that the budget received by a DMU should be directly related to its performance, particularly when dealing with the expenditures of the public sector [62] review the empirical literature analyzing the effectiveness of performance budgeting, i.e. the use of performance information to relate the funds provided to public institutions and their outputs. These authors conclude that the literature has shown that performance information can be used when allocating budgets to improve both allocative and productive efficiency.

However, it has also been recognized that the allocation of public resources based on the performance of DMUs is subject to multiple political, economic, legal, and organizational factors, as [49] show for US state governments. Indeed, the current literature acknowledges the multiple factors and actors affecting the performance of any public DMU as well as the biased evaluation incentives of the different groups monitoring and affected by the performance of public sector activities.

For example [31], illustrate how the interest in performance information varies significantly among DMs such as politicians and senior managers, who must actually use the corresponding information to make relevant budget decisions. Moreover, when

measuring the relative efficiency of different rail transit systems [54], conclude that highly subsidized systems are, on average, less efficient than those less highly subsidized. However, and more importantly, they believe that these subsidies are mainly transferred to the passengers in the form of reduced fares.

At the same time, there has been a considerable increase in the availability and spread of software allowing for a direct implementation of DEA [42] compares the outcome of four alternative DEA models using empirical data from 90 primary schools in the State of Geneva, Switzerland. He describes the divergent results delivered by these models and concludes that DMs may be prone to select the model that fits better with their own preferences, leading to ineffective decisions.

Given the fact that biased data can be provided in order to influence regulatory processes [2], implement super-efficiency as an outlier detector method when dealing with strategic reporting in the presence of data and model uncertainty. Super-efficiency is incorporated in our model so as to allow DMs to adjust different input and output objectives in a straightforward manner when improving the efficiency of a given DMU while being subject to budgetary restrictions. Note that this is a highly subjective process subject to strategic incentives on the side of the DMs, which become particularly relevant when dealing with unreliable data in underdeveloped areas.

Therefore, we conclude by emphasizing that one of the main findings of [12] is the higher likelihood of closings exhibited by rural schools. In this regard, the proposed model can be directly applied within an economic development setting. For example, the World Bank poverty reports have consistently highlighted the limited access of the poor to secondary education as one of the main causes of the increase in inequality observed between urban and rural areas [77]. Based on this empirical evidence [45], applied DEA to study the inequalities arising from the rural and urban divide in Thailand. Given the stricter budgetary restrictions suffered by developing countries and, in particular, the rural areas within them, our integrated model can be used to design efficient educational and schooling policies while accounting for the existing budget differentials across countries.

5. Concluding remarks

In this paper, we have established an equivalence relationship between MOLP problems and combined-oriented DEA models using a direction distance function designed to account for desirable and undesirable inputs and outputs together with uncontrollable variables. The resulting hybrid DEA-MOLP model constitutes the basis on which to apply the STOM interactive technique so as to locate the MPS along the efficient frontier for each DMU. The MPS generated using the STOM method provides rich insights into the performance assessment and the efficiency analysis of each DMU while accounting for realistic and technically feasible target values that incorporate the value judgments of DMs.

Among the main contributions of the hybrid DEA-MOLP method introduced in this paper, we should highlight that it extends the standard DEA analysis so as to allow the DM to consider different efficiency improvement strategies when subject to budgetary restrictions. Our case study concentrates on the education area, whose institutions are generally subject to pecuniary constraints, but we could have also analyzed hospitals or multinational firms that must consider the relative efficiency of their subsidiaries when distributing resources among them.

It should be emphasized that even though we have used extended strongly free disposability to handle undesirable factors, there are many practical situations, such as banking and environment performance evaluation, requiring weakly free disposability,

i.e. a reduction in the desirable outputs proportional to that in the undesirable ones. Thus, it may be more practical to establish an equivalence between DEA and MOLP models for situations where the inputs and outputs follow weakly free disposability. Moreover, the approach applied to deal with both desirable and undesirable factors is based on the BCC-DEA model of [8]; which cannot be used when dealing with constant-return to scale technologies. Our proposed method should therefore be extended to account for problems based on constant-return to scale technologies.

In future research, this hybrid method should be developed to consider strategic environments in which the values of inputs and outputs are uncertain. Moreover, further research should be performed to compare the results obtained using the current method with those following from the implementation of other interactive MOLP methods such as STEM, G-D-F, Wierzbicki, and Zionts-Wallenius. These comparisons should aim at understanding the applicability of different interactive MOLP methods to specific data sets and DM preferences.

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Appendix. Proof of Theorem 1

Using Equations (42)–(45) and Equations (51)–(54), the general combined-oriented DEA Model (36) can be rewritten as:

$$\begin{aligned}
 & \max \quad \psi_p \\
 & \text{s.t.} \\
 & \psi_p \frac{1}{w_i} - f_i(\lambda) \leq 0, \quad i = 1, 2, \dots, m_1, \\
 & \psi_p \frac{1}{\bar{w}_k} - \bar{f}_k(\lambda) \leq 0, \quad k = 1, 2, \dots, m_2, \\
 & \psi_p \frac{1}{v_r} - g_r(\lambda) \leq 0, \quad r = 1, 2, \dots, s_2, \\
 & \psi_p \frac{1}{\bar{v}_t} - \bar{g}_t(\lambda) \leq 0, \quad t = 1, 2, \dots, s_2, \\
 & \lambda \in \mathcal{A}_p
 \end{aligned} \tag{A1}$$

The first m_1 constraints in (A1) can be equivalently represented as follows:

$$\begin{aligned}
 & \psi_p \frac{1}{w_i} - f_i(\lambda) \leq 0 \Leftrightarrow -w_i f_i(\lambda) \leq -\psi_p \\
 & \Leftrightarrow T^{\max} - w_i f_i(\lambda) \leq T^{\max} - \psi_p \\
 & \Leftrightarrow w_i \left(\frac{T^{\max}}{w_i} - f_i(\lambda) \right) \leq T^{\max} - \psi_p \\
 & \Leftrightarrow w_i (f_i^{\text{ref}} - f_i(\lambda)) \leq \sigma
 \end{aligned} \tag{A2}$$

Similarly, the second m_2 constraints, the third s_1 constraints and the fourth s_2 constraints in (A1) can be respectively rewritten as follows:

$$\psi_p \frac{1}{\bar{w}_k} - \bar{f}_k(\lambda) \leq 0 \Leftrightarrow \bar{w}_k (\bar{f}_k^{\text{ref}} - \bar{f}_k(\lambda)) \leq \sigma \tag{A3}$$

$$\psi_p \frac{1}{v_r} - g_r(\lambda) \leq 0 \Leftrightarrow v_r (g_r^{\text{ref}} - g_r(\lambda)) \leq \sigma \tag{A4}$$

$$\psi_p \frac{1}{\bar{v}_t} - \bar{g}_t(\lambda) \leq 0 \Leftrightarrow \bar{v}_t (\bar{g}_t^{\text{ref}} - \bar{g}_t(\lambda)) \leq \sigma \tag{A5}$$

Also, the objective function of Model (A1) becomes:

$$\max \quad \psi_p = -\min (T^{\max} - \psi_p) = -\min \sigma \tag{A6}$$

Since $\hat{f}_{ip} = f_i(\lambda^*)$, Equation (55) implies that for any $\lambda \in \mathcal{A}_p$ we have:

$$f_i^{\text{ref}} = \frac{T^{\max}}{w_i} \geq \frac{w_i \hat{f}_{ip}}{w_i} = \hat{f}_{ip} = \max_{\lambda \in \mathcal{A}_p} f_i(\lambda), \quad i = 1, 2, \dots, m_1, \tag{A7}$$

In a similar way, from Equations (56)–(58) we have, respectively:

$$\bar{f}_k^{\text{ref}} = \frac{T^{\max}}{\bar{w}_k} \geq \frac{w_k \hat{f}_{kp}}{w_k} = \hat{f}_{kp} = \max_{\lambda \in \mathcal{A}_p} \bar{f}_k(\lambda), \quad k = 1, 2, \dots, m_2, \tag{A8}$$

$$g_r^{\text{ref}} = \frac{T^{\max}}{v_r} \geq \frac{v_r \hat{g}_{rp}}{v_r} = \hat{g}_{rp} = \max_{\lambda \in \mathcal{A}_p} g_r(\lambda), \quad r = 1, 2, \dots, s_1, \tag{A9}$$

$$\bar{g}_t^{\text{ref}} = \frac{T^{\max}}{\bar{v}_t} \geq \frac{\bar{v}_t \hat{g}_{tp}}{\bar{v}_t} = \hat{g}_{tp} = \max_{\lambda \in \mathcal{A}_p} \bar{g}_t(\lambda), \quad t = 1, 2, \dots, s_2, \tag{A10}$$

Equation (A7)–(A10) imply that for any $\lambda \in \mathcal{A}_p$:

$$f_i^{\text{ref}} - f_i(\lambda) \geq 0, \quad i = 1, 2, \dots, m_1, \tag{A11}$$

$$\bar{f}_k^{\text{ref}} - \bar{f}_k(\lambda) \geq 0, \quad k = 1, 2, \dots, m_2, \tag{A12}$$

$$g_r^{\text{ref}} - g_r(\lambda) \geq 0, \quad r = 1, 2, \dots, s_1, \tag{A13}$$

$$\bar{g}_t^{\text{ref}} - \bar{g}_t(\lambda) \geq 0, \quad t = 1, 2, \dots, s_2, \tag{A14}$$

Thus, for any $\lambda \in \mathcal{A}_p$, we have

$$\begin{aligned}
 \sigma &= T^{\max} - \psi_p \geq T^{\max} - w_i f_i(\lambda) \geq w_i \hat{f}_{ip} - w_i f_i(\lambda) \\
 &= w_i (\hat{f}_{ip} - f_i(\lambda)) \geq 0, \quad i = 1, 2, \dots, m_1,
 \end{aligned} \tag{A15}$$

$$\begin{aligned}
 \sigma &= T^{\max} - \psi_p \geq T^{\max} - \bar{w}_k \bar{f}_k(\lambda) \geq w_k \hat{f}_{kp} - w_k \bar{f}_k(\lambda) \geq 0, \\
 k &= 1, 2, \dots, m_2,
 \end{aligned} \tag{A16}$$

$$\begin{aligned}
 \sigma &= T^{\max} - \psi_p \geq T^{\max} - v_r g_r(\lambda) \geq v_r \hat{g}_{rp} - v_r g_r(\lambda) \geq 0, \\
 r &= 1, 2, \dots, s_1,
 \end{aligned} \tag{A17}$$

$$\begin{aligned}
 \sigma &= T^{\max} - \psi_p \geq T^{\max} - \bar{v}_t \bar{g}_t(\lambda) \geq \bar{v}_t \hat{g}_{tp} - \bar{v}_t \bar{g}_t(\lambda) \geq 0, \\
 t &= 1, 2, \dots, s_2,
 \end{aligned} \tag{A18}$$

Since Equations (A2)–(A6) hold true, the equivalence model between the general combined-oriented DEA Model (36) and the super-ideal point Model (41) can be established.

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