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A new two-stage Stackelberg fuzzy data envelopment analysis model

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ABSTRACT

Data Envelopment Analysis (DEA) is a widely used mathematical programming approach for evaluating the relative efficiency of Decision Making Units (DMUs). Conventional DEA methods treat DMUs as "black boxes", focusing entirely on their relative efficiencies. We propose an efficient two-stage fuzzy DEA model to calculate the efficiency scores for a DMU and its sub-DMUs. We use the Stackelberg (leader–follower) game theory approach to prioritize and sequentially decompose the efficiency score of the DMU into a set of efficiency scores for its sub-DMUs. The proposed models are linear and independent of the α-cut variables. The linear feature allows for a quick identification of the global optimum solution and the α-cut independency feature allows for a significant reduction in the computational efforts. Monte Carlo simulation is used to discriminately rank the efficiencies in the proposed method. We also present a case study to exhibit the efficacy of the procedures and to demonstrate the applicability of the proposed method to a two-stage performance evaluation problem in the banking industry.

1. Introduction

Data Envelopment Analysis (DEA) is a non-parametric mathematical programming approach that evaluates a group of Decision Making Units (DMUs) with comparative efficiencies. DEA generalizes the Farrell [24] single-input single-output ratio to a multiple-input multiple-output ratio by using a ratio of the weighted sum of outputs to the weighted sum of inputs [15]. Although DEA can evaluate the relative efficiency of a set of DMUs, it cannot identify the sources of inefficiency in the DMUs because conventional DEA models view DMUs as "black boxes" that consume a set of inputs to produce a set of outputs. A two-stage DEA model extends the basic DEA methodology by considering two sub-DMUs in a main DMU and allowing a "look inside" the DMU to study the sources of inefficiencies [59]. The sub-DMUs are related through a series relation and all the outputs of the first sub-DMU in Stage 1 are used as inputs of the second sub-DMU in Stage 2. The outputs of the first sub-DMU, known as "intermediate measures", are considered as the inputs of the second sub-DMU and no extra inputs are supplied for the second sub-DMU except for the outputs of the first sub-DMU [19,55].

In addition, the conventional DEA methods such as CCR [14] and BBC [7] require accurate measurement of both the

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inputs and outputs. However, the observed values of the input and output data in real-world problems are often imprecise or vague. Imprecise or vague data may be the result of unquantifiable, incomplete, and/or non-obtainable information. In recent years, many researchers have formulated DEA models to deal with the uncertain input and output data. One way to manipulate uncertain data in DEA is via probability distributions. However, probability distributions require either *a priori* predictable regularity or *a posteriori* frequency determinations which are difficult to construct. Cooper et al. [20] has studied this problem in the context of interval data. However, many real-life problems use linguistic variables that cannot be represented with interval values. An alternative approach is to represent the imprecise or vague values by membership functions of fuzzy set theory. Fuzzy set algebra developed by Zadeh [67] is the formal body of theory that allows the treatment of imprecise estimates in uncertain environments.

In this study, we propose an efficient method for solving two-stage fuzzy DEA problems and demonstrate the applicability of the proposed method in the banking industry. The proposed model has the following unique features: (1) it is independent of the α-cut variables (minimizing the computational efforts); (2) it does not require the step-size of α-cut variables based on heuristic rules and trial and error; (3) it is linear (producing global optimum solutions); and (4) it uses a procedure based on the Monte Carlo simulation method to discriminately rank the efficient DUMs and sub-DMUs.

In Section 2, we review performance measurement in the banking industry and examine the two-stage DEA models in the literature. In Section 3, we review the conventional two-stage DEA method. In Section 4, we present the details of the two-stage fuzzy DEA method proposed in this study. In Section 5, we present a case study to exhibit the efficacy of the procedures and to demonstrate the applicability of the proposed method to a two-stage performance evaluation problem in the banking industry. In Section 6, we conclude with our conclusions and future research directions.

2. Literature review

Evaluating the overall performance and monitoring the financial condition of commercial banks has been the focus of numerous research studies since the early works of Greenbaum [28] and Benston [10]. DEA has been widely used to measure the relative efficiency of a set of bank branches that possess shared functional goals with disproportionate inputs and outputs [3–6,16,18,35,46,49,50,55,57,58,63]. Rho and An [55] extended two-stage DEA models by considering input and output slacks. They applied their model to the data from the banking industry and compared the results with those of the previous two-stage DEA models. Akther et al. [2] proposed a two-stage DEA model with a slacks-based inefficiency measure and directional technology distance function for evaluating the performance of private commercial banks in Bangladesh. Paradi et al. [53] proposed a two-stage DEA model for simultaneously benchmarking the performance of bank branches along different dimensions. They used a modified slacks-based measure to aggregate the obtained efficiency scores from stage one and generated a composite performance index for each bank branch. Their approach improved the validity of the performance assessment method and enabled branch managers to clearly identify the strengths and weaknesses in their operations. See Liu et al. [47] for a recent survey of DEA application.

Two fundamental input–output systems are used to calculate the bank efficiencies [30]: the production approach [22,27,54] and the intermediation approach [8,9,26,31,32,48,60,61]. There is no commonly accepted approach for measuring the efficiency in the banking industry, which is why different efficiency scores are obtained using similar data [11].

Favero and Papi [25] used a sample of 174 banks and they tried to determine which of the two DEA models are more appropriate for describing the efficiency level in banking: constant returns to scale (CRS) or variable returns to scale (VRS). Pastor et al. [52] used DEA and a non-parametric approach to estimate the efficiency in the European and U.S. banking systems. They chose three outputs (loans, other productive assets, and deposits) and two inputs (non-interest expenses and personal expenses) to estimate the efficiency level in their study. Yildirim [65] used the DEA methodology to study the technical and scale efficiencies of the Turkish commercial banks. Casu and Girardone [12] used DEA to study the efficiency of the Italian banking system. Isık and Hassan [33] showed that DEA methodology could be utilized to analyze the performance of banks in transition countries. Özkan-Günay and Tektas [51] used nonparametric DEA methodology to conduct a similar study. Yildirim and Philippatos [66] evaluated the efficiency level of commercial banks in central and East-European countries by employing the stochastic frontier approach (SFA) and DFA techniques. Halkos and Tzeremes [29] proposed a bootstrapped DEA-based procedure to pre-calculate and pre-evaluate the short-run operating efficiency gains of a potential bank merger or acquisition.

Conventional DEA models consider the system under study as a black box which consumes inputs and produces outputs [4]. In such cases, using single-stage DEA may result in inaccurate efficiency evaluation [55]. In contrast, a two-stage DEA model allows one to further investigate the structure and processes inside the DMU, to identify the misallocation of inputs among sub-DMUs and generate insights about the sources of inefficiency within the DMU [23,44].

Seiford and Zhu [56] developed a two-stage DEA method characterized by profitability and marketability for evaluating commercial banks. In the first stage, they measured profitability by using labor and assets as inputs, and profits and revenue as outputs. In the second stage, they measured marketability by using the profits and revenue (outputs from Stage 1) as inputs, and market value, returns and earnings per share as outputs. Zhu [68] applied the same two-stage process to the Fortune Global 500 companies. Casu and Molyneux [13] used the non-parametric DEA approach to investigate the efficiency degree of the European banking system. They used the intermediation approach to specify two outputs (total loans and other earning assets) and two inputs (total costs and total deposits) for their study. Kao and Hwang [36] developed a two-stage DEA modeling that considered the series relationship of the two sub-processes within the whole process. The efficiency of a DMU was decomposed into efficiencies of the two sub-DMUs through their framework. Cook et al. [19] classified the two-stage DEA models into four categories: the standard DEA approach, the efficiency decomposition approach, the network-DEA approach, and the game-theoretic approach.

Abtahi and Khalili-Damghani [1] proposed a mathematical formulation for measuring the performance of agility in supply chains using single-stage fuzzy DEA. Khalili-Damghani et al. [38] applied the proposed formulation of Abtahi and Khalili-Damghani [1] to measure the efficiency of agility in supply chains and used simulation to rank the interval efficiency scores of Abtahi and Khalili-Damghani [1]. Khalili-Damghani and Taghavifard [39] proposed a fuzzy two-stage DEA approach for performance measurement in supply chains. They used linguistic terms parameterized with fuzzy sets to model qualitative and vague criteria in their proposed fuzzy two-stage approach designed for agility performance measurement in supply chains.

Khalili-Damghani et al. [40] used the ordinal Likert-based data in a new two-stage DEA approach for agility performance and illustrated the efficacy of their approach in a supply chain. Khalili-Damghani and Taghavifard [41] performed sensitivity and stability analysis in two-stage DEA models with fuzzy data. They proposed several models for calculating the stability radius in DEA problems with considerable input and output variations and uncertainties. Recently, Khalili-Damghani and Tavana [42] proposed a new network DEA model for measuring the performance of agility in supply chains. The uncertainty of the input and output data were modeled with linguistic terms and the proposed model was used to measure the performance of agility in a real-life case study in the dairy industry.

Kao and Liu [37] proposed a method for solving fuzzy two-stage DEA problems by extending the two-stage DEA efficiency decomposition model of Kao and Hwang [36]. They used a two-level optimization model to overcome the problem of intermediate measures. Although their model was able to decompose efficiency in two-stage DEA models, it had several drawbacks:

First, their proposed method used non-linear mathematical programming for the lower-bound efficiency calculations and could generate local optimum solutions in practice. Second, their method used α-cut dependent models which could be solved for different α-cuts. This resulted in an enormous number of calculations in real-world problems because of the lack of a known rule for determining the best step-size for the α-cut values. Third, they used a non-linear programming model for the lower-bound of efficiency calculation which was not straightforward. The decision variables could not be achieved directly and they referred to the optimality theorems of linear programming in which the reduced costs of the slack variables in the dual model were equal to the values of the decision variables in the primal model. Fourth, although they used a linear programming model for the upper-bound of efficiency calculation, they did not propose any optimization model for the sub-DMUs. They used the optimum values of the decision variables in the main DMU model and calculated the efficiency scores of the sub-DMUs. As a result, different combinations of the efficiencies scores for the sub-DMUs could result in a single efficiency score for the main DMU because it was possible to have multiple-optimum solutions for the decision variable values. Therefore, their model could not verify whether the values of the decision variables were unique or not. Fifth, while the knowledge of the final rankings of the DMUs and sub-DMUs is useful in real-world problems, they did not suggest any formal procedures for finding these rankings.

Cook et al. [19] have shown that all two-stage DEA models can be categorized as using either Stackelberg (leader–follower) or cooperative game concepts. We propose an efficient two-stage fuzzy DEA method that addresses the aforementioned shortfalls and can be classified as a Stackelberg (leader–follower) model [19]. We decompose the efficiency score of a two-stage DMU and use the Stackelberg (leader–follower) game [45] to calculate the efficiency scores of its sub-DMUs.

3. Conventional two-stage DEA method

In this section we briefly review the two-stage DEA method of Kao and Hwang [36] depicted in Fig. 1:

Let us consider a two-stage DEA system and assume that each DMU$_j$ ($j = 1, 2, \ldots, n$) consumes m inputs $x_{ij}(i = 1, 2, \ldots, m)$ to produce D outputs $z_{dj}(d = 1, 2, \ldots, D)$ in the first stage. The D outputs of the first stage are then used as inputs in the second stage to produce $y_{rj}(r = 1, 2, \ldots, s)$ outputs. For a given DMU$_j$, $e_j, e_j^1,$ and e_j^2 denotes the overall efficiency score, the efficiency score of the first stage, and the efficiency score of the second stage, respectively. By using the CRS input-oriented DEA model proposed by Charnes et al. [14] and assuming equal multipliers for all intermediate measures at each stage, the following linear program is presented:

$$
\begin{aligned}
& e_o = \text{Max} \sum_{r=1}^{s} u_r y_{ro} \\
& \text{s.t.} \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} \leqslant 0, \quad j = 1, 2, \ldots, n; \\
& \quad \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \leqslant 0, \quad j = 1, 2, \ldots, n; \\
& \quad \sum_{i=1}^{m} v_i x_{io} = 1; \\
& \quad w_d \geqslant \varepsilon, \quad d = 1, 2, \ldots, D; \\
& \quad v_i \geqslant \varepsilon, \quad i = 1, 2, \ldots, m; \\
& \quad u_r \geqslant \varepsilon, \quad r = 1, 2, \ldots, s.
\end{aligned}
\tag{1}
$$

Model (1) proposed by Kao and Hwang [36] represents the overall efficiency of the two-stage DEA system depicted in Fig. 1. By assuming a unique solution for Model (1), e_j^1 and e_j^2can be calculated. However, in cases where the optimal

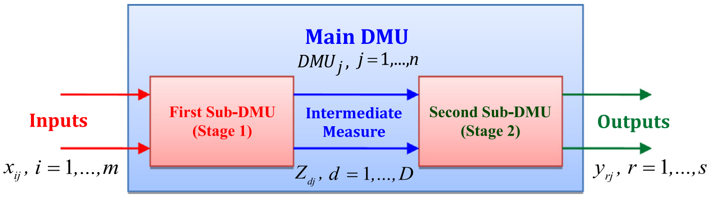

Fig. 1. Two-stage DEA process with crisp parameters.

multipliers of Model (1) are not unique, Kao and Hwang [36] have proposed a couple of models for calculating the maximum achievable values for e_j^1 and e_j^2, and testing whether the efficiency of the sub-DMU in each stage is unique or not. The minimum values of e_j^1 and $e_j^2$are calculated as $e_j^{2-} = e_j/e_j^{1+}$ and, $e_j^{1-} = e_j/e_j^{2+}$, respectively. It is clear that the efficiency values of the sub-DMUs are uniquely determined through Model (1) if and only if $e_j^{1-} = e_j^{1+}$ or $e_j^{2-} = e_j^{2+}$. Otherwise, an alternative decomposition of e_j^1 and e_j^2 can be achieved [45].

There are other alternative procedures in the two-stage DEA modeling literature. Wang and Chin [62] and Li et al. [44] have addressed the most well-known strategies in decomposing the efficiency score of a DMU into the efficiency scores of its sub-DMUs. Kao and Hwang [36] defined the overall efficiency of a two-stage process as the product of the efficiencies of two individual stages. Chen et al. [17] modeled the overall efficiency of a two-stage process as a weighted sum of the efficiencies of the two individual stages. Wang and Chin [62] modeled the overall efficiency of a two-stage process as a weighted harmonic mean of the efficiencies of the two individual stages. The two-stage DEA method proposed in this study can be customized for all three aforementioned models (i.e., multiplication, weighted sum, and weighted harmonic mean). In other words, the assumption of decomposing the efficiency score of a DMU does not invalidate the method proposed in this study.

In addition, in many real-life cases, all inputs in the first stage are not necessarily used in the second stage, and/or some extra inputs may be added to the system in the second the stage. The method proposed here can be easily customized to handle cases where all inputs in the first stage are not used in the second stage, and/or some extra inputs are sometimes added to the system in the second stage.

4. Proposed fuzzy two-stage DEA method

Fig. 2 shows a two-stage process, where each DMU is composed of two sub-DMUs in series, and intermediate products by the sub-DMU in stage 1 is consumed by the sub-DMU in stage 2.

This two-stage process is composed of fuzzy inputs, fuzzy intermediate measures and fuzzy outputs. Without loss of generality, trapezoidal fuzzy numbers (TrFNs) are used throughout the entire process. Let X be the universe of discourse, $X = \{x_1, x_2, \ldots, x_n\}$. A fuzzy set $\widetilde{A}$ of X is a set of order pairs $\left\{x_1, \mu_{\widetilde{A}}(x_1), x_2, \mu_{\widetilde{A}}(x_2), \ldots, x_n, \mu_{\widetilde{A}}(x_n)\right\}$ where $\mu_{\widetilde{A}} : X \rightarrow [0,1]$ is the membership function of $\widetilde{A}$, and $\mu_{\widetilde{A}}(x_i)$ represents the membership degree of x_i in $\widetilde{A}$. Let us consider the following definitions:

Definition 4.1 [69]. The α-cut $\widetilde{A}_\alpha$ and strong α-cut $\widetilde{A}_{\alpha+}$ of fuzzy set $\widetilde{A}$ in the universe of discourse X is defined by $\widetilde{A}_\alpha = \left\{x_i : \mu_{\widetilde{A}}(x_i) \geqslant \alpha, x_i \in X\right\}$, where $\alpha \in [0,1]$ and $\widetilde{A}_{\alpha+} = \left\{x_i : \mu_{\widetilde{A}}(x_i) > \alpha, x_i \in X\right\}$, where $\alpha \in [0, 1]$, respectively.

Definition 4.2 [69]. A trapezoidal fuzzy number (TrFN) can be defined as $\tilde{x} = (x^1, x^2, x^3, x^4)$, where the membership function $\mu_{\tilde{m}}$ of $\tilde{m}$is given as follows:

$$\mu(x) = \begin{cases} \frac{x-x^1}{x^2-x^1} \ (x^1 \leqslant x \leqslant x^2) \\ 1 \quad (x^2 \leqslant x \leqslant x^3) \\ \frac{x^4-x}{x^4-x^3} \ (x^3 \leqslant x \leqslant x^4) \end{cases}$$

where $[x^2, x^3]$ is called the *mode interval* of $\tilde{x}$, and x^1 and x^2 are called the *lower and upper limits* of $\tilde{x}$, respectively. As stated earlier, TrFNs are considered throughout this paper and all calculations are extended by this assumption.

Among the various types of fuzzy numbers, triangular and trapezoidal fuzzy numbers are the most important. We chose trapezoidal fuzzy numbers for this study because they are more general and most often used for characterizing imprecise, vague and ambiguous information in practical applications [43,64]. The frequent use of TrFNs is mainly attributed to their simplicity in both concept and computation. The proposed models can easily be extended for other well-known fuzzy membership functions such as triangular fuzzy numbers, Gaussian fuzzy numbers, generalized fuzzy numbers, and left–right fuzzy numbers.

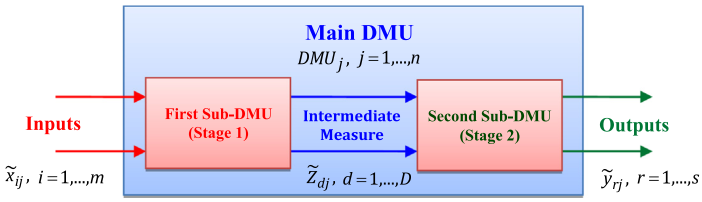

Fig. 2. Two-stage DEA process with fuzzy parameters.

4.1. Formulation of the two-stage fuzzy DEA model

In this study, the imprecise single-stage DEA model of Despotis and Smirlis [21], the two-stage DEA model of Kao and Hwang [36], and the two-stage fuzzy DEA model of Kao and Liu [37] are used to develop a comprehensive two-stage fuzzy DEA framework. Consider the TrFNs in the left and right spread format as inputs, intermediate measures, and outputs of n DMUs with two-stage processes. Each DMU$_j$ (j = 1, 2, ..., n) consumes m fuzzy inputs $\tilde{x}_{ij} = \left(x^1_{ij}, x^2_{ij}, x^3_{ij}, x^4_{ij}\right), i = 1, 2, \ldots, m$ to produce D intermediate measures $\tilde{z}_{dj} = \left(z^1_{dj}, z^2_{dj}, z^3_{dj}, z^4_{dj}\right), d = 1, 2, \ldots, d$ in the first stage. All D intermediate measures are then used as inputs in the second stage to produce s outputs $\tilde{y}_{rj} = \left(y^1_{rj}, y^2_{rj}, y^3_{rj}, y^4_{rj}\right), r = 1, 2, \ldots, s.$ Using an arbitrary α-cut for the inputs, the intermediate measures, the outputs, and the lower and upper-bounds of the membership functions are calculated as follows:

$$\left(x^L_{ij}\right)_{\alpha_i} = x^1_{ij} + \alpha_i\left(x^2_{ij} - x^1_{ij}\right), \quad \alpha_i \in [0,1], \quad i = 1, \ldots, m; \quad j = 1, \ldots, n \tag{2}$$

$$\left(x^U_{ij}\right)_{\alpha_i} = x^4_{ij} - \alpha_i\left(x^4_{ij} - x^3_{ij}\right), \quad \alpha_i \in [0,1], \quad i = 1, \ldots, m; \quad j = 1, \ldots, n \tag{3}$$

$$\left(z^L_{dj}\right)_{\alpha_r} = z^1_{dj} + \alpha_d\left(z^2_{dj} - z^1_{dj}\right), \quad \alpha_d \in [0,1], \quad d = 1, \ldots, D; \quad j = 1, \ldots, n \tag{4}$$

$$\left(z^U_{rj}\right)_{\alpha_r} = z^4_{dj} - \alpha_d\left(z^4_{dj} - z^3_{dj}\right), \quad \alpha_d \in [0,1], \quad d = 1, \ldots, D; \quad j = 1, \ldots, n \tag{5}$$

$$\left(y^L_{rj}\right)_{\alpha_r} = y^1_{rj} + \alpha_r\left(y^2_{rj} - y^1_{rj}\right), \quad \alpha_r \in [0,1], \quad r = 1, \ldots, s; \quad j = 1, \ldots, n \tag{6}$$

$$\left(y^U_{rj}\right)_{\alpha_r} = y^4_{rj} - \alpha_r\left(y^4_{rj} - y^3_{rj}\right), \quad \alpha_r \in [0,1], \quad r = 1, \ldots, s; \quad j = 1, \ldots, n \tag{7}$$

4.2. Upper-bound of the efficiency values for the main DMU

Model (8) is proposed to calculate the upper bound (e^U_o) of the efficiency values for the main DMU as follows:

$$\begin{aligned}
\text{Max } & e^U_o = \sum_{r=1}^{s} u_r \left(y^U_{ro}\right)_{\alpha_r} \\
\text{s.t. } & \sum_{r=1}^{s} u_r \left(y^L_{rj}\right)_{\alpha_r} - \sum_{d=1}^{D} w_d z_{dj} \leqslant 0, \quad j = 1, 2, \ldots, n; \quad j \neq o; \\
& \sum_{r=1}^{s} u_r \left(y^U_{ro}\right)_{\alpha_r} - \sum_{d=1}^{D} w_d z_{do} \leqslant 0; \\
& \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i \left(x^U_{ij}\right)_{\alpha_i} \leqslant 0, \quad j = 1, 2, \ldots, n; \quad j \neq o; \\
& \sum_{d=1}^{D} w_d z_{do} - \sum_{i=1}^{m} v_i \left(x^L_{io}\right)_{\alpha_i} \leqslant 0; \\
& \sum_{i=1}^{m} v_i \left(x^L_{io}\right)_{\alpha_i} = 1; \\
& v_i \geqslant \varepsilon, \quad i = 1, 2, \ldots, m; \\
& w_d \geqslant \varepsilon, \quad d = 1, 2, \ldots, D; \\
& u_r \geqslant \varepsilon, \quad r = 1, 2, \ldots, s.
\end{aligned} \tag{8}$$

In Model (8), the input variables take the lower-bound and the output variables take the upper-bound for the DMU under consideration. For all other DMUs, the input variables take the upper-bound and the output variables take the lower-bound.

The intermediate measures emerge in two different sets of constraints in Model (8) and therefore cannot be determined using a single-level optimization model. In addition, the intermediate measures, z_{dj}, are the outputs of the first stage and the inputs of the second stage concurrently. As a result, according to the optimistic view point, the output of a given DMU should be set equal to the maximum possible value (i.e., z^U_{dj}). Similarly, the input of a given DMU should be set equal to the minimum possible value (i.e., z^L_{dj}) according to this optimistic view point. Since the intermediate measures, z_{dj}, are outputs in the first stage, they should be set equal to z^U_{dj}. Moreover, as intermediate measures, z_{dj}, are inputs in the second stage, and therefore should be set equal to z^L_{dj}. This leads to a logical contradiction in which the proper values for the intermediate measures, z_{dj}, cannot be determined uniquely in a single-level optimization model.

Kao and Liu [37] proposed the two-level optimization Model (9) for determining the optimum values of the intermediate measures in which the objective function of Model (8) is at its highest possible value (for an arbitrary α-cut level).

$$\underset{\left(z^L_{dj}\right)_{\alpha_d} \leqslant z_{dj} \leqslant \left(z^U_{dj}\right)_{\alpha_d}, \forall j,d}{\text{Max}} \left\{ \begin{aligned}
\text{Max } & e^U_o = \sum_{r=1}^{s} u_r \left(y^U_{ro}\right)_{\alpha_r} \\
\text{s.t. } & \sum_{r=1}^{s} u_r \left(y^L_{rj}\right)_{\alpha_r} - \sum_{d=1}^{D} w_d z_{dj} \leqslant 0, \quad j = 1, 2, \ldots, n; \quad j \neq o; \\
& \sum_{r=1}^{s} u_r \left(y^U_{ro}\right)_{\alpha_r} - \sum_{d=1}^{D} w_d z_{do} \leqslant 0; \\
& \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i \left(x^U_{ij}\right)_{\alpha_i} \leqslant 0, \quad j = 1, 2, \ldots, n; \quad j \neq o; \\
& \sum_{d=1}^{D} w_d z_{do} - \sum_{i=1}^{m} v_i \left(x^L_{io}\right)_{\alpha_i} \leqslant 0; \\
& \sum_{i=1}^{m} v_i \left(x^L_{io}\right)_{\alpha_i} = 1; \\
& v_i \geqslant \varepsilon, \quad i = 1, 2, \ldots, m; \\
& w_d \geqslant \varepsilon, \quad d = 1, 2, \ldots, D; \\
& u_r \geqslant \varepsilon, \quad r = 1, 2, \ldots, s.
\end{aligned} \right. \tag{9}$$

Z_{dj}, $\forall j, d$ are the decision variables for the outer-level optimization problem and considered as the constant multipliers for the inner-level optimization problem. Hence, the two-level optimization (9) cannot be solved in its current form and should be reduced to a single-level optimization model. Fortunately, the orientation of both objective functions in Model (9) is maximization and this model can be replaced with the following single-level optimization (maximization) Model (10):

$$\begin{aligned}
\text{Max } & e^U_o = \sum_{r=1}^{s} u_r \left(y^U_{ro}\right)_{\alpha_r} \\
\text{s.t. } & \sum_{r=1}^{s} u_r \left(y^L_{rj}\right)_{\alpha_r} - \sum_{d=1}^{D} w_d z_{dj} \leqslant 0, \quad j = 1, 2, \ldots, n; \quad j \neq o;
\end{aligned}$$

$$
\begin{aligned}
&\sum_{r=1}^{s} u_r (y_{ro}^U)_{\alpha_r} - \sum_{d=1}^{D} w_d z_{do} \leqslant 0;\\
&\sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i \left(x_{ij}^U\right)_{\alpha_i} \leqslant 0,\ j=1,2,\ldots,n;\ j \neq o;\\
&\sum_{d=1}^{D} w_d z_{do} - \sum_{i=1}^{m} v_i (x_{io}^L)_{\alpha_i} \leqslant 0;\\
&\sum_{i=1}^{m} v_i (x_{io}^L)_{\alpha_i} = 1;\\
&\left(z_{dj}^L\right)_{\alpha_d} \leqslant z_{dj} \leqslant \left(z_{dj}^U\right)_{\alpha_d},\ j=1,2,\ldots,n;\ j\neq o;\\
&\quad d=1,2,\ldots,D;\\
&(z_{do}^L)_{\alpha_d} \leqslant z_{do} \leqslant (z_{do}^U)_{\alpha_d},\ d=1,2,\ldots,D;\\
&v_i \geqslant \varepsilon,\ i=1,2,\ldots,m;\\
&w_d \geqslant \varepsilon,\ d=1,2,\ldots,D;\\
&u_r \geqslant \varepsilon,\ r=1,2,\ldots,s.
\end{aligned}
\tag{10}
$$

Model (10) is a non-linear mathematical programming model and its global optimum cannot be found easily. Moreover, Model (10) is dependent on the α-cut and should be solved for different α-cut levels with a pre-determined step-size. We first replace the values of Eqs. (2)–(7) in Model (10) and construct Model (11) as follows:

$$
\begin{aligned}
\text{Max } e_o^U = &\sum_{r=1}^{s} u_r (y_{ro}^4 - \alpha_r (y_{ro}^4 - y_{ro}^3))\\
\text{s.t.}\quad &\sum_{r=1}^{s} u_r \left(y_{rj}^1 + \alpha_r (y_{rj}^2 - y_{rj}^1)\right) - \sum_{d=1}^{D} w_d z_{dj} \leqslant 0,\\
&\quad j=1,2,\ldots,n;\quad j\neq o;\\
&\sum_{r=1}^{s} u_r (y_{ro}^4 - \alpha_r (y_{ro}^4 - y_{ro}^3)) - \sum_{d=1}^{D} w_d z_{do} \leqslant 0;\\
&\sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i \left(x_{ij}^4 - \alpha_i (x_{ij}^4 - x_{ij}^3)\right) \leqslant 0,\\
&\quad j=1,2,\ldots,n;\quad j\neq o;\\
&\sum_{d=1}^{D} w_d z_{do} - \sum_{i=1}^{m} v_i (x_{io}^1 + \alpha_i (x_{io}^2 - x_{io}^1))_{\alpha_i} \leqslant 0;\\
&\sum_{i=1}^{m} v_i (x_{io}^1 + \alpha_i (x_{io}^2 - x_{io}^1)) = 1;\\
&\left(z_{dj}^1 + \alpha_d (z_{dj}^2 - z_{dj}^1)\right) \leqslant z_{dj}\\
&\quad \leqslant \left(z_{dj}^4 - \alpha_d \left(z_{dj}^4 - z_{dj}^3\right)\right),\quad j=1,2,\ldots,n;\\
&\qquad j\neq o;\ d=1,2,\ldots,D;\\
&(z_{do}^1 + \alpha_d (z_{do}^2 - z_{do}^1)) \leqslant z_{do}\\
&\quad \leqslant (z_{do}^4 - \alpha_d (z_{do}^4 - z_{do}^3)),\\
&d=1,2,\ldots,D;\\
&v_i \geqslant \varepsilon,\quad i=1,2,\ldots,m;\\
&w_d \geqslant \varepsilon,\quad d=1,2,\ldots,D;\\
&u_r \geqslant \varepsilon,\quad r=1,2,\ldots,s.
\end{aligned}
\tag{11}
$$

We then construct Model (12) by interchanging variables $\lambda_i = \alpha v_i, i=1,\ldots,m,$ where $0 \leqslant \lambda_i \leqslant v_i$, and $\eta_r = \beta u_r$, $r=1,\ldots,s$, where $0 \leqslant \eta_r \leqslant u_r$:

$$
\begin{aligned}
e_o^U = \text{Max} &\sum_{r=1}^{s} (u_r y_{ro}^4 - \eta_r (y_{ro}^4 - y_{ro}^3))\\
\text{s.t.}\quad &\sum_{r=1}^{s} \left(u_r y_{rj}^1 + \eta_r (y_{rj}^2 - y_{rj}^1)\right) - \sum_{d=1}^{D} w_d z_{dj} \leqslant 0,\\
&\quad j=1,2,\ldots,n;\quad j\neq o;\\
&\sum_{r=1}^{s} (u_r y_{ro}^4 - \eta_r (y_{ro}^4 - y_{ro}^3)) - \sum_{d=1}^{D} w_d z_{do} \leqslant 0;\\
&\sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} \left(v_i x_{ij}^4 - \lambda_i (x_{ij}^4 - x_{ij}^3)\right) \leqslant 0,\\
&\quad j=1,2,\ldots,n;\quad j\neq o;\\
&\sum_{d=1}^{D} w_d z_{do} - \sum_{i=1}^{m} (v_i x_{io}^1 + \lambda_i (x_{io}^2 - x_{io}^1)) \leqslant 0;\\
&\sum_{i=1}^{m} (v_i x_{io}^1 + \lambda_i (x_{io}^2 - x_{io}^1)) = 1;\\
&\left(z_{dj}^1 + \alpha_d (z_{dj}^2 - z_{dj}^1)\right) \leqslant z_{dj} \leqslant \left(z_{dj}^4 - \alpha_d (z_{dj}^4 - z_{dj}^3)\right),\\
&\quad j=1,2,\ldots,n;\ j\neq o;\quad d=1,2,\ldots,D;\\
&w_d \geqslant \varepsilon,\quad d=1,2,\ldots,D;\\
&v_i \geqslant \lambda_i \geqslant 0,\quad i=1,2,\ldots,m;\\
&u_r \geqslant \eta_r \geqslant 0,\quad r=1,2,\ldots,s.
\end{aligned}
\tag{12}
$$

Model (13) is also non-linear (due to the terms $w_d z_{dj}$ and $w_d z_{do}$) and dependent on the α-cut type variables α_d. These two concerns are resolved by using the following procedure. We multiply the sides of the intermediate measure inequalities by the positive value w_d:

$$
\begin{aligned}
&w_d \left(z_{dj}^1 + \alpha_d (z_{dj}^2 - z_{dj}^1)\right) \leqslant w_d z_{dj} \leqslant w_d \left(z_{dj}^4 - \alpha_d (z_{dj}^4 - z_{dj}^3)\right),\\
&\quad j=1,2,\ldots,n;\quad d=1,2,\ldots,D
\end{aligned}
$$

Model (13) is achieved by interchanging variables $\theta_d = \alpha_d w_d,\ d=1,2,\ldots,D$ where $0 \leqslant \theta_d \leqslant w_d$ and $\bar{z}_{dj} = w_d z_{dj};\ d=1,2,\ldots,D$ and $j = 1, 2, \ldots, n$ on the previous set of constraints in Model (13) as follows:

$$
\begin{aligned}
e_o^U = \text{Max} &\sum_{r=1}^{s} (u_r y_{ro}^4 - \eta_r (y_{ro}^4 - y_{ro}^3))\\
\text{s.t.}\quad &\sum_{r=1}^{s} \left(u_r y_{rj}^1 + \eta_r \left(y_{rj}^2 - y_{rj}^1\right)\right) - \sum_{d=1}^{D} \bar{z}_{dj} \leqslant 0,\\
&\quad j=1,2,\ldots,n;\quad j\neq o;\\
&\sum_{r=1}^{s} (u_r y_{ro}^4 - \eta_r (y_{ro}^4 - y_{ro}^3)) - \sum_{d=1}^{D} \bar{z}_{do} \leqslant 0;\\
&\sum_{d=1}^{D} \bar{z}_{dj} - \sum_{i=1}^{m} \left(v_i x_{ij}^4 - \lambda_i \left(x_{ij}^4 - x_{ij}^3\right)\right) \leqslant 0,\\
&\quad j=1,2,\ldots,n;\quad j\neq o;\\
&\sum_{d=1}^{D} \bar{z}_{do} - \sum_{i=1}^{m} (v_i x_{io}^1 + \lambda_i (x_{io}^2 - x_{io}^1)) \leqslant 0;\\
&\sum_{i=1}^{m} (v_i x_{io}^1 + \lambda_i (x_{io}^2 - x_{io}^1)) = 1;
\end{aligned}
$$

$$
\begin{aligned}
& w_d z^1_{dj} + \theta_d\left(z^2_{dj} - z^1_{dj}\right) \leqslant \bar{z}_{dj} \leqslant w_d z^4_{dj} - \theta_d\left(z^4_{dj} - z^3_{dj}\right), \\
& \quad j = 1,2,\ldots,n; \quad d = 1,2,\ldots,D; \\
& w_d \geqslant \theta_d \geqslant 0, \quad d = 1,2,\ldots,D; \\
& v_i \geqslant \lambda_i \geqslant 0, \quad i = 1,2,\ldots,m; \\
& u_r \geqslant \eta_r \geqslant 0, \quad r = 1,2,\ldots,s; \\
& \bar{z}_{dj} \geqslant 0, \quad d = 1,2,\ldots,D; \quad j = 1,2,\ldots,n.
\end{aligned} \tag{13}
$$

We can now find the global optimal solution for the single-stage linear optimization model (13) because it is independent of the α-cut levels. In addition, the problems with Kao and Liu's [37] method (i.e., the high volume of computational efforts, the determination of a proper step-size for the α-cut, the conflicting interval efficiency scores, and the conflicting DMU rankings) no longer exist in Model (13).

Theorem #1. *Model (13) is always feasible and bounded, and its optimal objective function is equal to unity.*

Proof. The dual form of Model (13) can be written as follows by defining a proper set of variables for each set of constraints:

$$
\omega = \text{Min}\left[\theta - \varepsilon\left(\sum_{i=1}^{m} s_i + \sum_{r=1}^{s} s_r + \sum_{d=1}^{D} s_d\right)\right]
$$

$$
\begin{aligned}
\text{s.t.} \quad & \sum_{\substack{j=1 \\ j\neq o}}^{n} p_j y^1_{rj} + p_o y^4_{ro} - s_r - \rho_r \geqslant y^4_{ro}, \quad r = 1,2,\ldots,s; \\
& \sum_{\substack{j=1 \\ j\neq o}}^{n} p_j\left(y^2_{rj} - y^1_{rj}\right) - p_o\left(y^4_{ro} - y^3_{ro}\right) - \gamma_r + \rho_r \\
& \qquad \geqslant -\left(y^4_{ro} - y^3_{ro}\right), \qquad r = 1,2,\ldots,s; \\
& -\sum_{\substack{j=1 \\ j\neq o}}^{n} \tau_j x^4_{ij} - \tau_o x^1_{io} + \theta x^1_{io} - s_i - \phi_i \geqslant 0, \quad i = 1,2,\ldots,m; \\
& \sum_{\substack{j=1 \\ j\neq o}}^{n} \tau_j\left(x^4_{ij} - x^3_{ij}\right) - \tau_o\left(x^2_{io} - x^1_{io}\right) + \theta\left(x^2_{io} - x^1_{io}\right) - \alpha_i + \phi_i \geqslant 0, \\
& \qquad i = 1,2,\ldots,m; \\
& \sum_{d=1}^{D}\sum_{j=1}^{n} q_{dj} z^1_{dj} - \sum_{d=1}^{D}\sum_{j=1}^{n} m_{dj} z^4_{dj} - s_d - \mu_d \geqslant 0, \quad d = 1,2,\ldots,D; \\
& \sum_{d=1}^{D}\sum_{j=1}^{n} q_{dj}\left(z^2_{dj} - z^1_{dj}\right) + \sum_{d=1}^{D}\sum_{j=1}^{n} m_{dj}\left(z^4_{dj} - z^3_{dj}\right) - \beta_d + \mu_d \geqslant 0, \\
& \qquad d = 1,2,\ldots,D; \\
& -\sum_{\substack{j=1 \\ j\neq o}}^{n} p_j + \sum_{\substack{j=1 \\ j\neq o}}^{n} \tau_j - \sum_{d=1}^{D}\sum_{j=1}^{n} q_{dj} + \sum_{d=1}^{D}\sum_{j=1}^{n} m_{dj} - \psi_{dj} \geqslant 0, \\
& \qquad d = 1,2,\ldots,D; \quad j = 1,2,\ldots,n; \\
& p_j, \tau_j \geqslant 0, \quad j = 1,2,\ldots,n; \\
& q_{dj}, m_{dj}, \psi_{dj} \geqslant 0, \quad d = 1,2,\ldots,D; \quad j = 1,2,\ldots,n; \\
& s_i, \alpha_i, \phi_i \geqslant 0, \quad i = 1,2,\ldots,m; \\
& s_d, \beta_d, \mu_d \geqslant 0, \quad d = 1,2,\ldots,D; \\
& s_r, \rho_r, \gamma_r \geqslant 0, \quad r = 1,2,\ldots,s; \quad \theta \text{ free}. \quad \square
\end{aligned} \tag{14}
$$

Consider an arbitrary solution for Model (14) as follows:

$$
\begin{aligned}
& p_j = \tau_j = 0, \quad j = 1,2,\ldots,n; \quad j \neq o; \\
& p_o = \tau_o = 1; \\
& q_{dj} = m_{dj} = \psi_{dj} = 0, \quad d = 1,2,\ldots,D, \quad j = 1,2,\ldots,n, \quad j \neq o; \\
& q_{do} = m_{do} = \psi_{do} = 1, \quad d = 1,2,\ldots,D; \\
& s_i = \alpha_i = \phi_i = 0, \quad i = 1,2,\ldots,m; \\
& s_d = \beta_d = \mu_d = 0, \quad d = 1,2,\ldots,D; \\
& s_r = \rho_r = \gamma_r = 0, \quad r = 1,2,\ldots,s; \\
& \theta = 1.
\end{aligned}
$$

This solution is always feasible because it satisfies all the constraints in Model (14) and is independent of the inputs, intermediate measures, outputs, and the α-cut level values. Appendix A shows how this solution satisfies the 6-th and 7-th set of constraints. Consequently, the dual form of Model (14), i.e., Model (13), always has a feasible solution. Clearly, in the above feasible solution, $\theta = 1$ and the slack variables are equal to zero. Hence, the optimum value of the objective function of Model (14) is less than or equal to1 (i.e., $\omega^* \leqslant 1$). Moreover, by the virtue of the dual theorem in linear programming, the optimal values of the objectives in Models (13) and (14) are equal (i.e., $\omega^* = e^{U^*}_o$). Therefore $\omega^* = e^{U^*}_o \leqslant 1$ and the optimal solution for Model (13) is also bounded. This completes the proof.

4.3. Lower-bound of the efficiency values for the main DMU

Model (15) is proposed for calculating the lower-bound (e^L_o) of the efficiency values for an arbitrary α-cut level.

$$
\begin{aligned}
& \text{Max}\, e^L_o = \sum_{r=1}^{s} u_r (y^L_{ro})_{\alpha_r} \\
\text{s.t.} \quad & \sum_{r=1}^{s} u_r (y^U_{rj})_{\alpha_r} - \sum_{d=1}^{D} w_d z_{dj} \leqslant 0, \quad j = 1,2,\ldots,n; \quad j \neq o; \\
& \sum_{r=1}^{s} u_r (y^L_{ro})_{\alpha_r} - \sum_{d=1}^{D} w_d z_{do} \leqslant 0; \\
& \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i (x^L_{ij})_{\alpha_i} \leqslant 0, \quad j = 1,2,\ldots,n; \quad j \neq o; \\
& \sum_{d=1}^{D} w_d z_{do} - \sum_{i=1}^{m} v_i (x^U_{io})_{\alpha_i} \leqslant 0; \\
& \sum_{i=1}^{m} v_i (x^U_{io})_{\alpha_i} = 1; \\
& v_i \geqslant \varepsilon, \quad i = 1,2,\ldots,m; \\
& w_d \geqslant \varepsilon, \quad d = 1,2,\ldots,D; \\
& u_r \geqslant \varepsilon, \quad r = 1,2,\ldots,s.
\end{aligned} \tag{15}
$$

In Model (15), the input variables assume the upper-bound and the output variables assume the lower-bound for the DMU under consideration. For all other DMUs, the input variables assume the lower-bound and the output variables assume the upper-bound. Model (15) cannot be solved using a single-level optimization model because of the emerging intermediate measures in the constraint sets. Again, similar to the procedure in the previous section, the following two-level optimization model is used to determine the optimum values of the intermediate measures where the objective function of Model (15) assumes the lowest possible values for an arbitrary α-cut level.

$$
\underset{(z_{dj}^{L})_{\alpha_d} \leqslant z_{dj} \leqslant (z_{dj}^{U})_{\alpha_d}, \forall j,d}{\text{Min}}
\left\{
\begin{array}{l}
\text{Max } e_o^L = \sum_{r=1}^{s} u_r (y_{ro}^L)_{\alpha_r} \\
\text{s.t.} \quad \sum_{r=1}^{s} u_r (y_{rj}^U)_{\alpha_r} - \sum_{d=1}^{D} w_d z dj \leqslant 0, \\
\qquad\qquad j = 1,2,\ldots,n; \quad j \neq o; \\
\quad \sum_{r=1}^{s} u_r (y_{ro}^L)_{\alpha_r} - \sum_{d=1}^{D} w_d z_{do} \leqslant 0; \\
\quad \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i (x_{ij}^L)_{\alpha_i} \leqslant 0, \\
\qquad\qquad j = 1,2,\ldots,n; \quad j \neq o; \\
\quad \sum_{d=1}^{D} w_d z_{do} - \sum_{i=1}^{m} v_i (x_{io}^U)_{\alpha_i} \leqslant 0; \\
\quad \sum_{i=1}^{m} v_i (x_{io}^U)_{\alpha_i} = 1; \\
\quad v_i \geqslant \varepsilon, \quad i = 1,2,\ldots,m; \\
\quad w_d \geqslant \varepsilon, \quad d = 1,2,\ldots,D; \\
\quad u_r \geqslant \varepsilon, \quad r = 1,2,\ldots,s.
\end{array}
\right. \tag{16}
$$

The two-level optimization Model (16) cannot be solved in its current form and should be reduced to a single-level optimization model. Unfortunately, some transformations are required before reducing Model (16) to a single-level optimization model because the two objective functions in this model are in the opposing direction. More formally, there is an interval parameter in the inner optimization model, i.e., z_{dj}, which is assumed as a decision variable for the outer optimization model. In addition, the objective functions of the inner and the outer optimizations models in the two-level optimization Model (16) are not the same. Thus a scenario may evolve in which we cannot be sure if the optimum value for the parameter/decision variable is achievable through a single optimization model. The reason is because when the direction of the optimization of the inner and outer optimization models are not the same, the best value for the parameter z_{dj} may not guarantee the optimum value for the decision variable z_{dj}. Therefore, we cannot simply add the extra constraints to the inner optimization model similar to the procedure used for Model (14). We construct Model (17) by replacing Eqs. (2)–(7) in the inner optimization problem of Model (16):

$$
\begin{array}{l}
\text{Max} e_o^L = \sum_{r=1}^{s} u_r \left(y_{ro}^1 + \alpha_r (y_{ro}^2 - y_{ro}^1)\right) \\
\text{s.t.} \ \sum_{r=1}^{s} u_r \left(y_{rj}^4 - \alpha_r (y_{rj}^4 - y_{rj}^3)\right) - \sum_{d=1}^{D} w_d z_{dj} \leqslant 0, \ j = 1,2,\ldots,n; \ j \neq o; \\
\quad \sum_{r=1}^{s} u_r \left(y_{ro}^1 + \alpha_r (y_{ro}^2 - y_{ro}^1)\right) - \sum_{d=1}^{D} w_d z_{do} \leqslant 0; \\
\quad \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i \left(x_{ij}^1 + \alpha_i \left(x_{ij}^2 - x_{ij}^1\right)\right) \leqslant 0, \ j = 1,2,\ldots,n; \ j \neq o; \\
\quad \sum_{d=1}^{D} w_d z_{do} - \sum_{i=1}^{m} v_i \left(x_{io}^4 - \alpha_i (x_{io}^4 - x_{io}^3)\right)_{\alpha_i} \leqslant 0; \\
\quad \sum_{d=1}^{D} w_d z_{do} - \sum_{i=1}^{m} v_i \left(x_{io}^4 - \alpha_i (x_{io}^4 - x_{io}^3)\right)_{\alpha_i} \leqslant 0; \\
\quad \sum_{d=1}^{D} w_d z_{do} - \sum_{i=1}^{m} v_i \left(x_{io}^4 - \alpha_i (x_{io}^4 - x_{io}^3)\right)_{\alpha_i} \leqslant 0; \\
\quad \sum_{i=1}^{m} v_i \left(x_{io}^4 - \alpha_i (x_{io}^4 - x_{io}^3)\right) = 1; \ \ v_i \geqslant \varepsilon, \ i = 1,2,\ldots,m; \\
\quad w_d \geqslant \varepsilon, \ d = 1,2,\ldots,D; \ \ u_r \geqslant \varepsilon, \ r = 1,2,\ldots,s
\end{array} \tag{17}
$$

Model (18) is then obtained by interchanging variables $\lambda_i = \alpha_i v_i, i = 1,\ldots,m$, where $0 \leqslant \lambda_i \leqslant v_i$; and $\eta_r = \alpha_i u_r$, $r = 1,\ldots,s$, where $0 \leqslant \eta_r \leqslant u_r$.

$$
\begin{array}{l}
e_o^L = \text{Max} \sum_{r=1}^{s} \left(u_r y_{ro}^1 + \eta_r (y_{ro}^2 - y_{ro}^1)\right) \\
\text{s.t.} \quad \sum_{r=1}^{s} \left(u_r y_{rj}^4 - \eta_r (y_{rj}^4 - y_{rj}^3)\right) - \sum_{d=1}^{D} w_d z_{dj} \leqslant 0, \\
\qquad j = 1,2,\ldots,n; \quad j \neq o; \\
\quad \sum_{r=1}^{s} \left(u_r y_{ro}^1 + \eta_r (y_{ro}^2 - y_{ro}^1)\right) - \sum_{d=1}^{D} w_d z_{do} \leqslant 0; \\
\quad \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} \left(v_i x_{ij}^1 + \lambda_i (x_{ij}^2 - x_{ij}^1)\right) \leqslant 0, \\
\qquad j = 1,2,\ldots,n; \quad j \neq o; \\
\quad \sum_{d=1}^{D} w_d z_{do} - \sum_{i=1}^{m} \left(v_i x_{io}^4 - \lambda_i (x_{io}^4 - x_{io}^3)\right) \leqslant 0; \\
\quad \sum_{i=1}^{m} \left(v_i x_{io}^4 - \lambda_i (x_{io}^4 - x_{io}^3)\right) = 1; \\
\quad w_d \geqslant \varepsilon, \quad d = 1,2,\ldots,D; \\
\quad v_i \geqslant \lambda_i \geqslant 0, \quad i = 1,2,\ldots,m; \\
\quad u_r \geqslant \eta_r \geqslant 0, \quad r = 1,2,\ldots,s.
\end{array} \tag{18}
$$

Model (18) is a single-level optimization linear programming model. However, if it is considered as a part of the two-level optimization Model (16), it is non-linear because of the terms $w_d z_{dj}$ and $w_d z_{do}$. We consider Model (18) as a single-level linear programming optimization model and define its dual form as follows:

$$
\begin{array}{l}
\text{Min} \quad z = \theta - \varepsilon \left(\sum_{i=1}^{m} s_i^- + \sum_{d=1}^{D} \phi_d + \sum_{r=1}^{s} s_r^+ \right) \\
\text{s.t.} \quad \sum_{\substack{j=1 \\ j \neq o}}^{n} y_{rj}^4 \alpha_j + y_{ro}^1 \alpha_o - \mu_r - s_r^+ \geqslant y_{ro}^1, \\
\qquad r = 1,2,\ldots,s; \\
\quad -\sum_{\substack{j=1 \\ j \neq o}}^{n} \left(y_{rj}^4 - y_{rj}^3\right) \alpha_j + (y_{ro}^2 - y_{ro}^1) \alpha_o - \rho_r + \mu_r \\
\qquad \geqslant (y_{ro}^2 - y_{ro}^1), \quad r = 1,2,\ldots,s; \\
\quad -\sum_{\substack{j=1 \\ j \neq o}}^{n} x_{ij}^1 \tau_j - x_{io}^4 \tau_o + x_{io}^4 \theta - s_i^- - \beta_i \\
\qquad \geqslant 0, \quad i = 1,2,\ldots,m; \\
\quad -\sum_{\substack{j=1 \\ j \neq o}}^{n} \left(x_{ij}^2 - x_{ij}^1\right) \tau_j + (x_{io}^4 - x_{io}^3) \tau_o - (x_{io}^4 - x_{io}^3) \theta \\
\qquad - \sigma_i + \beta_i \geqslant 0, \quad i = 1,2,\ldots,m; \\
\quad -\sum_{\substack{j=1 \\ j \neq o}}^{n} z_{dj} \alpha_j - z_{do} \alpha_o + \sum_{\substack{j=1 \\ j \neq o}}^{n} z_{dj} \tau_j + z_{do} \tau_o - \phi_d \\
\qquad \geqslant 0, \quad d = 1,2,\ldots,D; \\
\quad \alpha_j, \tau_j \geqslant 0, \quad j = 1,2,\ldots,n; \\
\quad \sigma_i, \beta_i, s_i^- \geqslant 0, \quad i = 1,2,\ldots,m; \\
\quad \phi_d \geqslant 0, \quad d = 1,2,\ldots,D; \\
\quad \mu_r, \rho_r, s_r^+ \geqslant 0, \quad r = 1,2,\ldots,s; \quad \theta \ \ \textit{free}.
\end{array} \tag{19}
$$

If we show that Model (19) is always feasible and bounded, we can claim that its objective function is equal to the objective function of Model (18) in optimality.

Theorem #2. *Model (19) is always feasible and bounded, and its optimal objective function is equal to unity.*

Proof. Consider the following arbitrary solution for Model (19):

$$\begin{aligned}
&\alpha_j = \tau_j = 0, \quad j = 1,2,\ldots,n; \quad j \neq o;\\
&\alpha_o = \tau_o = 1;\\
&\sigma_i = \beta_i = s_i^- = 0, \quad i = 1,2,\ldots,m;\\
&\phi_d = 0, \quad d = 1,2,\ldots,D;\\
&\mu_r = \rho_r = s_r^+ = 0, \quad r = 1,2,\ldots,s;\\
&\theta = 1;\\
&\bar{z}_{dj} = 0, \quad d = 1,2,\ldots,D; \quad j = 1,2,\ldots,n. \qquad \square
\end{aligned}$$

This solution is always feasible because it satisfies all the constraints in Model (19) and is independent of the inputs, intermediate measures, outputs, and the α-cut level values. Consequently, the dual form of Model (19), i.e., Model (18), always has a feasible solution. Clearly, in the above feasible solution, $\theta = 1$ and the slack variables are equal to zero. Hence, the optimum value of the objective function of Model (19) is less than or equal to1 (i.e., $z^* \leqslant 1$). Moreover, by the virtue of the dual theorem in linear programming, the optimal values of the objectives in Models (18) and (19) are equal (i.e., $z^* = e_o^{L^*}$). Therefore $z^* = e_o^{L^*} \leqslant 1$ and the optimal solution for Model (18) is also bounded. This completes the proof.

Given $z^* = e_o^{L^*} \leqslant 1$, we can construct Model (20) by using Model (19) instead of the inner optimization model in the two-level optimization programming Model (16):

$$\min_{(z_{dj}^L)_{\alpha_d} \leqslant z_{dj} \leqslant (z_{dj}^U)_{\alpha_d}, \forall j,d} \left\{ \begin{aligned}
&\text{Min } z = \theta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{d=1}^{D}\phi_d + \sum_{r=1}^{s} s_r^+\right)\\
&\text{s.t. } \sum_{\substack{j=1\\ j\neq o}}^{n} y_{rj}^4\alpha_j + y_{ro}^1\alpha_o - \mu_r - s_r^+ \geqslant y_{ro}^1,\\
&\qquad r = 1,2,\ldots,s;\\
&\quad -\sum_{\substack{j=1\\ j\neq o}}^{n}\left(y_{rj}^4 - y_{rj}^3\right)\alpha_j + (y_{ro}^2 - y_{ro}^1)\alpha_o - \rho_r\\
&\qquad + \mu_r \geqslant (y_{ro}^2 - y_{ro}^1), \quad r = 1,2,\ldots,s;\\
&\quad -\sum_{\substack{j=1\\ j\neq o}}^{n} x_{ij}^1\tau_j - x_{io}^4\tau_o + x_{io}^4\theta - s_i^- - \beta_i \geqslant 0,\\
&\qquad i = 1,2,\ldots,m;\\
&\quad -\sum_{\substack{j=1\\ j\neq o}}^{n}\left(x_{ij}^2 - x_{ij}^1\right)\tau_j + (x_{io}^4 - x_{io}^3)\tau_o\\
&\qquad -(x_{io}^4 - x_{io}^3)\theta - \sigma_i + \beta_i \geqslant 0,\\
&\qquad i = 1,2,\ldots,m;\\
&\quad -\sum_{\substack{j=1\\ j\neq o}}^{n} z_{dj}\alpha_j - z_{do}\alpha_o + \sum_{\substack{j=1\\ j\neq o}}^{n} z_{dj}\tau_j + z_{do}\tau_o\\
&\qquad -\phi_d \geqslant 0, d = 1,2,\ldots,D;\\
&\quad \alpha_j, \tau_j \geqslant 0, \quad j = 1,2,\ldots,n;\\
&\quad \sigma_i, \beta_i, s_i^- \geqslant 0, \quad i = 1,2,\ldots,m;\\
&\quad \phi_d \geqslant 0, \quad d = 1,2,\ldots,D;\\
&\quad \mu_r, \rho_r, s_r^+ \geqslant 0, \quad r = 1,2,\ldots,s; \quad \theta \textit{ free.}
\end{aligned}\right. \tag{20}$$

Because the inner and outer optimization objectives in Model (20) are in the form of minimization, we can reduce the two-level optimization programming Model (20) to a single-level optimization programming Model (21):

$$\begin{aligned}
&\text{Min } z = \theta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{d=1}^{D}\phi_d + \sum_{r=1}^{s} s_r^+\right)\\
&\text{s.t. } \sum_{\substack{j=1\\ j\neq o}}^{n} y_{rj}^4\alpha_j + y_{ro}^1\alpha_o - \mu_r - s_r^+ \geqslant y_{ro}^1, \quad r = 1,2,\ldots,s;\\
&\quad -\sum_{\substack{j=1\\ j\neq o}}^{n}\left(y_{rj}^4 - y_{rj}^3\right)\alpha_j + (y_{ro}^2 - y_{ro}^1)\alpha_o - \rho_r + \mu_r \geqslant (y_{ro}^2 - y_{ro}^1),\\
&\qquad r = 1,2,\ldots,s;\\
&\quad -\sum_{\substack{j=1\\ j\neq o}}^{n} x_{ij}^1\tau_j - x_{io}^4\tau_o + x_{io}^4\theta - s_i^- - \beta_i \geqslant 0, \quad i = 1,2,\ldots,m;\\
&\quad -\sum_{\substack{j=1\\ j\neq o}}^{n}\left(x_{ij}^2 - x_{ij}^1\right)\tau_j + (x_{io}^4 - x_{io}^3)\tau_o - (x_{io}^4 - x_{io}^3)\theta - \sigma_i + \beta_i \geqslant 0,\\
&\qquad i = 1,2,\ldots,m;\\
&\quad -\sum_{\substack{j=1\\ j\neq o}}^{n} z_{dj}\alpha_j - z_{do}\alpha_o + \sum_{\substack{j=1\\ j\neq o}}^{n} z_{dj}\tau_j + z_{do}\tau_o - \phi_d \geqslant 0,\\
&\qquad d = 1,2,\ldots,D;\\
&\quad (z_{dj}^L)_{\alpha_d} \leqslant z_{dj} \leqslant (z_{dj}^U)_{\alpha_d}, j = 1,2,\ldots,n; \quad d = 1,2,\ldots,D;\\
&\quad \alpha_j, \tau_j \geqslant 0, \quad j = 1,2,\ldots,n;\\
&\quad \sigma_i, \beta_i, s_i^- \geqslant 0, \quad i = 1,2,\ldots,m;\\
&\quad \phi_d \geqslant 0, \quad d = 1,2,\ldots,D;\\
&\quad \mu_r, \rho_r, s_r^+ \geqslant 0, \quad r = 1,2,\ldots,s; \quad \theta \textit{ free.}
\end{aligned} \tag{21}$$

Model (21) is non-linear and dependent on the α-cut. In order to resolve the non-linearity, we first multiply the sides of the constraints concerning intermediate measures by positive values of w_d. The following constraints are generated through this multiplication:

$$w_d\left(z_{dj}^1 + \alpha_d\left(z_{dj}^2 - z_{dj}^1\right)\right) \leqslant w_d z_{dj} \leqslant w_d\left(z_{dj}^4 - \alpha_d(z_{dj}^4 - z_{dj}^3)\right), \quad j = 1,2,\ldots,n; \quad d = 1,2,\ldots,D$$

The resulting $w_d z_{dj}$ term as well as $z_{dj} \times \alpha_j, j = 1, 2, \ldots, n; d = 1, 2, \ldots, D$ and $z_{dj} \times \tau_j, j = 1, 2, \ldots, n; d = 1, 2, \ldots, D$ are non-linear. The following variable interchanges are performed to resolve this non-linearity:

$$\begin{aligned}
&\theta_d = \alpha_d w_d, \quad d = 1,2,\ldots,D; \quad 0 \leqslant \theta_d \leqslant w_d\\
&\bar{z}_{dj} = w_d z_{dj}, \quad j = 1,2,\ldots,n; \quad d = 1,2,\ldots,D\\
&z'_{dj} = z_{dj} \times \tau_j, \quad j = 1,2,\ldots,n; \quad d = 1,2,\ldots,D\\
&z''_{dj} = z_{dj} \times \alpha_j, \quad j = 1,2,\ldots,n; \quad d = 1,2,\ldots,D
\end{aligned}$$

Consequently Model (22) is achieved as follows:

$$\begin{aligned}
&\text{Min } z = \theta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{d=1}^{D}\phi_d + \sum_{r=1}^{s} s_r^+\right)\\
&\text{s.t. } \sum_{\substack{j=1\\ j\neq o}}^{n} y_{rj}^4\alpha_j + y_{ro}^1\alpha_o - \mu_r - s_r^+ \geqslant y_{ro}^1, \quad r = 1,2,\ldots,s;
\end{aligned}$$

$$-\sum_{\substack{j=1\\ j\neq o}}^{n}\left(y_{rj}^{4}-y_{rj}^{3}\right)\alpha_j+(y_{ro}^{2}-y_{ro}^{1})\alpha_o-\rho_r+\mu_r\geqslant(y_{ro}^{2}-y_{ro}^{1}),$$
$$r=1,2,\ldots,s;$$
$$-\sum_{\substack{j=1\\ j\neq o}}^{n}x_{ij}^{1}\tau_j-x_{io}^{4}\tau_o+x_{io}^{4}\theta-s_i^{-}-\beta_i\geqslant 0,\ \ i=1,2,\ldots,m;$$
$$-\sum_{\substack{j=1\\ j\neq o}}^{n}\left(x_{ij}^{2}-x_{ij}^{1}\right)\tau_j+(x_{io}^{4}-x_{io}^{3})\tau_o-(x_{io}^{4}-x_{io}^{3})\theta-\sigma_i+\beta_i\geqslant 0,$$
$$i=1,2,\ldots,m;$$
$$-\sum_{\substack{j=1\\ j\neq o}}^{n}z'_{dj}-z'_{do}+\sum_{\substack{j=1\\ j\neq o}}^{n}z''_{dj}+z''_{do}-\phi_d\geqslant 0,\ \ d=1,2,\ldots,D;$$
$$w_d z_{dj}^{1}+\theta_d\left(z_{dj}^{2}-z_{dj}^{1}\right)\leqslant\bar{z}_{dj}\leqslant w_d z_{dj}^{4}-\theta_d\left(z_{dj}^{4}-z_{dj}^{3}\right),$$
$$j=1,2,\ldots,n;d=1,2,\ldots,D$$
$$Z'_{dj},Z''_{dj}\geqslant 0,\ \ j=1,2,\ldots,n;\ \ d=1,2,\ldots,D;$$
$$\sigma_i,\beta_i,s_i^{-}\geqslant 0,\ \ i=1,2,\ldots,m;$$
$$w_d,\phi_d\geqslant 0,\ \ d=1,2,\ldots,D;$$
$$w_d\geqslant\theta_d\geqslant 0,\ \ d=1,2,\ldots,D;$$
$$\mu_r,\rho_r,s_r^{+}\geqslant 0,\ \ r=1,2,\ldots,s;$$
$$\bar{z}_{dj}\geqslant 0,\ \ d=1,2,\ldots,D;\ \ j=1,2,\ldots,n;\ \ \theta\ \text{free}. \tag{22}$$

The resulting Model (22) is linear and independent of the α-cut variables. Therefore, it can achieve the global optimum for the lower-bound of the DMUs' efficiency values.

Theorem #3. *Model (22) is always feasible and bounded. Its optimal objective function is also equal to unity.*

Proof. Consider the following arbitrary solution for Model (22):

$$Z'_{dj}=Z''_{dj}=0,\quad j=1,2,\ldots,n;\quad j\neq o,\quad d=1,2,\ldots,D;$$
$$Z'_{do}=Z''_{do}=1,\quad d=1,2,\ldots,D;$$
$$\sigma_i=\beta_i=s_i^{-}=0,\quad i=1,2,\ldots,m;$$
$$\theta_d=w_d=\phi_d=0,\quad d=1,2,\ldots,D;$$
$$\mu_r=\rho_r=s_r^{+}=0,\quad r=1,2,\ldots,s;$$
$$\theta=1;$$
$$\bar{z}_{dj}=0,\quad d=1,2,\ldots,D;\quad j=1,2,\ldots,n.$$

This solution is always feasible because it satisfies all the constraints in Model (22) and is independent of the inputs, intermediate measures, outputs, and the α-cut level values. Clearly, in the above feasible solution, $\theta=1$ and slack variables are equal to zero. Hence, the optimum value of the objective function of Model (22) is less than or equal to 1 (i.e., $z^*\leqslant 1$). Therefore the optimal solution for Model (22) is also bounded. This completes the proof. □

4.4. Maximum achievable value of the upper-bound of efficiency for the sub-DMU in stage 1

Model (23) is proposed to calculate the maximum achievable value of the upper-bound of efficiency for the sub-DMU in Stage 1as Model (23).

$$\text{Max }\left[e_o^{1+}\right]^U=\sum_{d=1}^{D}w_d z_{do}$$
$$\text{s.t.}\quad \sum_{r=1}^{s}u_r(y_{ro}^{U})_{\alpha_r}=e_o^{U^*};$$
$$\sum_{r=1}^{s}u_r(y_{rj}^{L})_{\alpha_r}-\sum_{d=1}^{D}w_d z_{dj}\leqslant 0,\ \ j=1,2,\ldots,n;\ \ j\neq o;$$
$$\sum_{r=1}^{s}u_r(y_{ro}^{U})_{\alpha_r}-\sum_{d=1}^{D}w_d z_{do}\leqslant 0;$$
$$\sum_{d=1}^{D}w_d z_{dj}-\sum_{i=1}^{m}v_i(x_{ij}^{U})_{\alpha_i}\leqslant 0,\ \ j=1,2,\ldots,n;\ \ j\neq o;$$
$$\sum_{d=1}^{D}w_d z_{do}-\sum_{i=1}^{m}v_i(x_{io}^{L})_{\alpha_i}\leqslant 0;$$
$$\sum_{i=1}^{m}v_i(x_{io}^{L})_{\alpha_i}=1;\quad v_i\geqslant\varepsilon,\ \ i=1,2,\ldots,m;$$
$$w_d\geqslant\varepsilon,\ \ d=1,2,\ldots,D;\quad u_r\geqslant\varepsilon,\ \ r=1,2,\ldots,s. \tag{23}$$

Next, we decompose the efficiency score of the main DMU into the efficiency scores of its sub-DMUs. Therefore, the constraint which satisfies the upper-bound of the efficiency score in the main DMU, achieved by Model (15), is considered as $\sum_{r=1}^{s}u_r(y_{ro}^{U})_{\alpha_r}=e_o^{U^*}$ in Model (23). Moreover, we use the Stackelberg game to calculate the efficiency scores of the sub-DMUs. We assume that the first sub-DMU is the leader and calculate its efficiency scores. We then calculate the efficiency scores of the second sub-DMU (i.e., the follower), based on the achieved efficiency scores for the first the leader.

Model (23) cannot be solved in its current form. The optimum value of the intermediate measures in Model (23) should be determined using an upstream optimization model for different α-cut levels. Therefore, the following two-level optimization model (24) is proposed:

$$\max_{(z_{dj}^{L})_{\alpha_d}\leqslant z_{dj}\leqslant(z_{dj}^{U})_{\alpha_d},\forall j,d}\left\{\begin{array}{l}\text{Max}\left[e_o^{1+}\right]^U=\sum_{d=1}^{D}w_d z_{do}\\ \text{s.t.}\quad \sum_{r=1}^{s}u_r(y_{ro}^{U})_{\alpha_r}=e_o^{U^*};\\ \qquad\sum_{r=1}^{s}u_r(y_{rj}^{L})_{\alpha_r}-\sum_{d=1}^{D}w_d z_{dj}\leqslant 0,\\ \qquad\quad j=1,2,\ldots,n;\quad j\neq o;\\ \qquad\sum_{r=1}^{s}u_r(y_{ro}^{U})_{\alpha_r}-\sum_{d=1}^{D}w_d z_{do}\leqslant 0;\\ \qquad\sum_{d=1}^{D}w_d z_{dj}-\sum_{i=1}^{m}v_i(x_{ij}^{U})_{\alpha_i}\leqslant 0,\\ \qquad\quad j=1,2,\ldots,n;\quad j\neq o;\\ \qquad\sum_{d=1}^{D}w_d z_{do}-\sum_{i=1}^{m}v_i(x_{io}^{L})_{\alpha_i}\leqslant 0;\\ \qquad\sum_{i=1}^{m}v_i(x_{io}^{L})_{\alpha_i}=1;\\ \quad v_i\geqslant\varepsilon,\quad i=1,2,\ldots,m;\\ \quad w_d\geqslant\varepsilon,\quad d=1,2,\ldots,D;\\ \quad u_r\geqslant\varepsilon,\quad r=1,2,\ldots,s.\end{array}\right. \tag{24}$$

The two-level optimization Model (24) can be reduced to the single-level optimization Model (25) because both objective functions in Model (24) are in the maximization form:

$$
\begin{aligned}
&\text{Max } \left[e_o^{1+}\right]^U = \sum_{d=1}^{D} w_d z_{do}\\
&\text{s.t.} \ \sum_{r=1}^{s} u_r (y_{ro}^U)_{\alpha_r} = e_o^{U^*};\\
&\quad \sum_{r=1}^{s} u_r (y_{rj}^L)_{\alpha_r} - \sum_{d=1}^{D} w_d z_{dj} \leqslant 0, \ j=1,2,\ldots,n; \ j \neq o;\\
&\quad \sum_{r=1}^{s} u_r (y_{ro}^U)_{\alpha_r} - \sum_{d=1}^{D} w_d z_{do} \leqslant 0;\\
&\quad \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i (x_{ij}^U)_{\alpha_i} \leqslant 0, \ j=1,2,\ldots,n; \ j \neq o;\\
&\quad \sum_{d=1}^{D} w_d z_{do} - \sum_{i=1}^{m} v_i (x_{io}^L)_{\alpha_i} \leqslant 0;\\
&\quad \sum_{i=1}^{m} v_i (x_{io}^L)_{\alpha_i} = 1;\\
&(z_{dj}^L)_{\alpha_d} \leqslant z_{dj} \leqslant (z_{dj}^U)_{\alpha_d}, \ j=1,2,\ldots,n; \ j \neq o; \ d=1,2,\ldots,D;\\
&(z_{do}^L)_{\alpha_d} \leqslant z_{do} \leqslant (z_{do}^U)_{\alpha_d}, \ d=1,2,\ldots,D;\\
&v_i \geqslant \varepsilon, \ i=1,2,\ldots,m;\\
&w_d \geqslant \varepsilon, \ d=1,2,\ldots,D;\\
&u_r \geqslant \varepsilon, \ r=1,2,\ldots,s.
\end{aligned}
\tag{25}
$$

Model (25) is non-linear and dependent on the α-cut variables. Similar to the previous model, a variable exchange is performed to resolve these issues. As a result, Model (26) which is linear and independent of α-cut variables is developed. We should note that the procedure for proposing Model (26) is the same as the procedure for proposing Model (14). Therefore, for the sake of brevity only the final Model (26) is shown here:

$$
\begin{aligned}
&\text{Max}\left[e_o^{1+}\right]^U = \sum_{d=1}^{D} \bar{z}_{do}\\
&\text{s.t.} \ \sum_{r=1}^{s} \left(u_r y_{ro}^4 - \eta_r (y_{ro}^4 - y_{ro}^3)\right) = e_o^{U^*};\\
&\quad \sum_{r=1}^{s} \left(u_r y_{rj}^1 + \eta_r \left(y_{rj}^2 - y_{rj}^1\right)\right) - \sum_{d=1}^{D} \bar{z}_{dj} \leqslant 0,\\
&\qquad j=1,2,\ldots,n; \quad j \neq o;\\
&\quad \sum_{r=1}^{s} \left(u_r y_{ro}^4 - \eta_r (y_{ro}^4 - y_{ro}^3)\right) - \sum_{d=1}^{D} \bar{z}_{do} \leqslant 0;\\
&\quad \sum_{d=1}^{D} \bar{z}_{dj} - \sum_{i=1}^{m} \left(v_i x_{ij}^4 - \lambda_i (x_{ij}^4 - x_{ij}^3)\right) \leqslant 0,\\
&\qquad j=1,2,\ldots,n; \quad j \neq o;\\
&\quad \sum_{d=1}^{D} \bar{z}_{do} - \sum_{i=1}^{m} \left(v_i x_{io}^1 + \lambda_i (x_{io}^2 - x_{io}^1)\right) \leqslant 0;\\
&\quad \sum_{i=1}^{m} \left(v_i x_{io}^1 + \lambda_i (x_{io}^2 - x_{io}^1)\right) = 1;\\
&w_d z_{dj}^1 + \theta_d \left(z_{dj}^2 - z_{dj}^1\right) \leqslant \bar{z}_{dj} \leqslant w_d z_{dj}^4 - \theta_d \left(z_{dj}^4 - z_{dj}^3\right),\\
&\qquad j=1,2,\ldots,n; \quad d=1,2,\ldots,D;\\
&w_d \geqslant \theta_d \geqslant 0, \quad d=1,2,\ldots,D;\\
&v_i \geqslant \lambda_i \geqslant 0, \quad i=1,2,\ldots,m;\\
&u_r \geqslant \eta_r \geqslant 0, \quad r=1,2,\ldots,s;\\
&\bar{z}_{dj} \geqslant 0, \quad d=1,2,\ldots,D; \quad j=1,2,\ldots,n.
\end{aligned}
\tag{26}
$$

Theorem #4. *Model (26) is always feasible and bounded. Its optimal objective function is also equal to unit.*

Proof. The dual form of Model (26) can be written as Model (27) by defining the proper set of variables for each set of constraints in Model (26). □

$$
\begin{aligned}
&\omega = \text{Min}\left[\theta + \left(e_o^{U^*} \times \delta\right) - \varepsilon\left(\sum_{i=1}^{m} s_i + \sum_{r=1}^{s} s_r + \sum_{d=1}^{D} s_d\right)\right]\\
&\text{s.t.} \ \sum_{\substack{j=1\\ j \neq o}}^{n} p_j y_{rj}^1 + p_o y_{ro}^4 + \delta y_{ro}^4 - s_r - \rho_r \geqslant 0, \ r=1,2,\ldots,s;\\
&\quad \sum_{\substack{j=1\\ j \neq o}}^{n} p_j \left(y_{rj}^2 - y_{rj}^1\right) - p_o (y_{ro}^4 - y_{ro}^3) - \delta (y_{ro}^4 - y_{ro}^3) - \gamma_r + \rho_r \geqslant 0,\\
&\qquad r=1,2,\ldots,s;\\
&\quad -\sum_{\substack{j=1\\ j \neq o}}^{n} \tau_j x_{ij}^4 - \tau_o x_{io}^1 + \theta x_{io}^1 - s_i - \phi_i \geqslant 0, \ i=1,2,\ldots,m;\\
&\quad \sum_{\substack{j=1\\ j \neq o}}^{n} \tau_j \left(x_{ij}^4 - x_{ij}^3\right) - \tau_o (x_{io}^2 - x_{io}^1) + \theta (x_{io}^2 - x_{io}^1) - \alpha_i + \phi_i \geqslant 0,\\
&\qquad i=1,2,\ldots,m;\\
&\quad \sum_{d=1}^{D}\sum_{j=1}^{n} q_{dj} z_{dj}^1 - \sum_{d=1}^{D}\sum_{j=1}^{n} m_{dj} z_{dj}^4 - s_d - \mu_d \geqslant 0, \ d=1,2,\ldots,D;\\
&\quad \sum_{d=1}^{D}\sum_{j=1}^{n} q_{dj} \left(z_{dj}^2 - z_{dj}^1\right) + \sum_{d=1}^{D}\sum_{j=1}^{n} m_{dj} \left(z_{dj}^4 - z_{dj}^3\right) - \beta_d + \mu_d \geqslant 0,\\
&\qquad d=1,2,\ldots,D;\\
&\quad -\sum_{j=1}^{n} p_j + \sum_{j=1}^{n} \tau_j - \sum_{d=1}^{D}\sum_{j=1}^{n} q_{dj} + \sum_{d=1}^{D}\sum_{j=1}^{n} m_{dj} - \psi_{dj} \geqslant 1,\\
&\qquad d=1,2,\ldots,D; \ j=1,2,\ldots,n;\\
&p_j, \tau_j \geqslant 0, \ j=1,2,\ldots,n;\\
&q_{dj}, m_{dj}, \psi_{dj} \geqslant 0, \ d=1,2,\ldots,D; \ j=1,2,\ldots,n;\\
&s_i, \alpha_i, \phi_i \geqslant 0, \ i=1,2,\ldots,m;\\
&s_d, \beta_d, \mu_d \geqslant 0, \ d=1,2,\ldots,D;\\
&s_r, \rho_r, \gamma_r \geqslant 0, \ r=1,2,\ldots,s; \ \theta, \delta \ \textit{free}.
\end{aligned}
\tag{27}
$$

Consider the following arbitrary solution for Model (27):

$$
\begin{aligned}
&p_j = 0, \ j=1,2,\ldots,n;\\
&\tau_j = 0, \ j=1,2,\ldots,n; \ j \neq o;\\
&\tau_o = 1;\\
&q_{dj} = m_{dj} = \psi_{dj} = 0, \ d=1,2,\ldots,D; \ j=1,2,\ldots,n; \ j \neq o;\\
&q_{do} = m_{do} = \psi_{do} = 1, \ d=1,2,\ldots,D;\\
&s_i = \alpha_i = \phi_i = 0, \ i=1,2,\ldots,m;\\
&s_d = \beta_d = \mu_d = 0, \ d=1,2,\ldots,D;\\
&s_r = \rho_r = \gamma_r = 0, \ r=1,2,\ldots,s;\\
&\theta = 1;\\
&\delta = 0.
\end{aligned}
$$

This solution is always feasible because it satisfies all the constraints in Model (27) and is independent of the inputs, intermediate measures, outputs, and the α-cut level values. Consequently, the dual form of Model (27), i.e., Model (26), always has a feasible solution. Clearly, in the above feasible solution, $\theta = 1$, $\delta = 0$, and the slack variables are equal to zero. This means that for an arbitrary feasible solution, the value of the objective function is equal to unity. Hence, the optimum value of the objective function of Model (27) is less than or equal to1 (i.e., $\omega^* \leqslant 1$). Moreover, by the virtue of the dual theorem in linear programming, the optimal values of the objectives in Models (26) and (27) are equal (i.e., $[e_o^{1+}]^{U^*} = \omega^*$). Therefore, we can conclude that $[e_o^{1+}]^{U^*} = \omega^* \leqslant 1$ and the optimal solution for Model (26) is also bounded. This completes the proof.

4.5. Maximum achievable value of the lower-bound of efficiency for the sub-DMU in stage 1

Model (28) is proposed to calculate the maximum achievable value of the lower-bound of efficiency for the sub-DMU in stage 1 as follows:

$$\begin{aligned}
&\text{Max } [e_o^{+1}]^L = \sum_{d=1}^{D} w_d z_{do} \\
&\text{s.t.} \sum_{r=1}^{s} u_r (y_{ro}^L)_{\alpha_r} = e_o^{L^*}; \\
&\quad \sum_{r=1}^{s} u_r (y_{rj}^U)_{\alpha_r} - \sum_{d=1}^{D} w_d z_{dj} \leqslant 0, \quad j = 1,2,\ldots,n; \quad j \neq o; \\
&\quad \sum_{r=1}^{s} u_r (y_{ro}^L)_{\alpha_r} - \sum_{d=1}^{D} w_d z_{do} \leqslant 0; \\
&\quad \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i (x_{ij}^L)_{\alpha_i} \leqslant 0, \quad j = 1,2,\ldots,n; \quad j \neq o; \\
&\quad \sum_{d=1}^{D} w_d z_{do} - \sum_{i=1}^{m} v_i (x_{io}^U)_{\alpha_i} \leqslant 0; \\
&\quad \sum_{i=1}^{m} v_i (x_{io}^U)_{\alpha_i} = 1; \\
&\quad v_i \geqslant \varepsilon, \quad i = 1,2,\ldots,m; \\
&\quad w_d \geqslant \varepsilon, \quad d = 1,2,\ldots,D.
\end{aligned} \tag{28}$$

Similar to the case of the upper-bound of efficiency score, i.e., Model (23), the constraint which satisfies the lower-bound of the efficiency score for the main DMU, achieved by Model (22), is considered as $\sum_{r=1}^{s} u_r (y_{ro}^L)_{\alpha_r} = e_o^{L^*}$ in Model (28) based on the decomposition concept. The Stackelberg game is again used to calculate the efficiency scores of the sub-DMUs

Model (28) cannot be solved in its current form. The optimum value of the intermediate measures of Model (28) which minimizes the maximum values of the objective function of the inner optimization model in each α-cut level should be determined using an upstream optimization model for different α-cut levels. Therefore, the following two-level optimization model (29) is proposed:

$$\min_{(z_{dj}^L)_{\alpha_d} \leqslant z_{dj} \leqslant (z_{dj}^U)_{\alpha_d}, \forall j,d} \left\{ \begin{aligned}
&\text{Max} [e_o^{+1}]^L = \sum_{d=1}^{D} w_d z_{do} \\
&\text{s.t.} \sum_{r=1}^{s} u_r (y_{ro}^L)_{\alpha_r} = e_o^{L^*}; \\
&\quad \sum_{r=1}^{s} u_r (y_{rj}^U)_{\alpha_r} - \sum_{d=1}^{D} w_d zdj \leqslant 0, \\
&\qquad j = 1,2,\ldots,n; \; j \neq o; \\
&\quad \sum_{r=1}^{s} u_r (y_{ro}^L)_{\alpha_r} - \sum_{d=1}^{D} w_d z_{do} \leqslant 0; \\
&\quad \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i (x_{ij}^L)_{\alpha_i} \leqslant 0, \\
&\qquad j = 1,2,\ldots,n; \; j \neq o; \\
&\quad \sum_{d=1}^{D} w_d z_{do} - \sum_{i=1}^{m} v_i (x_{io}^U)_{\alpha_i} \leqslant 0; \\
&\quad \sum_{i=1}^{m} v_i (x_{io}^U)_{\alpha_i} = 1; \\
&\quad v_i \geqslant \varepsilon, \; i = 1,2,\ldots,m; \\
&\quad w_d \geqslant \varepsilon, \; d = 1,2,\ldots,D; \\
&\quad u_r \geqslant \varepsilon, \; r = 1,2,\ldots,s.
\end{aligned} \right. \tag{29}$$

Model (29) cannot be easily reduced to a single-level optimization model because the two objective functions in this model are in opposing directions. We construct Model (30) as follows by replacing Eqs. (2)–(7) in the inner optimization problem of Model (29):

$$\begin{aligned}
&\text{Max} [e_o^{+1}]^L = \sum_{d=1}^{D} w_d z_{do} \\
&\text{s.t.} \; \sum_{r=1}^{s} (u_r y_{ro}^1 + \alpha_r (y_{ro}^2 - y_{ro}^1)) = e_o^{L^*}; \\
&\quad \sum_{r=1}^{s} u_r \left(y_{rj}^4 - \alpha_r \left(y_{rj}^4 - y_{rj}^3 \right) \right) - \sum_{d=1}^{D} w_d z_{dj} \leqslant 0, \; j = 1,2,\ldots,n; \\
&\qquad j \neq o; \\
&\quad \sum_{r=1}^{s} u_r (y_{ro}^1 + \alpha_r (y_{ro}^2 - y_{ro}^1)) - \sum_{d=1}^{D} w_d z_{do} \leqslant 0; \\
&\quad \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i \left(x_{ij}^1 + \alpha_i (x_{ij}^2 - x_{ij}^1) \right) \leqslant 0, \; j = 1,2,\ldots,n; \\
&\qquad j \neq o; \\
&\quad \sum_{d=1}^{D} w_d z_{do} - \sum_{i=1}^{m} v_i (x_{io}^4 - \alpha_i (x_{io}^4 - x_{io}^3))_{\alpha_i} \leqslant 0; \\
&\quad \sum_{i=1}^{m} v_i (x_{io}^4 - \alpha_i (x_{io}^4 - x_{io}^3)) = 1; \\
&v_i \geqslant \varepsilon, \; i = 1,2,\ldots,m; \\
&w_d \geqslant \varepsilon, \; d = 1,2,\ldots,D; \\
&u_r \geqslant \varepsilon, \; r = 1,2,\ldots,s.
\end{aligned} \tag{30}$$

Model (31) is developed by using the required variable interchange as $\lambda_i = \alpha_i v_i, i = 1, \ldots, m$, where $0 \leqslant \lambda_i \leqslant v_i$ and $\eta_r = \alpha_r u_r$, $r = 1, \ldots, s$, where $0 \leqslant \eta_r \leqslant u_r$:

$$
\begin{aligned}
&\text{Max}\left[e_o^{+1}\right]^L = \sum_{d=1}^{D} w_d z_{do}\\
&\text{s.t.}\quad \sum_{r=1}^{s}\left(u_r y_{ro}^1 + \alpha_r\left(y_{ro}^2 - y_{ro}^1\right)\right) = e_o^{L^*};\\
&\qquad \sum_{r=1}^{s} u_r\left(y_{rj}^4 - \alpha_r\left(y_{rj}^4 - y_{rj}^3\right)\right) - \sum_{d=1}^{D} w_d z_{dj} \leqslant 0,\\
&\qquad\quad j = 1,2,\ldots,n;\quad j \neq o;\\
&\qquad \sum_{r=1}^{s} u_r\left(y_{ro}^1 + \alpha_r\left(y_{ro}^2 - y_{ro}^1\right)\right) - \sum_{d=1}^{D} w_d z_{do} \leqslant 0;\\
&\qquad \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i\left(x_{ij}^1 + \alpha_i\left(x_{ij}^2 - x_{ij}^1\right)\right) \leqslant 0,\\
&\qquad\quad j = 1,2,\ldots,n;\quad j \neq o;\\
&\qquad \sum_{d=1}^{D} w_d z_{do} - \sum_{i=1}^{m} v_i\left(x_{io}^4 - \alpha_i\left(x_{io}^4 - x_{io}^3\right)\right)_{\alpha_i} \leqslant 0;\\
&\qquad \sum_{i=1}^{m} v_i\left(x_{io}^4 - \alpha_i\left(x_{io}^4 - x_{io}^3\right)\right) = 1;\\
&\quad v_i \geqslant \varepsilon,\quad i = 1,2,\ldots,m;\\
&\quad w_d \geqslant \varepsilon,\quad d = 1,2,\ldots,D;\\
&\quad u_r \geqslant \varepsilon,\quad r = 1,2,\ldots,s.
\end{aligned}
\tag{31}
$$

Although Model (31) is a single-level linear programming optimization model, if it is considered as a part of the two-level optimization Model (29), it is non-linear because of the terms $w_d z_{dj}$ and $w_d z_{do}$. We consider Model (31) as a single-stage linear programming optimization model and define the dual of Model (31) as Model (32):

$$
\begin{aligned}
&\text{Min}\ \ z = \left[\theta + e_o^{L^*} \times \delta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{d=1}^{D}\phi_d + \sum_{r=1}^{s} s_r^+\right)\right]\\
&\text{s.t.}\quad \sum_{\substack{j=1\\ j\neq o}}^{n} y_{rj}^4\alpha_j + y_{ro}^1\delta + y_{ro}^1\alpha_o - \mu_r - s_r^+ \geqslant 0,\quad r = 1,2,\ldots,s;\\
&\qquad -\sum_{\substack{j=1\\ j\neq o}}^{n}\left(y_{rj}^4 - y_{rj}^3\right)\alpha_j + \left(y_{ro}^2 - y_{ro}^1\right)\alpha_o + \left(y_{ro}^2 - y_{ro}^1\right)\delta - \rho_r + \mu_r\\
&\qquad\quad \geqslant 0,\quad r = 1,2,\ldots,s;\\
&\qquad -\sum_{\substack{j=1\\ j\neq o}}^{n} x_{ij}^1\tau_j - x_{io}^4\tau_o + x_{io}^4\theta - s_i^- - \beta_i \geqslant 0,\quad i = 1,2,\ldots,m;\\
&\qquad -\sum_{\substack{j=1\\ j\neq o}}^{n}\left(x_{ij}^2 - x_{ij}^1\right)\tau_j + \left(x_{io}^4 - x_{io}^3\right)\tau_o - \left(x_{io}^4 - x_{io}^3\right)\theta - \sigma_i + \beta_i\\
&\qquad\quad \geqslant 0,\quad i = 1,2,\ldots,m;\\
&\qquad -\sum_{\substack{j=1\\ j\neq o}}^{n} z_{dj}\alpha_j - z_{do}\alpha_o + \sum_{\substack{j=1\\ j\neq o}}^{n} z_{dj}\tau_j + z_{do}\tau_o - \phi_d \geqslant z_{do},\\
&\qquad d = 1,2,\ldots,D;\\
&\quad \alpha_j, \tau_j \geqslant 0,\quad j = 1,2,\ldots,n;\\
&\quad \sigma_i, \beta_i, s_i^- \geqslant 0,\quad i = 1,2,\ldots,m;\\
&\quad \phi_d \geqslant 0,\quad d = 1,2,\ldots,D;\\
&\quad \mu_r, \rho_r, s_r^+ \geqslant 0,\quad r = 1,2,\ldots,s;\quad \theta, \delta\ \textit{free}.
\end{aligned}
\tag{32}
$$

If we can show that Model (32) is always feasible and bounded, then its objective function is equal to the same as the objective function of model (31) in optimality. Therefore, we may replace Model (32) with the inner optimization problem in the two-level optimization programming Model (29).

Theorem #5. *Model (32) is always feasible and bounded. Its optimal objective function is also equal to unity.*

Proof. Consider the following arbitrary solution for Model (32):

$$
\begin{aligned}
&\alpha_j = \tau_j = 0,\quad j = 1,2,\ldots,n;\quad j \neq o;\\
&\alpha_0 = \tau_0 = 1;\\
&\sigma_i = \beta_i = s_i^- = 0,\quad i = 1,2,\ldots,m;\\
&\phi_d = 0,\quad d = 1,2,\ldots,D;\\
&\mu_r = \rho_r = s_r^+ = 0,\quad r = 1,2,\ldots,s;\\
&\theta = 1;\\
&\delta = 0.\qquad \square
\end{aligned}
$$

This solution is always feasible because it satisfies all the constraints in Model (32) and is independent from the inputs, intermediate measures, outputs, and the α-cut level values. Clearly, in the above feasible solution, $\theta = 1$, $\delta = 0$, and the slack variables are equal to zero. Hence, the optimum value of the objective function of Model (32) is less than or equal to 1 (i.e., $z^* \leqslant 1$). Therefore the optimal solution for Model (32) is also bounded. This completes the proof.

Given $\left[e_o^{+1}\right]^L = Z^* \leqslant 1$, we can replace Model (32) with the inner optimization model in the two-level optimization programming Model (29). This will result in the following two-level optimization Model (33):

$$
\begin{aligned}
&\underset{(z_{dj}^L)_{\alpha_d} \leqslant z_{dj} \leqslant (z_{dj}^U)\alpha_d, \forall j,d}{\text{Min}}\\
&\times\left\{
\begin{aligned}
&\text{Min}\ \ z = \left[\theta + e_o^{L^*} \times \delta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{d=1}^{D}\phi_d + \sum_{r=1}^{s} s_r^+\right)\right]\\
&\text{s.t.}\quad \sum_{\substack{j=1\\ j\neq o}}^{n} y_{rj}^4\alpha_j + y_{ro}^1\delta + y_{ro}^1\alpha_o - \mu_r - s_r^+ \geqslant 0,\quad r = 1,2,\ldots,s;\\
&\qquad -\sum_{\substack{j=1\\ j\neq o}}^{n}\left(y_{rj}^4 - y_{rj}^3\right)\alpha_j + \left(y_{ro}^2 - y_{ro}^1\right)\alpha_o + \left(y_{ro}^2 - y_{ro}^1\right)\delta - \rho_r\\
&\qquad\quad + \mu_r \geqslant 0, r = 1,2,\ldots,s;\\
&\qquad -\sum_{\substack{j=1\\ j\neq o}}^{n} x_{ij}^1\tau_j - x_{io}^4\tau_o + x_{io}^4\theta - s_i^- - \beta_i \geqslant 0,\quad i = 1,2,\ldots,m;\\
&\qquad -\sum_{\substack{j=1\\ j\neq o}}^{n}\left(x_{ij}^2 - x_{ij}^1\right)\tau_j + \left(x_{io}^4 - x_{io}^3\right)\tau_o - \left(x_{io}^4 - x_{io}^3\right)\theta - \sigma_i\\
&\qquad\quad + \beta_i \geqslant 0, i = 1,2,\ldots,m;\\
&\qquad -\sum_{\substack{j=1\\ j\neq o}}^{n} z_{dj}\alpha_j - z_{do}\alpha_o + \sum_{\substack{j=1\\ j\neq o}}^{n} z_{dj}\tau_j + z_{do}\tau_o - \phi_d \geqslant z_{do},\\
&\qquad\quad d = 1,2,\ldots,D;\\
&\quad \alpha_j, \tau_j \geqslant 0,\quad j = 1,2,\ldots,n;\\
&\quad \sigma_i, \beta_i, s_i^- \geqslant 0,\quad i = 1,2,\ldots,m;\\
&\quad \phi_d \geqslant 0,\quad d = 1,2,\ldots,D;\\
&\quad \mu_r, \rho_r, s_r^+ \geqslant 0,\quad r = 1,2,\ldots,s;\\
&\quad \theta, \delta\ \textit{free}.
\end{aligned}
\right.
\end{aligned}
\tag{33}
$$

Next, we can reduce the two-level optimization programming Model (33) to the following single-level optimization programming Model (34) because both inner and outer optimization problems in Model (33) are in the form of minimization:

$$\begin{aligned}
\text{Min}\ \ & z=\left[\theta+e_o^{L^*}\times\delta-\varepsilon\left(\sum_{i=1}^{m}s_i^-+\sum_{d=1}^{D}\phi_d+\sum_{r=1}^{s}s_r^+\right)\right]\\
\text{s.t.}\ \ & \sum_{\substack{j=1\\ j\neq o}}^{n}y_{rj}^4\alpha_j+y_{ro}^1\delta+y_{ro}^1\alpha_o-\mu_r-s_r^+\geqslant 0,\ \ r=1,2,\ldots,s;\\
& -\sum_{\substack{j=1\\ j\neq o}}^{n}\left(y_{rj}^4-y_{rj}^3\right)\alpha_j+(y_{ro}^2-y_{ro}^1)\alpha_o+(y_{ro}^2-y_{ro}^1)\delta-\rho_r+\mu_r\\
&\qquad\geqslant 0,\ \ r=1,2,\ldots,s;\\
& -\sum_{\substack{j=1\\ j\neq o}}^{n}x_{ij}^1\tau_j-x_{io}^4\tau_o+x_{io}^4\theta-s_i^--\beta_i\geqslant 0,\ \ i=1,2,\ldots,m;\\
& -\sum_{\substack{j=1\\ j\neq o}}^{n}\left(x_{ij}^2-x_{ij}^1\right)\tau_j+(x_{io}^4-x_{io}^3)\tau_o-(x_{io}^4-x_{io}^3)\theta-\sigma_i+\beta_i\\
&\qquad\geqslant 0,\ \ i=1,2,\ldots,m;\\
& -\sum_{\substack{j=1\\ j\neq o}}^{n}z_{dj}\alpha_j-z_{do}\alpha_o+\sum_{\substack{j=1\\ j\neq o}}^{n}z_{dj}\tau_j+z_{do}\tau_o-\phi_d\geqslant z_{do},\\
&\qquad d=1,2,\ldots,D;\\
& (z_{dj}^L)_{\alpha_d}\leqslant zdj\leqslant (z_{dj}^U)_{\alpha_d},\ \ j=1,2,\ldots,n;\ \ d=1,2,\ldots,D;\\
& \alpha_j,\tau_j\geqslant 0,\ \ j=1,2,\ldots,n;\\
& \sigma_i,\beta_i,s_i^-\geqslant 0,\ \ i=1,2,\ldots,m;\\
& \phi_d\geqslant 0,\ \ d=1,2,\ldots,D;\\
& \mu_r,\rho_r,s_r^+\geqslant 0,\ \ r=1,2,\ldots,s;\ \ \theta,\delta\ \ \textit{free}.
\end{aligned}\tag{34}$$

Model (34) is non-linear and dependent on the α-cut variables. In order to resolve these two issues, we first multiply the sides of the constraints concerning the intermediate measures by the positive value w_d. The resulting w_dz_{dj} term as well as $z_{dj}\times\alpha_j$, $j = 1, 2, \ldots, n$; $d = 1, 2, \ldots, D$ and $z_{dj}\times\tau_j$, $j = 1, 2, \ldots, n$; $d = 1, 2, \ldots, D$ are non-linear. Unfortunately, Model (34) cannot be transformed into a linear programming model. Therefore, the following non-linear Model (35) which is independent of the α-cut is achieved:

$$\begin{aligned}
\text{Min}\ \ & z=\left[\theta+e_o^{L^*}\times\delta-\varepsilon\left(\sum_{i=1}^{m}s_i^-+\sum_{d=1}^{D}\phi_d+\sum_{r=1}^{s}s_r^+\right)\right]\\
\text{s.t.}\ \ & \sum_{\substack{j=1\\ j\neq o}}^{n}y_{rj}^4\alpha_j+y_{ro}^1\delta+y_{ro}^1\alpha_o-\mu_r-s_r^+\geqslant 0,\ \ r=1,2,\ldots,s;\\
& -\sum_{\substack{j=1\\ j\neq o}}^{n}\left(y_{rj}^4-y_{rj}^3\right)\alpha_j+(y_{ro}^2-y_{ro}^1)\alpha_o+(y_{ro}^2-y_{ro}^1)\delta\\
&\qquad-\rho_r+\mu_r\geqslant 0,\quad r=1,2,\ldots,s;\\
& -\sum_{\substack{j=1\\ j\neq o}}^{n}x_{ij}^1\tau_j-x_{io}^4\tau_o+x_{io}^4\theta-s_i^--\beta_i\geqslant 0,\ \ i=1,2,\ldots,m;\\
& -\sum_{\substack{j=1\\ j\neq o}}^{n}\left(x_{ij}^2-x_{ij}^1\right)\tau_j+(x_{io}^4-x_{io}^3)\tau_o-(x_{io}^4-x_{io}^3)\theta-\sigma_i+\beta_i\\
&\qquad\geqslant 0,\quad i=1,2,\ldots,m;\\
& -\sum_{\substack{j=1\\ j\neq o}}^{n}z_{dj}\alpha_j-z_{do}\alpha_o+\sum_{\substack{j=1\\ j\neq o}}^{n}z_{dj}\tau_j+z_{do}\tau_o-\phi_d\geqslant z_{do},\\
&\qquad d=1,2,\ldots,D;\\
& w_dz_{dj}^1+\theta_d\left(z_{dj}^2-z_{dj}^1\right)\leqslant\bar{z}_{dj}\leqslant w_dz_{dj}^4-\theta_d\left(z_{dj}^4-z_{dj}^3\right),\\
&\qquad j=1,2,\ldots,n;\ \ d=1,2,\ldots,D;\\
& \alpha_j,\tau_j\geqslant 0,\ \ j=1,2,\ldots,n;\\
& \sigma_i,\beta_i,s_i^-\geqslant 0,\ \ i=1,2,\ldots,m;\\
& w_d,\phi_d\geqslant 0,\ \ d=1,2,\ldots,D;\\
& w_d\geqslant\theta_d\geqslant 0,\ \ d=1,2,\ldots,D;\\
& \mu_r,\rho_r,s_r^+\geqslant 0,\ \ r=1,2,\ldots,s;\\
& \bar{z}_{dj}\geqslant 0,\ \ d=1,2,\ldots,D;\ \ j=1,2,\ldots,n;\ \theta,\delta\ \ \textit{free}.
\end{aligned}\tag{35}$$

Theorem #6. *Model (35) is always feasible and bounded. Its optimal objective function is also equal to unity*

Proof. *According to Theorems #1 to #5, the proof is clear.* □

4.6. Maximum achievable value of the upper and lower-bound of efficiency for the sub-DMU in stage 2

According to the proposed procedure for modeling the lower and upper bounds for the main DMU (Steps 3.2 and 3.3), and the proposed procedure for modeling the maximum achievable value of the upper and lower bound of efficiency for the first sub-DMU (Steps 3.4 and 3.5), the procedure for modeling the maximum achievable value of the upper and lower bounds of efficiency for the sub-DMU in stage 2 is straightforward and is not shown here for the sake of brevity.

4.7. Classification and ranking of the DMUs under uncertainty

4.7.1. Classification of the efficiency scores

The uncertainty of inputs, intermediate measures and outputs has been modeled through fuzzy sets. In addition to the flexibility in choosing weights, the levels of inputs, intermediate measures and outputs of the DMUs could be adjusted with the fuzzy membership functions. Moreover, the variable exchange process allows each DUM to select the most preferred α-cut value as well as the lower and upper bound values of the inputs, intermediate measures, and the outputs. Therefore, further discrimination of the efficient units becomes more essential in the two-step fuzzy DEA process. Eqs. (36)–(38) can be used to categorize the resulting interval efficiency scores of the main DMU, the first sub-DMUs, and the sub-DMU in stage 2 as follows:

$$\begin{aligned}
& DMU_p\\
& E^{++}=\left\{j\in J\middle|e_j^L=1\right\}\\
& E^{+}=\left\{j\in J\middle|e_j^L<1\ \text{and}\ e_j^*=1\right\}\\
& E^{-}=\left\{j\in J\middle|e_j^*<1\right\}
\end{aligned}\tag{36}$$

$$
\begin{aligned}
&Sub-DMU_{p1}\\
&E^{++}=\left\{j\in J\middle|\left[e_o^{1-}\right]^L=1\right\}\\
&E^{+}=\left\{j\in J\middle|\left[e_o^{1-}\right]^L<1 \text{ and } \quad e_o^{1-}=1\right\}\\
&E^{-}=\{j\in J|e_o^{1-}<1\}
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
&Sub-DMU_{p2}\\
&E^{++}=\left\{j\in J\middle|\left[e_o^{2-}\right]^L=1\right\}\\
&E^{+}=\left\{j\in J\middle|\left[e_o^{2-}\right]^L<1 \text{ and } \quad e_o^{2-}=1\right\}\\
&E^{-}=\{j\in J|e_o^{2-}<1\}
\end{aligned}
\tag{38}
$$

where J is the set of DMUs with cardinality n (i.e., $|J| = n$), $e_j^*=Max\left(e_j^L,e_j^U\right)$, $e_j^{1+}=\text{Max}\left(\left[e_o^{1+}\right]^L,\left[e_o^{1+}\right]^U\right)$, $e_j^{1-}=\text{Max}\left(\left[e_o^{1-}\right]^L,\left[e_o^{1-}\right]^U\right)$, $e_j^{2+}=Max\left(\left[e_o^{2+}\right]^L,\left[e_o^{2+}\right]^U\right)$, and $e_j^{2-}=\text{Max}\left(\left[e_o^{2-}\right]^L,\left[e_o^{2-}\right]^U\right)$.

4.7.2. Ranking the efficient DMUs through Monte Carlo simulation

In order to rank the efficient DMUs (i.e., those who belong to the E^{++} and E^+ sets), we use a simulation method proposed by Jahanshahloo et al. [34] for interval DEA ranking. We should note that the proposed procedure in this paper is for fuzzy data which is a general form of interval data. Monte Carlo simulations are used in two-stage models to reflect the inference procedure in finite samples [59].

Definition 2.1. The Region of Exclusive Domination (RED) for an efficient DMU_o with interval inputs/outputs is a sub-region of Possibility Production Set (PPS) such that DMU_o dominates all other DMUs, but none of the other DMUs dominate DMU_o.The RED measure is a criterion for ranking efficient DMUs. The bigger the RED measure is for a given DMU, the further away that DMU is from the other DMUs.

Consider an illustrative example with four interval DMUs. Each DMU has one interval input and one interval output and the PPS has been assumed to have Variable Return to Scale (VRS). DMU_A and DMU_B are efficient in this instance. The gray area in Fig. 3 represents the RED for the illustrative instance while W^+ and W^- represent the bounds in the PPS.

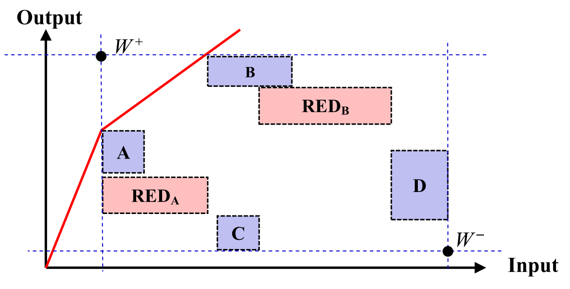

Fig. 3. Region of exclusive domination (RED) in a two-dimensional space.

Eqs. (39) and (40) can be used to calculate W^+ and W^-, respectively in a multi-dimensional PPS. W^+ and W^- help to bound the RED values for each DMU in the multi-dimensional PPS.

$$
\begin{aligned}
&W^+=\left(x_1^{\min},x_2^{\min},\ldots,x_m^{\min},y_1^{\max},y_2^{\max},\ldots,y_s^{\max}\right),\\
&\qquad i=1,\ldots,m,\quad r=1,\ldots,s,\\
&\text{where}\quad x_i^{\min}=\min_{j=1,\ldots,n}\left(x_{ij}^L\right);\quad y_r^{\max}=\max_{j=1,\ldots,n}\left(y_{rj}^U\right).
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
&W^-=\left(x_1^{\max},x_2^{\max},\ldots,x_m^{\max},y_1^{\min},y_2^{\min},\ldots,y_s^{\min}\right),\\
&\qquad i=1,\ldots,m,\quad r=1,\ldots,s,\\
&\text{where}\quad x_i^{\max}=\max_{j=1,\ldots,n}\left(x_{ij}^U\right);y_r^{\min}=\min_{j=1,\ldots,n}\left(y_{rj}^L\right).
\end{aligned}
\tag{40}
$$

The measure for the whole multi-dimensional region is $V^*=\prod_{i=1}^m\left(x_i^{\max}-x_i^{\min}\right)\times\prod_{r=1}^s\left(y_r^{\max}-y_r^{\min}\right)$. Then, the random spots in the bounded multi-dimensional region are generated and the RED measure is calculated using a Monte Carlo simulation process with a large number of runs. The random spots are represented by $\overline{W}=(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_m,\bar{y}_1,\bar{y}_2,\ldots,\bar{y}_s)$ and $W^-\leqslant\overline{W}\leqslant W^+$ where $\bar{x}_i\sim U\left(x_i^{\min},x_i^{\max}\right),i=1,\ldots,m$ and $\bar{y}_r\sim U\left(y_r^{\min},y_r^{\max}\right)$, $r=1,\ldots,s$. We should note that the RED is a multi-dimensional measure for DMUs with more than two inputs and two outputs. On the other hand, a ratio is calculated for each efficient DMU. The ratio is equal to the division of the numbers of the random generated spots which are within the RED of an efficient DMU by the total numbers of random generated spots. More formally, the RED measure for a given efficient DMU can be calculated as follows:

$$
M_{\text{RED}_j}=\frac{N_{\text{Hit}_j}}{N_{\text{Hit}_j}+N_{\text{Miss}_j}}
\tag{41}
$$

where M_{RED_j} is equal to the measure of the multi-dimensional RED for the efficient DMU_j, N_{Hit_j} is the number of random generated spots within the RED for the efficient DMU_j, and N_{Miss_j} is the number of random generated spots outside the RED of the efficient DMU_j. It is worth noting that the procedure is the same for the first and second sub-DMUs considering their associated inputs and outputs (i.e., X, and Z for the sub-DMU in stage 1, and Z and Y for the sub-DMU in stage 2).

Table 1
Inputs, intermediate measures, and outputs used in the case study.

Inputs (Stage 1)	X_1	Operational costs – employee salaries
	X_2	Capital costs – buying equipment and or buildings
	X_3	Financing costs – interest expense for loans and bonds
	X_4	Information technology costs
Intermediate measures (Stage 1 outputs) (Stage 2 inputs)	Z_1	Checking deposits
	Z_2	Saving deposits
	Z_3	Money market deposits
	Z_4	Certificates of deposits
Outputs (Stage 2)	Y_1	Return on assets
	Y_2	User fees income
	Y_3	Interest income
	Y_4	Earnings quality

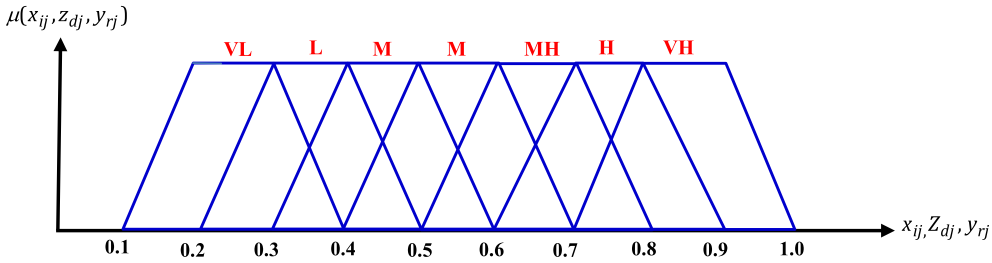

Fig. 4. Membership functions for the linguistic terms.

5. Efficiency measurement at Peoples Bank[1] branches

The primary role of a bank is to efficiently transform savings into investments. Successful investments build up the capital in the economy and foster future growth. Although banks are not the only financial institutions, they play a dominant role in the local and regional economy. Peoples Bank, the largest local bank in the State of East Virginia, is feeling the crunch in the southern part of the State. Bank lending has collapsed in the southern East Virginia counties and the Chief Financial Officer (CFO) of the bank plans to trim the headcount by closing some branches in the affected counties. He has put 20 bank branches in on the watch list and has decided to carefully analyze their overall efficiencies. We launched a pilot study for the bank to analyze the efficiency of each branch (DMU) using DEA. We proposed the two-stage model proposed in this study for this purpose.

In their first stage, the bank branches use operational costs (X_1), capital costs (X_2), financing costs (X_3), and information technology costs (X_4) as inputs to generate checking deposits (Z_1), saving deposits (Z_2), money market deposits (Z_3), and certificates of deposits (Z_4) as the outputs (intermediate measures). In the second stage, bank branches use the deposit dollars as a source of funds to invest in securities and to provide loans. Return on assets (Y_1), user fees income (Y_2), interest income (Y_3), and earnings quality (Y_4) are used as four outputs in the second stage. Table 1 presents the inputs, intermediate measures, and outputs used in this case study.

After a careful consideration of the available data and several brainstorming sessions with the branch managers and the senior management of the Peoples Bank, we noticed that there in a great deal of uncertainties and vagueness associated with the data concerning the inputs, intermediate measures, and the outputs. We decided to use fuzzy logic to capture the uncertainties associated with the input, intermediate measures and the output data. We developed the linguistic terms parameterized through TrFNs presented in Fig. 4 and Table 2 to measure the inputs, intermediate measures, and the outputs.

We met with each branch manager separately and asked them to provide us with a fuzzy measure for each input, intermediate measure, and output at their branch. We compiled a table for this data and confirmed these judgments with the bank's senior management. After a series of modifications to these scores, senior management at Peoples Bank approved Table 3 as the bank performance evaluation data to be used in this study.

The proposed two-stage fuzzy DEA models were coded using LINGO 11.0 software. Running the proposed models yielded the interval efficiency scores for the 20 branches (DMUs) and their stage 1 and 2 sub-DMUs. The results are summarized in Table 4.

Table 4 shows the lower and upper bound of the overall efficiency score (i.e., e^U and e^L), the lower and upper bound of the partial efficiency scores in the first stage (i.e., $[e^{+1}]^U$ and $[e^{+1}]^L$), and the lower and upper bound of the partial efficiency scores in the second stage (i.e., $[e^{+2}]^U$ and $[e^{+2}]^L$) for each bank branch. For instance, the efficiency score of the main DMU_5 is a value which belongs to the interval [0.585501, 0.813474]. The main DMU_5 is classified in the E^- group. This means that DMU_5 is inefficient as long as we are making a decision under uncertainty. We can further investigate the source of this inefficiency in the first or second stages. The efficiency score of the first Sub-DMU_5 is a value which belongs to the interval [1, 1]. This means that the first sub-DMU_5 is always efficient regardless of varying its inputs and outputs in uncertain situation. Therefore, it can be concluded that the source of variation in the efficiency score for the main DMU_5 is not its first sub-DMU. The efficiency score of the first Sub-DMU_5 is a value which belongs to the interval [0.585501, 0.813474]. This means that the source of inefficiencies of main DMU_5 can be found in its second sub-DMU.

We used this information to conclude that the lower and upper bounds of the overall efficiency scores of the branches (main DMUs) were less than unity. Therefore all branches were grouped in the E^- class. Further investigation into the efficiency scores of the first and second stages revealed the cause of these inefficiencies.

In the first stage, the upper bounds of the efficiency scores of all sub-DMUs were equal to unity except for sub-DMU_{18} which was equal to 0.856. The lower bound of the efficiency scores of sub-DMUs 2, 4, 5, and 13 was equal to unity while the lower bound of the efficiency scores of the remaining sub-DMUs were less than unity. Therefore, sub-DMUs 2, 4, 5, and 13 were grouped in the E^{++} class and the remaining sub-DMUs were grouped in the E^+ group. In the second stage, the upper bound of efficiency scores of all sub-DMUs was less than unity. The lower bound of efficiency scores of all sub-DMUs was also

[1] The name is changed to protect the anonymity of the bank.

Table 2
Linguistic terms and their associated TrFNs used in the case study.

Linguistic term	TrFN
Very Low (VL)	(0.1, 0.2, 0.3, 0.4)
Low(L)	(0.2, 0.3, 0.4, 0.5)
Medium Low (ML)	(0.3, 0.4, 0.5, 0.6)
Medium (M)	(0.4, 0.5, 0.6, 0.7)
Medium High (MH)	(0.5, 0.6, 0.7, 0.8)
High (H)	(0.6, 0.7, 0.8, 0.9)
Very High (VH)	(0.7, 0.8, 0.9, 1.0)

less than unity. Therefore, all sub-DMUs in the second stage were grouped in the E^- class.

In summary, the efficiency scores in the second stage were less than the efficiency scores in the first stage for all bank branches. In other words, the branches in general were less successful in converting their inputs (i.e., checking deposits, saving deposits, money market deposits, and certificates of deposits) to outputs (i.e., return on assets, user fees income, interest income, and earnings quality). We then used a Monte Carlo-based ranking procedure for each main DMU (bank branch) and their sub-DMUs in stages 1 and 2. The results are presented in Table 5.

The Monte Carlo model was run for 100,000 iterations. For instance, using the optimistic case, the dimensions of the generated random point were in the region of exclusive dominance (RED) of $DMU_1$18 times ($N_{Hit} = 18$) and the dimensions of the generated random point were outside the RED of $DMU_1$99982 times ($N_{Miss} = 99982$). Using the pessimistic case, the dimensions of the generated random point were inside the RED of $DMU_1$15 times ($N_{Hit} = 15$) and the dimensions

Table 3
Bank performance evaluation data.

Bank branch (DMU)	Inputs				Intermediate measures				Outputs			
	X_1	X_2	X_3	X_4	Z_1	Z_2	Z_3	Z_4	Y_1	Y_2	Y_3	Y_4
1	MH	M	M	L	L	MH	VL	MH	H	MH	VL	ML
2	H	H	ML	VH	H	VL	VH	ML	M	MH	VL	M
3	ML	L	M	MH	L	L	H	H	M	VH	H	L
4	MH	VH	L	M	VL	VH	MH	VL	M	VL	VL	H
5	L	VL	L	ML	L	VH	VH	ML	L	VL	VL	MH
6	ML	ML	MH	L	M	VL	VL	M	VL	ML	H	L
7	MH	ML	L	H	M	M	ML	VL	M	MH	H	H
8	VL	M	H	ML	MH	M	M	ML	L	ML	L	VL
9	L	VH	VL	ML	ML	ML	VL	ML	L	M	M	VL
10	VH	MH	M	ML	VL	MH	ML	MH	M	H	VL	VL
11	M	M	VL	MH	H	H	MH	VL	ML	MH	M	M
12	H	ML	VH	ML	M	M	MH	VL	M	H	H	ML
13	L	M	H	H	M	M	VH	H	L	ML	L	M
14	VH	L	MH	VL	M	M	VL	VL	H	MH	VH	L
15	VL	L	M	L	L	L	VL	VL	M	L	M	L
16	ML	MH	VL	M	VL	VL	ML	H	L	ML	ML	MH
17	ML	L	ML	VL	M	M	MH	H	H	MH	ML	L
18	M	H	M	ML	L	L	ML	L	VL	VL	M	H
19	H	VL	L	ML	L	L	M	M	VL	L	VH	VL
20	MH	M	ML	MH	M	M	H	VL	VL	VH	MH	M

Table 4
Results of the proposed models.

Bank branch (DMU)	Class	e^U	e^L	Stage 1 sub-DMU	$[e^{+1}]^U$	$[e^{+1}]^L$	Stage 2 sub-DMU	$[e^{+2}]^U$	$[e^{+2}]^L$
DMU_1	E^-	0.889914	0.512253	E^+	1	0.979045	E^-	0.889914	0.523217
DMU_2	E^-	0.882190	0.409598	E^{++}	1	1	E^-	0.882190	0.409598
DMU_3	E^-	0.897657	0.693975	E^+	1	0.999442	E^-	0.897657	0.694363
DMU_4	E^-	0.904627	0.530421	E^{++}	1	1	E^-	0.904628	0.530421
DMU_5	E^-	0.813474	0.585501	E^{++}	1	1	E^-	0.813474	0.585501
DMU_6	E^-	0.849191	0.461988	E^+	1	0.952381	E^-	0.849191	0.485087
DMU_7	E^-	0.857143	0.644336	E^+	1	0.952148	E^-	0.857143	0.676718
DMU_8	E^-	0.679437	0.461578	E^+	1	0.978934	E^-	0.679437	0.471511
DMU_9	E^-	0.750000	0.614151	E^+	1	0.914286	E^-	0.750000	0.671728
DMU_{10}	E^-	0.870827	0.469037	E^+	1	0.979592	E^-	0.870827	0.478808
DMU_{11}	E^-	0.851320	0.670826	E^+	1	0.999442	E^-	0.851320	0.671201
DMU_{12}	E^-	0.878730	0.467811	E^+	1	0.964286	E^-	0.878730	0.485137
DMU_{13}	E^-	0.768204	0.398024	E^{++}	1	1	E^-	0.768204	0.398024
DMU_{14}	E^-	0.857143	0.645679	E^+	1	0.952381	E^-	0.857143	0.677963
DMU_{15}	E^-	0.620000	0.617833	E^+	1	0.856306	E^-	0.600000	0.721510
DMU_{16}	E^-	0.784937	0.552501	E^+	1	0.996429	E^-	0.784937	0.554481
DMU_{17}	E^-	0.898326	0.800000	E^+	1	0.996429	E^-	0.898325	0.802867
DMU_{18}	E^-	0.569491	0.464667	E^-	0.856291	0.770662	E^-	0.665068	0.602946
DMU_{19}	E^-	0.793560	0.794890	E^+	1	0.952148	E^-	0.793560	0.834838
DMU_{20}	E^-	0.914139	0.566442	E^+	1	0.983232	E^-	0.914139	0.576102

Table 5
Monte Carlo simulation results for bank branches.

Measure of efficiency	Bank branch																		
	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5	DMU_6	DMU_7	DMU_8	DMU_9	DMU_{10}	DMU_{11}	DMU_{12}	DMU_{13}	DMU_{14}	DMU_{15}	DMU_{16}	DMU_{17}	DMU_{19}	DMU_{20}
Optimistic scenario																			
N_{Hit}	18	20	14	16	12	14	14	17	12	7	12	17	12	16	25	18	18	15	18
N_{Miss}	99982	99980	99986	99984	99988	99986	99986	99983	99988	99993	99988	99983	99988	99984	99975	99982	99982	99985	99982
M_{RED}	0.00018	0.0002	0.00014	0.00016	0.00012	0.00014	0.00014	0.00017	0.00012	0.00007	0.00012	0.00017	0.00012	0.00016	0.00025	0.00018	0.00018	0.00015	0.00018
Optimistic rank	3	2	11	8	14	11	11	6	14	18	14	6	14	8	1	3	3	10	3
Pessimistic scenario																			
N_{Hit}	15	18	15	21	5	16	6	8	20	13	19	19	20	17	15	15	23	16	17
N_{Miss}	99985	99982	99985	99979	99995	99984	99994	99992	99980	99987	99981	99981	99980	99983	99985	99985	99977	99984	99983
M_{RED}	0.00015	0.00018	0.00015	0.00021	0.00005	0.00016	0.00006	0.00008	0.0002	0.00013	0.00019	0.00019	0.0002	0.00017	0.00015	0.00015	0.00023	0.00016	0.00017
Pessimistic rank	12	7	12	2	19	10	18	17	3	16	5	5	3	8	12	12	1	10	8
Overall rank	12	17	4	16	2	5	3	6	10	1	8	14	10	9	18	12	19	7	15

of the generated random point were outside the RED of DMU_1 99985 times ($N_{Miss} = 99985$).

Using Eq. (41), the measure of RED is calculated for each DMU for both the optimistic and pessimistic cases. The optimistic and pessimistic rankings were used to calculate a final ranking for each DMU using the geometric mean. As shown in Table 5, the Monte Carlo simulation was able to discriminate among the 20 bank branches and provide us with an overall ranking of the bank branches. The overall rankings was shared with the bank's senior management who decided to further investigate the possible closure of the bottom 25% branches on this list which included DMU_{17}, DMU_{15}, DMU_2, DMU_4, and DMU_{20}.

6. Conclusions and future research directions

The conventional DEA models view DMUs as "black boxes" that consume a set of inputs to produce a set of outputs and do not take into consideration the intermediate measures within a DMU. In this paper we introduced a new fuzzy DEA approach for assessing the relative efficiency scores of a DMU and its sub-DMUs in a two-stage fuzzy system with uncertain inputs and outputs. The uncertainty of the input and output data is modeled with linguistic terms parameterized with fuzzy sets. We decomposed the efficiency score of a two-stage DMU and used the Stackelberg (leader–follower) game to calculate the efficiency scores of its sub-DMUs. The lower and upper bounds of the efficiency scores are calculated using the proposed two-stage fuzzy DEA method. The proposed model has the following unique features: (1) it is independent of the α-cut variables (minimizing the computational efforts); (2) it does not require the step-size of α-cut variables based on heuristics rules and trial and error; (3) it is linear (producing global optimum solutions); and (4) it uses a Monte Carlo simulation procedure to discriminately rank the efficient DUMs and sub-DMUs.

We presented a case study and exhibited the efficacy of the procedures and demonstrated the applicability of the proposed method to a two-stage performance evaluation problem in the banking industry. We used operational costs, capital costs, financing costs, and information technology costs as inputs in the first stage to generate checking deposits, saving deposits, money market deposits, and certificates of deposits as the outputs (intermediate measures). These deposit dollars (intermediate measures) were used as a source of funds to invest in securities and to provide loans in the second stage. Return on assets, user fees income, interest income, and earnings quality were used as outputs in the second stage.

The results from the case study were promising and the computations were straightforward. The proposed method can be used to solve various real-world problems in the public and private sectors.

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Appendix A

Consider an arbitrary solution for Model (14) as follows:

$$\begin{aligned}
&p_j=\tau_j=0,\ \ j=1,2,\ldots,n;\ \ j\neq o;\\
&p_o=\tau_o=1;\\
&q_{dj}=m_{dj}=\psi_{dj}=0,\ \ d=1,2,\ldots,D,\ \ j=1,2,\ldots,n,\ \ j\neq o;\\
&q_{do}=m_{do}=\psi_{do}=1,\ \ d=1,2,\ldots,D;\\
&s_i=\alpha_i=\phi_i=0,\ \ i=1,2,\ldots,m;\\
&s_d=\beta_d=\mu_d=0,\ \ d=1,2,\ldots,D;\\
&s_r=\rho_r=\gamma_r=0,\ \ r=1,2,\ldots,s;\\
&\theta=1.
\end{aligned}$$

The 6-th set of constraints, as shown in Model (14) is:

$$\sum_{d=1}^{D}\sum_{j=1}^{n}q_{dj}\left(z_{dj}^{2}-z_{dj}^{1}\right)+\sum_{d=1}^{D}\sum_{j=1}^{n}m_{dj}\left(z_{dj}^{4}-z_{dj}^{3}\right)-\beta_d+\mu_d \geqslant 0,\quad d=1,2,\ldots,D;$$

This set of constraints can be reduced to the following relations considering the proposed arbitrary solution:

$$\begin{aligned}
&q_{do}(z_{do}^{2}-z_{do}^{1})+m_{do}(z_{do}^{4}-z_{do}^{3})\geqslant 0\\
&(z_{do}^{2}-z_{do}^{1})+(z_{do}^{4}-z_{do}^{3})\geqslant 0
\end{aligned}$$

As in all TrFNs, we have $z_{do}^{i}\geqslant z_{do}^{j}|i\geqslant j,\quad \forall i,j$. Therefore, both the terms $(z_{do}^{2}-z_{do}^{1})$ and $(z_{do}^{4}-z_{do}^{3})$are greater than or equal to zero. Consequently, it is obvious that $(z_{do}^{2}-z_{do}^{1})+(z_{do}^{4}-z_{do}^{3})\geqslant 0$ is always true and this completes the justification.

Similarly, the 7-th set of constraints, as shown in Model (14) is:

$$-\sum_{\substack{j=1\\ j\neq o}}^{n}p_j+\sum_{\substack{j=1\\ j\neq o}}^{n}\tau_j-\sum_{d=1}^{D}\sum_{j=1}^{n}q_{dj}+\sum_{d=1}^{D}\sum_{j=1}^{n}m_{dj}-\psi_{dj}\geqslant 0,\quad d=1,2,\ldots,D;\quad j=1,2,\ldots,n;$$

This set of constraints can also be reduced to the following relations considering the proposed arbitrary solution:

$$\begin{aligned}
&q_{do}+m_{do}-\psi_{do}\geqslant 0\\
&1+1-1\geqslant 0
\end{aligned}$$

This completes the justification.

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