Computers & Industrial Engineering 87 (2015) 621–631

Contents lists available at ScienceDirect

Computers & Industrial Engineering

journal homepage: www.elsevier.com/locate/caie

Extended symmetric and asymmetric weight assignment methods in data envelopment analysis ☆

Adel Hatami-Marbini [a,*], Mohsen Rostamy-Malkhalifeh [b], Per J. Agrell [a], Madjid Tavana [c,d], Fatemeh Mohammadi [b]

[a] Louvain School of Management, Center of Operations Research and Econometrics (CORE), Universite catholique de Louvain, 34 Voie du Roman Pays, B-1348 Louvain-la-Neuve, Belgium
[b] Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran
[c] Business Systems and Analytics Department, Lindback Distinguished Chair of Information Systems and Decision Sciences, La Salle University, Philadelphia, PA 19141, USA
[d] Business Information Systems Department, Faculty of Business Administration and Economics, University of Paderborn, D-33098 Paderborn, Germany

ARTICLE INFO

Article history:
Received 5 December 2014
Received in revised form 29 March 2015
Accepted 21 June 2015
Available online 27 June 2015

Keywords:
Data envelopment analysis
Weight restriction
Symmetric weight assignment technique
District heating

ABSTRACT

Dual weight restrictions are commonly suggested as a remedy to the problem of low discriminatory power and absurd marginal prices in conventional Data Envelopment Analysis (DEA) models. However, weight restriction models also suffer from potential problems of infeasibility, lack of exogenous determination and ambiguous interpretations. The Symmetric Weight Assignment Technique (SWAT) addresses these concerns through a symmetric endogenous weight selection process. In this paper, we extend the SWAT method by proposing four new DEA models. Symmetric and asymmetric weights are rewarded and penalized, respectively, in the proposed models. The first model takes into account the symmetrical weights assigned to the outputs in the input-oriented model. The second model takes into account the symmetrical weights assigned to the inputs in the output-oriented model. The third and fourth models simultaneously take into account symmetric input–output weights in both the input and output orientations. We demonstrate the applicability of the proposed models and the efficacy of the procedures and algorithms with an application to Danish district heating plants.

1. Introduction

Any performance evaluation technique that results in a set of measures of lower dimensionality than the original production space must necessarily consider weighting the resources that are consumed and the outputs that are produced. The methodology for determining the relative costs or prices is one of the pivotal challenges in performance evaluation. Whereas market prices may be observed or elicited in certain circumstances, they may not necessarily reflect the social welfare effects due to externalities and horizon problems. Tradeoff rates may be inferred from preferences solicited from managers, although there is little incentive for managers to provide this information and it is likely to result in biased data. Engineering data may postulate costs for a given technology, but this may be doubtful in regulatory contexts as well as in the presence of technological innovation or process heterogeneity. Non-parametric frontier approaches such as the Data Envelopment Analysis (DEA) by Charnes, Cooper, and Rhodes (1978, 79), drawing on the seminal work of Farrell (1957)[1] address this issue by allocating sets of individual endogenous weights that put the individual unit in the best possible light. In this manner DEA provides the evaluator with a conservative performance estimate that is valid for a range of preference functions. Under a convex frontier specification, the analysis explicitly provides the evaluator with dual information that later may be used to refine the preference model of the evaluator by inserting additional constraints. In an open retrospective evaluation, where the modeling rests entirely at the discretion of the analyst or collectively of the units, such an

☆ This manuscript was processed by Area Editor Imed Kacem, Pr.

* Corresponding author. Tel.: +32 104 74319; fax: +32 104 74301.
E-mail addresses: adel.hatamimarbini@uclouvain.be (A. Hatami-Marbini), mohsen_rostamy@yahoo.com (M. Rostamy-Malkhalifeh), per.agrell@uclouvain.be (P.J. Agrell), tavana@lasalle.edu (M. Tavana), f.mohammadi89@yahoo.com (F. Mohammadi).
URL: http://tavana.us/ (M. Tavana).

[1] Whereas both Farrell (1957) and Charnes et al. (1978) are seminal contributions to the area of efficiency analysis and measurement, the viewpoints of the authors are different: Farrell defined a single-dimensional radial primal projection, Charnes et al. defined their metric from a productivity ratio-approach with an immediate dual approach in linear programming for the multi-dimensional case. Note that under the conventional assumptions for the productivity possibility set, the primal and dual approaches are equivalent under the duality theorem.

http://dx.doi.org/10.1016/j.cie.2015.06.014
0360-8352/

approach may support organizational learning and development. Unrestricted weights are relevant in the determination of technical efficiency, i.e., the overall transformation rate of inputs to outputs irrespective of input costs and output preferences. However, the endogenously determined weights in conventional DEA may also lead to estimations of the efficient frontier that imply absurd cost functions, where certain inputs and outputs seemingly have no or extremely low costs and values, respectively. Consequently, the technical efficiency estimates are lower bounds to the "true" technical efficiency when taking into account a more constrained set of marginal costs and prices. The subsequent rankings are also weaker, leading to poor discriminatory ability for small data sets.

The use of weight restrictions techniques has a long tradition in the DEA literature (for the early development see e.g. the survey in Allen, Athanassopoulos, Dyson, & Thanassoulis (1997)). Golany (1988) developed a DEA model with ordinal relations among the weights for subsets of the inputs or outputs. Ali, Cook, and Seiford (1991) later proved that weak ordinal relations need non-standard DEA models of the type in Golany (1988). Dyson and Thanassoulis (1988) proposed a procedure for determining weight restrictions through direct constraints, their interpretation and derivation through linear regression. An alternative approach, restricting the weights through their relative shares, was first launched in Wong and Beasley (1990). Kornbluth (1991) extended the direct dual constraint approach by using the cone ratio constraint approach originally developed in Charnes, Cooper, Wei, and Huang (1989). Roll, Cook, and Golany (1991) proposed a DEA model where absolute bounds were imposed on the factor weights. Their method specified the eligible bounds for the weights as well as introducing the notion of a Common Set of Weights (CSW). Cook, Roll, and Kazakov (1990) and Roll et al. (1991) presented ratio-based approaches based on the dual "weight matrix" obtained in an unrestricted DEA model onto which specific interval coefficients were applied to derive acceptable bounds for the dual weight variation.

Thompson, Langemeier, Lee, Lee, and Thrall (1990) proposed the Assurance Region (AR) method for factoring weight control by setting ratios to improve the discriminatory power of DEA. Thompson, Lee, and Thrall (1992) studied the AR-efficiencies of US independent oil and gas producers using the DEA ratio and convex models in the presence of weight bounds. Roll and Golany (1993) proposed a framework involving a number of method for controlling the input and output weights in DEA by setting their bounds. They also classified the weight bounding methods in the literature. Cook, Kress, and Seiford (1992) proposed an alternative DEA method by imposing distinct conditions on the weights.

Allen et al. (1997) categorized the weight restriction methods into three categories: (1) the Assurance Region I (ARI) methods first developed by Thompson, Singleton, Thrall, and Smith (1986); (2) the Assurance Region II (ARII) methods first proposed by Thompson et al. (1990), often called linked-cone assurance region methods; and (3) the absolute weight restriction method first introduced by Dyson and Thanassoulis (1988). The AR methods are different from the absolute weight restrictions methods since the ratios between the weights are imposed to be within given bounds instead of imposing the weights to be within given bounds. The ARI methods determine the ratios between the input and the output weights separately while the ARII methods determine the ratios that connect the input weights to the output weights.

Halme and Korhonen (2000) considered two types of preference information involving the most preferred inputs and outputs and the information on the weights of the inputs and outputs in a DEA problem using the value efficiency analysis presented by Halme, Joro, Korhonen, Salo, and Wallenius (1999). Podinovski (2004a) first proposed a modified approach by incorporating the production trade-offs into the DEA models that maintained all the principle properties of efficiency, particularly, the radial target of each inefficient DMU. This method was then applied in the multiplier DEA models with weight restrictions. Podinovski (2004b) discussed the problem of using absolute weight bounds in DEA and then determined certain types of non-homogeneous restrictions that do not result in the observed error. In the Podinovski (2004b) model, there are no lower bounds on the input weights and no upper bounds on the output weights to correctly measure the efficiency of the DMUs. Sarrico and Dyson (2004) incorporated a virtual weight restriction and a virtual AR instead of the absolute weight restrictions into the DEA model in order to supply a natural representation of the decision makers' preferences. They also showed that *proportional* weight restrictions can lead to infeasible solutions in DEA problems.

Bernroider and Stix (2007) proposed a DEA method using weight restrictions to provide more significant information on the stability and validity of the results. Estellita Lins, Moreira da Silva, and Lovell (2007) addressed the problem of infeasibility of the developed weight restrictions in LPs. They proposed a theorem to demonstrate the feasibility conditions for the DEA multiplier programs with weight restrictions. Cooper, Ruiz, and Sirvent (2007) proposed a two-step procedure for selecting the weights in conjunction with the efficient facets of the highest possible dimension of the frontier in the DEA multiplier model. They showed that optimal solutions of the multiplier DEA formulation have alternate optima for the weights. Kuosmanen, Cherchye, and Sipiläinen (2006) adopted the so-called Law of One Price to the DEA model with weight restrictions. Their method was able to handle firm-specific output weights and variable returns to scale along with maintaining the linearity of the original DEA model. Liu and Peng (2008) introduced a DEA method to find the most favorable CSW to discern the difference between the efficient DMUs in view of maximizing the group's efficiency score.

Wang, Chin, and Poon (2008) proposed the DEA-AR model for weight derivation in the analytic hierarchy process to overcome the shortcomings of illogical local weights, over-insensitivity to some comparisons, information loss, and the overestimation of some local weights. Meng, Zhang, Qi, and Liu (2008) developed a two-level nonlinear frontier model where inputs and outputs with similar characteristics are aggregated into input and output groups, respectively, in order to increase the discrimination power. The weights among different classes were obtained using the conventional DEA models while the weights within groups were identified by a weighted-average DEA method. Kao (2008) modified the model presented by Meng et al. (2008) by converting the nonlinear model into a linear model using a variable substitution method. Zhiani Rezai and Davoodi (2011) showed that the cone-ratio weight restriction method developed by Cooper, Seiford, and Tone (1999) is a general case of the two-level DEA model studied by Meng et al. (2008) and Kao (2008). Wang, Luo, and Liang (2009) proposed an alternative DEA ranking method by imposing the minimum weight restrictions on inputs and outputs when the factor weights are determined through a set of maximin problems. Liu and Peng (2009) developed a systematic procedure to search the CSW for preferable and robust rankings by using the virtual weights restriction.

Wu, Liang, and Yang (2009a) proposed a cross-efficiency evaluation method to assess the performance of the nations participating in the Olympic games, similar to an AR application in Li, Liang, Chen, and Morita (2008). They incorporated the weight restrictions into their model to assure ordinal valuations among the obtained medals as disaggregated outputs. Khalili, Camanho, Portela, and Alirezaee (2010) adjusted the ARII model by introducing a nonlinear model that overcomes the shortcomings of the conventional ARII involving underestimation of efficiency and infeasibility. Korhonen, Soleimani-damaneh, and Wallenius (2011) developed computationally efficient approaches for Returns To Scale (RTS) determination of a DMU embedded in a weight-restricted DEA framework. Wang, Luo, and Lan (2011) established two nonlinear regression models for seeking common weights in DEA to obtain a full ranking order of the DMUs. Wu, Sun, and Liang (2012) proposed a weight-balanced DEA model to solve non-uniqueness of cross-efficiency scores. Their contributions were to reduce the number of zero-input–output weights as well as to reduce the differences in the weighted inputs and weighted outputs through the assessment process. Silva and Milioni (2012) presented a parametric model to distribute an input that allowed the inclusion of weight restrictions.

Ramón, Ruiz, and Sirvent (2012) proposed a DEA method for ranking professional tennis players using a CSW. The aim behind this approach was to reduce the differences of the CSW from the DEA profiles of chosen weights without zeros of the efficient DMUs. Ruiz and Sirvent (2012) used a weighted average of cross-efficiencies instead of the widely used arithmetic mean to obtain the cross-efficiency scores in which the choice of DEA weights was obtained on the basis of Ramón, Ruiz, and Sirvent (2010a). Regarding the choice of more suitable weights in the cross-efficiency evaluation, apart from the benevolent and aggressive approaches (Sexton, Silkman, & Hogan, 1986; Doyle & Green, 1994), extensive development has been implemented in the literature by Liang, Wu, Cook, and Zhu (2008), Wang and Chin (2010a, 2010b), Wu, Liang, and Yang (2009b), Ramón, Ruiz, and Sirvent (2010b), Ramón, Ruiz, and Sirvent (2011). Hosseinzadeh Lotfi, Hatami-Marbini, Agrell, Aghayi, and Gholami (2013) and Hatami-Marbini, Tavana, Agrell, Lotfi, and Beigi (2015) proposed a common dual weights approach based on the goal programming concept for activity planning and resource allocation in DEA.

Dimitrov and Sutton (2010) recently proposed the Symmetric Weight Assignment Technique (SWAT) for weight restriction in DEA to assess the efficiency of the DMUs with respect to considering a symmetric managerial preference structure. Dimitrov and Sutton (2013) generalized the SWAT method, naming it g-SWAT, to consider non-symmetric managerial preferences in evaluating the performance of the DMUs. They also studied the impact of changing the preferences of the decision makers on the results. Jahanshahloo, Hosseinzadeh Lotfi, Jafari, and Maddahi (2011) proposed a variant of the SWAT in Dimitrov and Sutton (2010) with a secondary goal for cross-efficiency.

It is important to note that more recent weight restrictions models have addressed many of the weaknesses of previous weight restriction approaches related to issues of ambiguous interpretation and infeasibility. For example, Podinovski (2004b) and Sarrico and Dyson (2004) discussed some of the weaknesses of using absolute weight restrictions and suggested modifications that created a more accurate representation of the decision makers' preferences. Many of the works cited above such as Bernroider and Stix (2007), Estellita Lins et al. (2007), Wang et al. (2008), Meng et al. (2008), Liu and Peng (2009), Khalili et al. (2010), and Korhonen et al. (2011) noted weaknesses of traditional weight restriction models related to issues of stability, validity, feasibility, information loss, estimation, discrimination power, robustness, and computational efficiency. The SWAT technique, originally proposed by Dimitrov and Sutton (2010), has been one of the most successful methods in addressing the various weaknesses associated with using traditional weight restriction models, which is the main reason why we use this approach as the basis for our models.

In this paper, we propose four new DEA models based on the SWAT by preserving the linearity. We incorporate both the symmetric and asymmetric weights into the proposed models. Symmetric weights are rewarded while asymmetric weights are penalized in our models. The first model takes into account the symmetrical weights assigned to the outputs in the input-oriented model. The second model takes into account the symmetrical weights assigned to the inputs in the output-oriented model. The third and fourth models take into account symmetric input–output weights in both the input and output orientations, simultaneously. We demonstrate the efficacy of our models through a numerical example introduced by Dimitrov and Sutton (2010) and pinpoint the applicability of the proposed method through an application to district heating plants.

The remainder of this paper is organized as follows: In Section 2, we provide an overview of the mathematical DEA background for the performance analysis followed by a brief review of the SWAT in the context of weight restriction in DEA. In Section 4, we present the mathematical details of the four DEA models proposed in this study. In Section 5, we use the numerical example of Dimitrov and Sutton (2010) to demonstrate the efficacy of the proposed models and in Section 6 we present a case study to demonstrate the applicability of the proposed method. We conclude our paper in Section 7 with our conclusions and future research directions.

2. Measuring performance

Let us consider n DMUs such that $Y_j = (y_{rj}) = (y_{1j}, y_{2j}, \ldots, y_{sj})$ is the vector of s outputs, and $X_j = (x_{ij}) = (x_{1j}, x_{2j}, \ldots, x_{mj})$ is the vector of m inputs for DMU j. The production space $\{X, Y\}$ does not include the zero vector for X; $\{0, Y\}$, the conventional no-free-lunch assumption. The conventional radial input-oriented technical efficiency metric is defined as Charnes et al. (1978):

$$e_j = uY_j / vX_j = \sum_{r=1}^{s} u_r y_{rj} \Big/ \sum_{i=1}^{m} v_i x_{ij},$$

where $u = (u_1, u_2, \ldots, u_s)$ and $v = (v_1, v_2, \ldots, v_m)$ are the vectors of dual weights (multipliers) assigned to the outputs and inputs, respectively. The conventional dual formulation of the DEA under Constant Returns to Scale (CRS) is then:

$$\begin{aligned} e_p = \max_{u,v} \quad & uY_p \\ \text{s.t.} \quad & vX_p = 1, \\ & uY_j - vX_j \leqslant 0, \quad \forall j, \\ & u, v \geqslant 0. \end{aligned} \tag{1}$$

The conventional model imposes no constraints on the dual weights u, v that are set endogenously throughout the linear program.

DMU_p is [technically] input-efficient if the optimal value of the objective function, e_p^*, is equal to one; otherwise, it is [technically] input-inefficient. The following output-oriented technical efficiency is a measure of the potential output of DMU_p when the inputs are held constant:

$$\begin{aligned} f_p = \min_{u,v} \quad & vX_p \\ \text{s.t.} \quad & uY_p = 1, \\ & uY_j - vX_j \leqslant 0, \quad \forall j, \\ & u, v \geqslant 0. \end{aligned} \tag{2}$$

DMU_p is [technically] output-efficient if the optimal value of the objective function, f_p^*, is equal to one; otherwise, it is [technically] output-inefficient. Assuming that e_p^* is the optimal solution for the input-oriented model (1), then $f_p^* = 1/e_p^*$ under CRS.

3. Symmetric Weight Assignment Technique (SWAT)

In the conventional DEA models, the total weight flexibility is a potential limitation, limiting the usefulness of the results when dual prices contradict logical, systemic or economic conditions that pertain to the data set. Moreover, zero-valued dimensions contradict the fundamental assumption that all facets of the production space carry some information about the resources and results obtained from the process. In order to provide a balance between *rigidity* and *total flexibility*, a series of weight-restriction DEA models have been proposed. In this section, we provide a brief overview of the method in Dimitrov and Sutton (2010).

The symmetric and asymmetric weights can be rewarded and penalized, respectively, using the following absolute value function:

$$Z_{ij} = |u_i y_{ip} - u_j y_{jp}|, \quad \forall i,j,$$

where z_{ij} is the difference between the ith and jth output dimensions for DMU_p. A penalty function for the asymmetry measure, defined as the sum of z_{ij} (i.e., $\sum_{i,j} z_{ij} = e^t Ze$), is minimized by a scaling factor $\beta \geqslant 0$. As a result, the following output oriented CRS model is obtained by adding the symmetry equation to the objective function in (2):

$$\begin{aligned} O_p^{(O)}(\beta) = \min_{u,v,Z} \quad & vX_p + \min_{Z} \beta e^T Ze \\ \text{s.t.} \quad & uY_p = 1, \\ & uY_j - vX_j \leqslant 0, \quad \forall j, \\ & |u_i y_{ip} - u_j y_{jp}| = z_{ij}, \quad \forall i,j, \\ & u, v \geqslant 0, \end{aligned} \tag{3}$$

where the parameter $\beta \in [0,\infty)$ is a penalty term to impose a symmetric selection of the virtual weights of DMU_p. Naturally, at the lower bound for β, model (3) is equivalent to model (2). Asymptotically, as β approaches infinity, the virtual output weights converge for all the dimensions for all the DMUs. Note here that a large β value plays a neutral role when a DMU already uses fully symmetric virtual weight selection. Inversely, a small β value decreases the discriminatory power. Due to the term $\min \beta e^T Ze$ in the objective function, we can relax the equality constraints and reformulate the model as follows:

$$\begin{aligned} O_p^{(O)}(\beta) = \min_{u,v,Z} \quad & vX_p + \beta e^T Ze \\ \text{s.t.} \quad & uY_p = 1, \\ & uY_j - vX_j \leqslant 0, \quad \forall j, \\ & u_i y_{ip} - u_j y_{jp} \leqslant z_{ij}, \quad \forall i,j, \\ & -u_i y_{ip} + u_j y_{jp} \leqslant z_{ij}, \quad \forall i,j, \\ & u, v \geqslant 0. \end{aligned} \tag{4}$$

The optimal objective function value of model (4), $O_p^{(O)}(\beta) \in [1,\infty)$, is said to be the *SWAT score* and is qualitatively similar to a radial projection in a conventional output oriented DEA model, in which the smaller scores are more desirable. Notice that the added constraints in (4) in comparison with model (2) do not have an influence on the feasibility region.

Analogously, the input oriented DEA model in terms of the symmetric selection of the input weights can be formulated as follows (Dimitrov & Sutton, 2010):

$$\begin{aligned} I_p^{(I)}(\gamma) = \min_{u,v,Z} \quad & \gamma e^T Ze + \max_{u,v} uY_p \\ \text{s.t.} \quad & vX_p = 1, \\ & uY_j - vX_j \leqslant 0, \quad \forall j, \\ & v_i x_{ip} - v_j x_{jp} \leqslant z_{ij}, \quad \forall i,j, \\ & -v_i x_{ip} + v_j x_{jp} \leqslant z_{ij}, \quad \forall i,j, \\ & u, v \geqslant 0, \end{aligned} \tag{5}$$

where the parameter $\gamma \in (-\infty, 0]$ is used to penalize the asymmetric selection of the virtual weights of DMU_p. The objective function of (5) is a max–min problem. For the purpose of computational feasibility, Dimitrov and Sutton (2010) used a dualization of the inner maximization problem (5) for a fixed value of Z as follows:

$$\begin{aligned} d_p = \min_{\theta,\lambda,h^1,h^2} \quad & \theta + \sum_{i,j} \left(h_{ij}^1 + h_{ij}^2\right) z_{ij} \\ \text{s.t.} \quad & \theta X_p - \lambda^t X_j + \sum_{ij} h_{ij}^1 (x_{ip} - x_{jp}) + \sum_{ij} h_{ij}^2 (-x_{ip} + x_{jp}) \geqslant 0, \forall j, \\ & \lambda^t Y_j \geqslant Y_p, \quad \forall j, \\ & \lambda, h^1, h^2 \geqslant 0. \end{aligned} \tag{6}$$

The optimal objective function value of model (6), $d_p^* \in (0,1]$, is said to be the *SWAT score* and larger scores are more desirable. The parametric programming of (5) is monotone in Z. In fact, the resulting model is an example of bilinear programming when using the outer minimization with the inner minimization of model (5).

4. Proposed models

In this section, we first propose a new model to deal with the problem of model (5). We then develop four additional cases consisting of absolute value function.

4.1. Symmetrization of the input weights in the input-oriented model

As mentioned in the previous section, model (5) proposed by Dimitrov and Sutton (2010) involves a shortcoming in its linearization. To overcome this problem, we first rewrite model (1) as follows:

$$\begin{aligned} -e_p = \min_{u,v} \quad & -uY_p \\ \text{s.t.} \quad & vX_p = 1, \\ & uY_j - vX_j \leqslant 0, \quad \forall j, \\ & u, v \geqslant 0. \end{aligned} \tag{7}$$

Adding a symmetric weight constraint for the inputs and the penalty function to the above model results in the following program:

$$\begin{aligned} -I_p^{(I)}(\gamma) = \min_{u,v} \quad & -uY_p + \min_{Z} \gamma e^T Ze \\ \text{s.t.} \quad & vX_p = 1, \\ & uY_j - vX_j \leqslant 0, \quad \forall j, \\ & |v_i x_{ip} - v_j x_{jp}| = z_{ij}, \quad \forall i,j, \\ & u, v \geqslant 0, \end{aligned} \tag{8}$$

where the symmetric scaling factor γ is non-negative. Therefore, we can simply obtain the linear program (9) without using the dual program (6).

$$\begin{aligned} -I_p^{(I)}(\gamma) = \min_{u,v,Z} \quad & -uY_p + \gamma e^T Ze \\ \text{s.t.} \quad & vX_p = 1, \qquad \text{(I)} \\ & uY_j - vX_j \leqslant 0, \quad \forall j, \qquad \text{(II)} \\ & v_i x_{ip} - v_j x_{jp} \leqslant z_{ij}, \quad \forall i,j, \\ & -v_i x_{ip} + v_j x_{jp} \leqslant z_{ij}, \quad \forall i,j, \\ & u, v \geqslant 0. \end{aligned} \tag{9}$$

Theorem 1. Problem (9) always has a feasible solution in the domain $-1 \leqslant -I_p^{(I)}(\gamma) \leqslant 2\gamma(m-1)$.

Proof. See Appendix A. □

Definition 1. DMU_p is called *symmetrically input efficient* if the optimal solution of model (9) is equal to -1.

In Sections 4.2 and 4.3, we will propose two DEA models for symmetrizing the output weights and the input weights of the input-oriented and output-oriented models, respectively. Then, in Sections 4.4 and 4.5 we will propose two DEA models for symmetrizing the input–output weights simultaneously for the input-oriented and output-oriented models, respectively.

4.2. Symmetrization of the output weights in the input-oriented model

Model (8) is intended to reward the DMUs that make a symmetric selection of the input weights. However, the following proposed input-oriented CRS model considers a penalty for using a set of asymmetric weights and rewards the DMUs that select the symmetric output weights:

$$\begin{aligned} I_p^{(O)}(\beta) = -\min_{u,v,Z} \quad & -uY_p + \min_Z \beta e^T Ze \\ \text{s.t.} \quad & vX_p = 1, \\ & uY_j - vX_j \leqslant 0, \quad \forall j, \\ & |u_i y_{ip} - u_j y_{jp}| = z_{ij}, \quad \forall i,j, \\ & u, v \geqslant 0, \end{aligned} \tag{10}$$

where the symmetry penalty β is non-negative. The problem can be converted into a linear program by relaxing the equality constraints as well as merging the two minimization problems as follows:

$$\begin{aligned} -I_p^{(O)}(\beta) = \min_{u,v,Z} \quad & -uY_p + \beta e^T Ze \\ \text{s.t.} \quad & vX_p = 1, \\ & uY_j - vX_j \leqslant 0, \quad \forall j, \\ & u_i y_{ip} - u_j y_{jp} \leqslant z_{ij}, \quad \forall i,j, \\ & -u_i y_{ip} + u_j y_{jp} \leqslant z_{ij}, \quad \forall i,j, \\ & u, v \geqslant 0. \end{aligned} \tag{11}$$

Corollary 1. *Problem (11) always has a feasible solution and a lower bound for the objective function value at* -1.

Proof. Follows by extension from the proof of Theorem 1. □

Definition 2. DMU_p is called *symmetrically output efficient* if the optimal solution of model (11) is equal to -1.

Theorem 2. *There is an* $\beta_o > 0$ *in model (11) such that* $z_{ij}^* = 0$ *in an optimal solution.*

Proof. See Appendix B. □

4.3. Symmetrization of the input weights in the output-oriented model

The following output-oriented CRS model makes a symmetric choice of weights for the inputs:

$$\begin{aligned} O_p^{(I)}(\gamma) = \min_{u,v,Z} \quad & vX_p + \min_Z \gamma e^T Ze \\ \text{s.t.} \quad & uY_p = 1, \\ & uY_j - vX_j \leqslant 0, \quad \forall j, \\ & |v_i x_{ip} - v_j x_{jp}| = z_{ij}, \quad \forall i,j, \\ & u, v \geqslant 0, \end{aligned} \tag{12}$$

where the symmetry penalty γ is non-negative. Compared to model (3) in Dimitrov and Sutton (2010), model (12) imposes a symmetric choice of input weights while model (3) imposes a symmetric choice of output weights. We rewrite model (12) as the following linear program:

$$\begin{aligned} O_p^{(I)}(\gamma) = \min_{u,v,Z} \quad & vX_p + \gamma e^T Ze \\ \text{s.t.} \quad & uY_p = 1, \\ & uY_j - vX_j \leqslant 0, \quad \forall j, \\ & v_i x_{ip} - v_j x_{jp} \leqslant z_{ij}, \quad \forall i,j, \\ & -v_i x_{ip} + v_j x_{jp} \leqslant z_{ij}, \quad \forall i,j, \\ & u, v \geqslant 0. \end{aligned} \tag{13}$$

Model (13) is feasible and the value of the objective function in model (13) lies within $[1, \infty)$. Thus, a smaller value for the objective function of a given DMU for a given penalty γ implies a higher ranking.

Definition 3. DMU_p is called *symmetrically input efficient* if the optimal solution of model (13) is equal to 1.

Theorem 3.

(a) *In model (9)* $I_p^{(I)}(\gamma) = 1$, *if and only if* $O_p^{(I)}(\gamma) = 1$ *in model (13).*
(b) *In model (11)* $I_p^{(o)}(\beta) = 1$, *if and only if* $O_p^{(o)}(\beta) = 1$ *in model (4).*

Proof. See Appendix C. □

4.4. Symmetrization of the input–output weights in the input-oriented model

The purpose of the following model is to reward the DMUs that make a symmetric selection of the input and outputs weights by means of the penalty function:

$$\begin{aligned} -I_p^{(IO)}(\beta) = \min_{u,v,Z,T} \quad & -uY_p + \min_{Z,T} \beta(e^T Ze + e^T Te) \\ \text{s.t.} \quad & vX_p = 1, \\ & uY_j - vX_j \leqslant 0, \quad \forall j, \\ & |u_i y_{ip} - u_j y_{jp}| = z_{ij}, \quad \forall i,j, \\ & |v_i x_{ip} - v_j x_{jp}| = t_{ij}, \quad \forall i,j, \\ & u, v \geqslant 0, \end{aligned} \tag{14}$$

where the asymmetry penalty β is non-negative, here applied to the sum of the output and input weight deviations simultaneously. Similarly, model (14) can be linearized as follows:

$$\begin{aligned} -I_p^{(IO)}(\beta) = \min_{u,v,Z,T} \quad & -uY_p + \min_{Z,T} \beta(e^T Ze + e^T Te) \\ \text{s.t.} \quad & vX_p = 1, \\ & uY_j - vX_j \leqslant 0, \quad \forall j, \\ & u_i y_{ip} - u_j y_{jp} \leqslant z_{ij}, \quad \forall i,j, \\ & -u_i y_{ip} + u_j y_{jp} \leqslant z_{ij}, \quad \forall i,j, \\ & v_i x_{ip} - v_j x_{jp} \leqslant t_{ij}, \quad \forall i,j, \\ & -v_i x_{ip} + v_j x_{jp} \leqslant t_{ij}, \quad \forall i,j, \\ & u, v \geqslant 0. \end{aligned} \tag{15}$$

This program is always feasible and the optimal value of the objective function is less than or equal to 1 for a given β.

Definition 4. DMU_p is called *symmetrically input–output efficient* if the optimal solution of model (15) is the maximum across all DMUs.

4.5. Symmetrization of the input–output weights in the output-oriented model

In the output-oriented case, model (14) is formulated as follows:

$$\begin{aligned} O_p^{(IO)}(\beta) = \min_{u,v,Z,T} \quad & vX_p + \min_{Z,T} \beta(e^T Ze + e^T Te) \\ \text{s.t.} \quad & uY_p = 1, \\ & uY_j - vX_j \leqslant 0, \quad \forall j, \\ & |u_i y_{ip} - u_j y_{jp}| = z_{ij}, \quad \forall i,j, \\ & |v_i x_{ip} - v_j x_{jp}| = t_{ij}, \quad \forall i,j, \\ & u, v \geqslant 0, \end{aligned} \tag{16}$$

where the non-negative asymmetry penalty β. Analogously, this model can be changed to a linear program as follows:

$$\begin{aligned} O_p^{(IO)}(\beta) = \min_{u,v,Z,T} \quad & vX_p + \min_{Z,T} \beta(e^T Ze + e^T Te) \\ \text{s.t.} \quad & uY_p = 1, \\ & uY_j - vX_j \leqslant 0, \quad \forall j, \\ & u_i y_{ip} - u_j y_{jp} \leqslant z_{ij}, \quad \forall i,j, \\ & -u_i y_{ip} + u_j y_{jp} \leqslant z_{ij}, \quad \forall i,j, \\ & v_i x_{ip} - v_j x_{jp} \leqslant t_{ij}, \quad \forall i,j, \\ & -v_i x_{ip} + v_j x_{jp} \leqslant t_{ij}, \quad \forall i,j, \\ & u, v \geqslant 0. \end{aligned} \tag{17}$$

This model is feasible and its optimal value of the objective function belongs to $[1, \infty)$. It is possible in this case that all SWAT scores for a given DMU take a value bigger than one. We should note that incorporating symmetry of input–output weights in the output-oriented model (17) is a special case of the model in Section 3.3 of Dimitrov and Sutton (2013). Dimitrov and Sutton's (2013) model allows for non-symmetric managerial preferences, whereas model (17) is strictly based on symmetric managerial preferences. The detailed difference between model (17) and Dimitrov and Sutton's (2013) model is the addition of the parameters α_{ij} and α'_{ij} to the left hand side of the constraints as follows:

$$\begin{aligned} & u_i y_{ip} - \alpha_{ij} u_j y_{jp} \leqslant z_{ij}, \quad \forall i,j, \\ & -u_i y_{ip} + \alpha_{ij} u_j y_{jp} \leqslant z_{ij}, \quad \forall i,j, \\ & v_i x_{ip} - \alpha'_{ij} v_j x_{jp} \leqslant t_{ij}, \quad \forall i,j, \\ & -v_i x_{ip} + \alpha'_{ij} v_j x_{jp} \leqslant t_{ij}, \quad \forall i,j, \end{aligned}$$

where α_{ij} and α'_{ij} are equal to or larger than unity.

Definition 5. DMU_p is called *symmetrically input–output efficient* if the optimal solution of model (17) is the minimum across all DMUs.

5. Numerical example

In this section, we revisit the numerical example in Dimitrov and Sutton (2010) to illustrate the applicability of the proposed models. Let us consider the performance evaluation problem presented in Table 1 with six DMUs, one input variable, and four output variables.

The SWAT score of model (4) developed by Dimitrov and Sutton (2010) for different β is shown in Fig. 1.

This figure shows the SWAT scores of the six DMUs as a function of β in terms of the symmetric output weights in the output-oriented model (4) proposed by Dimitrov and Sutton (2010). In this paper, we generalize this approach by taking into account the symmetric output weights in the input-oriented model (11). The SWAT scores of the DMUs as a function of β is depicted in Fig. 2 by considering the symmetric output weights.

Table 1
The input and output values of the numerical example proposed by Dimitrov and Sutton (2010).

DMU	Inputs	Outputs			
	x_1	y_1	y_2	y_3	y_4
1	1	10	10	10	10
2	1	5	5	5	5
3	3	10	20	15	23
4	2	20	20	20	10
5	5	50	50	25	25
6	4	12	16	32	20

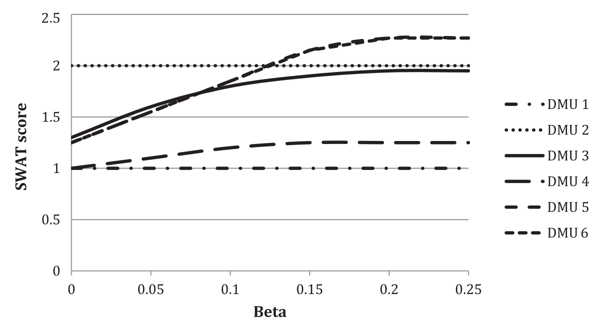

Fig. 1. The SWAT scores of the DMUs as a function of beta in terms of the symmetric output weights in the output-oriented model (4) proposed by Dimitrov and Sutton (2010).

In the output-oriented case, we take into consideration the symmetric input weights (see model (13)) to identify the SWAT score of each DMU. The result is plotted in Fig. 3.

It is interesting to determine the SWAT scores when the input–output weights are symmetric in both input and output orientations. We therefore apply models (15) and (17) to symmetrize the input–output weights in the input and output oriented cases as presented in Figs. 4 and 5, respectively.

Our results confirm the findings in Dimitrov and Sutton (2010) that the symmetry penalty parameter β has a significant impact on the SWAT score of the DMUs. Generally, a large β value imposes symmetry as the primary performance criterion. Inversely, the results for a small β value converge to those of the conventional DEA model (identical for $\beta = 0$). As β increases, i.e. as we impose symmetry, the respective SWAT scores may change. In particular, DMUs that exhibit highly asymmetric endogenous weights for low β will be increasingly penalized with β which may provoke changes in the ordinal rankings of the DMUs.

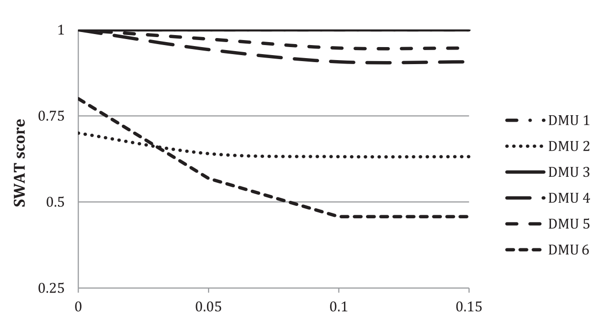

Fig. 2. The SWAT scores of the DMUs as a function of beta in terms of the symmetric output weights in the input-oriented model (11) proposed in this study (Note that DMU 1 overlaps with DMU 3).

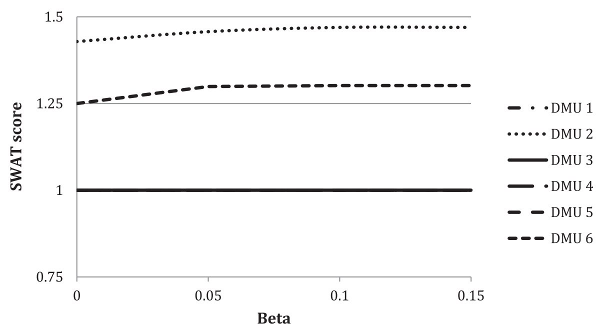

Fig. 3. The SWAT scores of the DMUs as a function of beta in terms of the symmetric input weights in the output-oriented model (13) proposed in this study (Note that DMUs 1, 3, 4 and 5 overlap each other at the SWAT score of 1).

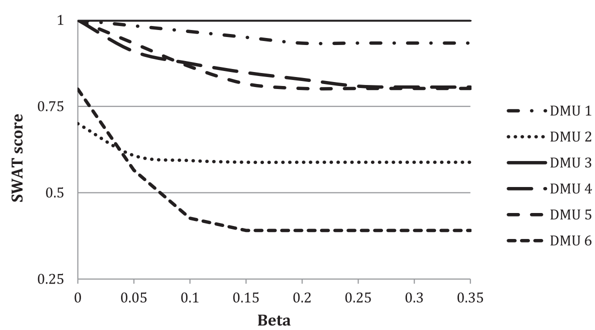

Fig. 4. The SWAT scores of the DMUs as a function of beta in terms of the symmetric input–output weights in the input-oriented model (15) proposed in this study.

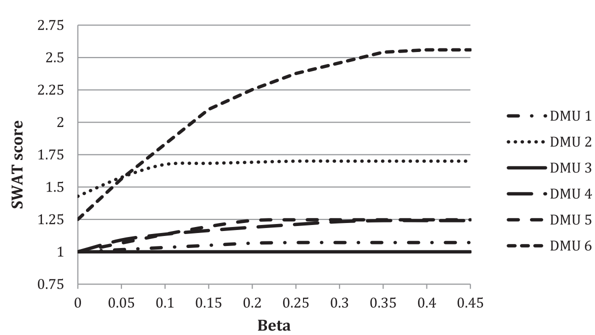

Fig. 5. The SWAT scores of the DMUs as a function of beta in terms of the symmetric input–output weights in the output-oriented model (17) proposed in this study.

On the other hand, in the case of a DMU that already has fully used symmetric dual weight selection, increasing β does not affect the corresponding SWAT score. In other cases, a sufficiently small β value allows a decision maker to potentially discriminate among the efficient DMUs and create a more clear ordinal rankings. When β approaches infinity, the weights, u and v, converge towards full symmetry. Hence, there exists a critical threshold $\beta(u)$ such that for all $\beta \geqslant \beta(u)$ output weights are totally symmetric. We call $\beta(u)$ the *cost of symmetry* as this corresponds to the optimal dual price of the symmetry constraint.

For large β values, we expect the ordinal rankings between the DMUs to be different from the ordinal rankings when the β values are small or zero. Taking these two extremes into consideration, we note that each condition is very important as $\beta = 0$ represents a ranking when symmetry is not relevant to the performance dimensions, while a large β represents a ranking when the output weights are imposed to be fully symmetric.

6. A case study

In this section, we apply the method to a panel of district heating plants in Denmark 2000–2001 presented in Agrell and Bogetoft (2005). We evaluate 286 plants for with two inputs and four outputs, presented in Table 2.

The selected input parameters, $\{x_1, x_2\}$, and output parameters, $\{y_1, y_2, y_3, y_4\}$, are listed below:

Table 2
Descriptive statistics for the district heating plants in Denmark.[a]

Variable	Unit of measurement	Mean	Median	Min	Max	Standard deviation
x_1	kDKK	6319	2301	138	226,039	18,580
x_2	GJ	314	84	10	11,919	1037
y_1	GJ	259	64	7	10,308	907
y_2	GW h	9	0	0	280	24
y_3	MW	78	21	2	2405	241
y_4	km	54	25	0	1597	127

[a] Agrell and Bogetoft (2005).

Table 3
Statistical results for Model (9).

Indicator	β				
	0 (CRS)	0.1	0.2	0.3	0.4
SWAT score					
Average	0.830	0.705	0.683	0.683	0.683
Median	0.82	0.689	0.655	0.655	0.655
Min	0.614	0.424	0.391	0.391	0.391
Max	1.000	1.000	1.000	1.000	1.000
Number of efficient DMUs	23	8	8	8	8

Table 4
Statistical results for Model (13).

Indicator	β					
	0 (CRS)	0.1	0.2	0.3	0.4	0.5
SWAT score						
Average	0.830	0.733	0.691	0.683	0.683	0.683
Median	0.820	0.720	0.666	0.655	0.655	0.820
Min	0.614	0.520	0.446	0.397	0.391	0.391
Max	1.000	1.000	1.000	1.000	1.000	1.000
Number of efficient DMUs	23	8	8	8	8	8

Table 5
Statistical results for Model (11).

Indicator	β						
	0 (CRS)	0.1	0.2	0.3	0.4	0.5	0.6
SWAT score							
Average	0.829	0.513	0.346	0.258	0.197	0.151	0.151
Median	0.820	0.496	0.309	0.208	0.105	0.000	0.000
Min	0.614	0.246	0	0	0	0	0
Max	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Number of efficient DMUs	23	1	1	1	1	1	1

Table 6
Statistical results for Model (4).

Indicator	β										
	0 (CRS)	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
SWAT score											
Average	0.829	0.621	0.521	0.462	0.420	0.388	0.362	0.342	0.324	0.310	0.299
Median	0.820	0.606	0.497	0.428	0.379	0.344	0.320	0.300	0.284	0.269	0.255
Min	0.614	0.449	0.353	0.292	0.248	0.216	0.191	0.172	0.156	0.142	0.131
Max	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Number of efficient DMUs	23	1	1	1	1	1	1	1	1	1	1

x_1: The operating expenditure (1000 Danish Crowns: kDKK),
x_2: The primary fuel input in gigajoules (GJ),
y_1: The heat energy delivered in gigajoules (GJ),
y_2: The electrical energy delivered in kilowatt hours (GW h),
y_3: The heat capacity utilized in megawatt (MW), and
y_4: The total length of pipelines in kilometers (km).

We first apply the CRS model to calculate the technical input-efficiency of the plants as presented in Table 3 ($\beta = 0$). Under total weight flexibility, the outcome yields a multitude of fully input-efficient DMUs. However, in this regulatory problem all measured dimensions have strictly positive resource prices and all outputs correspond to marketable services. In spite of a common regulation and technology, the resulting 23 efficient peers are not comparable and do not all correspond to economic, technical or environmental best practice. To deal with this dilemma, we calculate SWAT scores for distinct β values where symmetric and asymmetric weights are rewarded and penalized, respectively.

We first focus on the symmetric weights of inputs in the input and output orientations using models (9) and (13). The statistical descriptions of outcomes are reported in Tables 3 and 4. Note that the multiplicative inverse value of the SWAT scores in the output-oriented is shown in the tables. The results demonstrate that the inverse value of SWAT scores in the output-oriented model (13) is almost equal to the SWAT scores for the input-oriented model (9) for a given β level. In addition, the results mirror Theorem 3(a) for the DMU whose SWAT score is one. From these models, there are eight efficient DMUs and the discriminatory power increases 65% compared with CRS for a given $\beta(\neq 0)$ level. Interestingly, all DMUs in the input-oriented and output-oriented cases converge to the constant SWAT scores when $\beta \in [0.4, \infty]$ and $\beta \in [0.5, \infty]$, respectively. We may interpret this result as revealing the implicit cost of endogenous weights; masking the true sector inefficiency through a considerable number of "irrelevant" peers that distort the assessment of an otherwise stable and relatively homogenous technology.

We then consider the symmetric weights of outputs in the output and input orientations using models (4) and (11). Tables 5 and 6 present the descriptive statistics for the results of models (11) and (4), respectively. An important observation is that the inverse value of SWAT scores in output-oriented model (13) is not equal to SWAT scores in the input-oriented model (9) for a given $\beta(\neq 0)$ level. The SWAT scores of DMUs in input-oriented model (11) converge to the constant when $\beta \in [0.6, \infty]$ while we do not have any convergence for the SWAT scores of DMUs in output-oriented model (4). In comparison with CRS, the discriminatory power improves considerably for positive β as shown in Tables 5 and 6. In other words, DMU 19 is the only efficient DMU in the both cases (also see Theorem 3(b)).

We now take account of the symmetric weights of input–output in the input and output orientations using models (15) and (17). In this case the inverse value of SWAT scores in output-oriented model (17) is not equal to SWAT scores in the input-oriented model (15) for a given $\beta(\neq 0)$ level. The SWAT scores of DMUs in input-oriented model (15) converge to the constant when $\beta \in [0.6, \infty]$ while we do not have any convergence for the SWAT scores of DMUs in output-oriented model (17). As seen in Table 7 and 8, although the maximum of SWAT for models (15) and (17) is less than unity DMU 19 has still the best score for various β values. Note that the models (4), (9), (11) and (13) can be transformed to the conventional CRS models when $\beta = 0$. This approach is applicable in the case where the activity model exhaustively covers the assessed production, the activity dimensions are scaled in commensurate units and where the reference set is homogenous in both input and output technology. In this particular case, the extreme results confirm the intuition that the output dimensions for the application have varying marginal value. Naturally, the heat volume delivered must be more important for a district heating plant than e.g. the installed electricity generation capacity, which is a secondary output.

Table 7
Statistical results for Model (15).

Indicator	β						
	0 (CRS)	0.1	0.2	0.3	0.4	0.5	0.6
SWAT score							
Average	0.829	0.450	0.306	0.231	0.178	0.137	0.137
Median	0.820	0.4405	0.276	0.1785	0.091	0	0
Min	0.614	0.156	0	0	0	0	0
Max	1.000	0.991	0.982	0.977	0.977	0.977	0.977

Table 8
Statistical results for Model (17).

Indicator	β										
	0 (CRS)	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
SWAT score											
Average	0.830	0.568	0.472	0.421	0.387	0.359	0.337	0.319	0.303	0.290	0.280
Median	0.820	0.558	0.460	0.400	0.360	0.333	0.306	0.282	0.262	0.247	0.236
Min	0.614	0.397	0.290	0.232	0.202	0.180	0.162	0.148	0.136	0.126	0.117
Max	1.000	0.991	0.982	0.977	0.977	0.977	0.977	0.977	0.977	0.977	0.977

7. Conclusions and future research directions

One of the most enticing, yet frustrating, features of non-parametric frontier analysis with DEA is the endogenous dual weight determination. The absence of the requirement to impose an a priori functional form turns DEA into a powerful, informative and cautious performance assessment method. However, complete flexibility to assign dual weights may also lead to results that are nothing but mathematical abstractions, failing to detect best system practice among outliers and mavericks with little or no predictive value.

Weight restriction techniques address this dilemma by constraining the dual price space through various cuts and relations – frequently requiring exactly the type of a priori production information that DEA purports to exempt the analysts to possess. Opting for the endogenous quality in weight assessment, yet mitigating the undesirable effects of excessive weight flexibility, the Dimitrov and Sutton (2010) Symmetric Weight Assignment Technique (SWAT) is our starting point for this paper. We generalize the SWAT model by exploiting the degree of imposed symmetry as a scalarizing parameter, where the given SWAT model is a special case. Our approach extends and consolidates previous work in that we show the continuity between fully flexible weights in conventional DEA models and fully symmetric weights as a special case of the CSW methods. Using our framework, we can study and quantify the stability of frontier assessments using technological homogeneity assumptions, expressed through symmetry conditions.

The case study for the Danish district heating perfectly illustrates the use and novelty of the approach, since neither the conventional CRS approach (which overestimates technical efficiency), nor the previous specific SWAT cut (that may be sensitive to outliers) can fully capture the degree of homogeneity of the underlying technology.

The case study observations also suggest some promising directions for further research. The variation of the symmetry parameter allows the identification of outliers in the non-parametric sense, meaning units that have an impact on the frontier estimates, in both ends of the interval. It would be interesting to explore the statistical and computational properties of this technique, as well as comparisons with alternative techniques (cf. Agrell & Niknazar, 2014). Another interesting avenue leads to the endogenization of the flexibility parameter itself, e.g. through parametric estimates of the statistical stability in the marginal cost or benefit estimates. Such work would close the gap with a fully endogenous and robust performance assessment model, potentially improving the applicability of DEA in panels with small cross-sections.

Acknowledgements

They also would like to thank the anonymous reviewers for their insightful and constructive comments. Adel Hatami-Marbini also likes to thank the FRS-FNRS for the support he received as a “chargé de recherches” at the Universitè catholique de Louvain for this research.

Appendix A

Proof of Theorem 1. Feasibility is shown by construction. Since the X_p vector is not zero, it involves at least one positive component. Hence, we assume that x_{kp} is a positive input. An upper bound for the objective function would be to maximize the weight difference and cancel the output contribution:

$$\bar{v}_k = \frac{1}{x_{kp}},$$
$$\bar{v}_l = 0, \quad l = 1, \ldots, m, \; l \neq k,$$
$$u = 0,$$
$$e^T Z e = 2(m-1).$$

Analogously, a lower bound is found at -1 as the intersection of constraints (I) and (II) in model (9) when input contributions are fully symmetric. Thus, the value of the objective function always lies within $[-1, 2\gamma(m-1)]$. □

Appendix B

Proof of Theorem 2. Corollary 1 implies that model (11) has at least one feasible solution with the corresponding objective function value of 0. Owing to the fact that the objective function of (11) is minimizing, the feasible solution is an upper bound for an optimal solution:

$$-u^* Y_p + \beta e^T Z^* e \leqslant 0. \tag{I}$$

In addition, with respect to the constraints, we have:

$$\left.\begin{array}{l} vX_p = 1 \\ -vX_p + uY_p \leqslant 0 \end{array}\right\} \rightarrow uY_p \leqslant 1. \tag{II}$$

In (II), the non-negativity of u and Y_p result in:

$$0 \leqslant uY_p \leqslant 1 \rightarrow -1 \leqslant -uY_p \leqslant 0. \tag{III}$$

By adding $\beta e^T Z e$ to both sides of Eq. (III), $-1 \leqslant -uY_p$, the following equation is obtained:

$$-1 + \beta e^T Z e \leqslant -uY_p + \beta e^T Z e. \tag{IV}$$

The right hand side of (IV) is equivalent to the objective function of model (11), and β and Z are non-negative in the left hand side of (IV). It therefore implies the finite optimal solution of (11) and extreme optimal solutions (corner points).

Assuming that $\{(u^k, v^k, z^k_{ij}); k = 1, \ldots, l\}$ is a feasible solution (extreme point), the objective function of (11) is equivalent to $\max\, uY_p + \min_{\beta \leqslant 0} \beta e^T Z e$.

Let $\min\{e^T Z^i e; e^T Z^i e > 0; i = 1, \ldots, l\} = e^T Z^i e = \alpha > 0$ and $\beta_o = 1/\alpha$. Then, for each $\beta > \beta_o$ and extreme point with $e^T Z^i e > 0$, we have

$$\beta e^T Z^i e > \beta_o e^T Z^i e = \frac{1}{\alpha} e^T Z^i e > \frac{1}{\alpha} \underbrace{e^T Z^i e}_{\substack{\Downarrow \\ \min\{e^T Z^i e\}}} = \frac{1}{\alpha} \cdot \alpha = 1 \rightarrow \beta e^T Z^i e > 1. \tag{V}$$

So, we have $\beta e^T Z^i e > 1$ for each extreme point with $e^T Z^i e > 0$.

By considering (V) and $-1 \leqslant -uY_p$, the following equation is obtained:

$$-u^i Y_p + \beta e^T Z^i e > -1 + 1 = 0.$$

The above inequality contradicts the fact that the optimal value of the objective function is always less than or equal to 0 (see (I)). Therefore, this extreme point (u^k, v^k, z^k_{ij}) cannot be an optimal solution. In other words, for each $\beta > \beta_o$, the extreme point cannot be optimal unless $e^T Z^i e = 0$. As a result, for each extreme optimal solution, we always have $e^T Z^i e = 0$ since $z_{ij} \geqslant 0$, meaning that if (u^*, v^*, z^*_{ij}) is an extreme optimal solution, then $z^*_{ij} = 0$. □

Appendix C

Proof of Theorem 3. Assume that (u^*, v^*, γ^*) is an optimal solution for model (9) where $I_p^{(I)}(\gamma) = 1$. Considering the constraints in model (9), $v^*X_p = 1$ and $-v^*X + u^*Y \leqslant 0$, therefore:

$$u^*Y \leqslant 1, \tag{i}$$

and since $I_p^{(I)}(\gamma) = 1$, we have:

$$-u^*Y_p + \gamma e^t Ze = -1.$$

Moreover, with regards to the non-negativity of the penalty function and symmetry scaling factor γ, we have

$$-u^*Y_p + \beta e^t Ze + 1 \geqslant 1. \tag{ii}$$

From (i) and (ii), we can conclude that $u^*Y_p = 1$ and $\beta e^t Ze = 0$.

This optimal solution for (9) is a feasible solution of (13) since it satisfies the constraints of (13), therefore, $O_p^{(I)}(\gamma) = 1$. Since $O_p^{(I)}(\gamma)$ has the domain $[1, \infty)$, (u^*, v^*, γ^*) is an optimal solution to (13).

The proof of (b) is analogous to the proof of (a) and left to the reader. □

References

Agrell, P. J., & Bogetoft, P. (2005). Economic and environmental efficiency of district heating plants. *Energy Policy, 33*(10), 1351–1362.
Agrell, P. J., & Niknazar, P. (2014). Structural and behavioral robustness in applied best-practice regulation. *Socio-Economic Planning Sciences, 48*(1), 89–103.
Ali, A. I., Cook, W. D., & Seiford, L. M. (1991). Source strict vs. weak ordinal relations for multipliers in data envelopment analysis. *Management Science, 37*(6), 733–738.
Allen, R., Athanassopoulos, A., Dyson, R. G., & Thanassoulis, E. (1997). Weights restrictions and value judgments in data envelopment analysis: Evolution, development and future directions. *Annals of Operations Research, 73*, 13–34.
Bernroider, E., & Stix, V. (2007). A method using weight restrictions in data envelopment analysis for ranking and validity issues in decision making. *Computers & Operations Research, 34*, 2637–2647.
Charnes, A., Cooper, W. W., & Rhodes, E. L. (1978). Measuring the efficiency of decision making units. *European Journal of Operational Research, 2*(6), 429–444.
Charnes, A., Cooper, W. W., Wei, Q. L., & Huang, Z. M. (1989). Cone ratio data envelopment analysis and multi-objective programming. *International Journal of Systems Science, 20*(7), 1099–1118.
Cook, W. D., Kress, M., & Seiford, L. M. (1992). Prioritization models for frontier decision making units in DEA. *European Journal of Operational Research, 59*, 319–323.
Cook, W., Roll, Y., & Kazakov, A. (1990). A DEA model for measurement the relative efficiency of highway maintenance patrols. *Information Systems and Operational Research, 28*(2), 113–124.
Cooper, W. W., Ruiz, J. L., & Sirvent, I. (2007). Choosing weights from alternative optimal solutions of dual multiplier models in DEA. *European Journal of Operational Research, 180*, 443–458.
Cooper, W. W., Seiford, L. M., & Tone, K. (1999). Data envelopment analysis: A comprehensive text with models. In *Applications references and DEA-solver software*. Kluwer Academic Publishers.
Dimitrov, S., & Sutton, W. (2010). Promoting symmetric weight selection in data envelopment analysis: A penalty function approach. *European Journal of Operational Research, 200*(1), 281–288.
Dimitrov, S., & Sutton, W. (2013). Generalized symmetric weight assignment technique: Incorporating managerial preferences in data envelopment analysis using a penalty function. *Omega, 41*(1), 48–54.
Doyle, J. R., & Green, R. H. (1994). Efficiency and cross-efficiency in DEA: Derivations, meanings and uses. *Journal of the Operational Research Society, 45*(5), 567–578.
Dyson, R. G., & Thanassoulis, E. (1988). Reducing weight flexibility in DEA. *Journal of the Operations Research Society, 39*, 563–576.
Estellita Lins, M. P., Moreira da Silva, A. C., & Lovell, C. A. K. (2007). Avoiding infeasibility in DEA models with weight restrictions. *European Journal of Operational Research, 181*, 956–966.
Farrell, M. J. (1957). The measurement of productive efficiency. *Journal of the Royal Statistical Society, 120*, 253–281.
Golany, B. (1988). A note on including ordinal relations among multipliers in DEA. *Management Science, 34*, 1029–1033.
Halme, M., Joro, T., Korhonen, P., Salo, S., & Wallenius, J. (1999). A value efficiency approach to incorporating preference information in data envelopment analysis. *Management Science, 45*, 103–115.
Halme, M., & Korhonen, P. (2000). Restricting weights in value efficiency analysis. *European Journal of Operational Research, 126*, 175–188.
Hatami-Marbini, A., Tavana, M., Agrell, P. J., Lotfi, F. H., & Beigi, Z. G. (2015). A common-weights DEA model for centralized resource reduction and target setting. *Computers & Industrial Engineering, 79*, 195–203.
Hosseinzadeh Lotfi, F., Hatami-Marbini, A., Agrell, P. J., Aghayi, N., & Gholami, K. (2013). Allocating fixed resources and setting targets using a common-weights DEA approach. *Computers & Industrial Engineering, 64*(2), 631–640.
Jahanshahloo, G. R., Hosseinzadeh Lotfi, F., Jafari, Y., & Maddahi, R. (2011). Selecting symmetric weights as a secondary goal in DEA cross-efficiency evaluation. *Applied Mathematical Modelling, 35*, 544–549.
Kao, C. (2008). A linear formulation of the two-level DEA model. *Omega, 36*, 958–962.
Khalili, M., Camanho, A. S., Portela, M. C. A. S., & Alirezaee, M. R. (2010). The measurement of relative efficiency using data envelopment analysis with assurance regions that link inputs and outputs. *European Journal of Operational Research, 203*, 761–770.
Korhonen, P. J., Soleimani-damaneh, M., & Wallenius, J. (2011). Ratio-based RTS determination in weight-restricted DEA models. *European Journal of Operational Research, 215*, 431–438.
Kornbluth, J. (1991). Analysing policy effectiveness using cone restricted data envelopment analysis. *Journal of the Operational Research Society, 42*, 1097–1104.
Kuosmanen, T., Cherchye, L., & Sipiläinen, T. (2006). The law of one price in data envelopment analysis: Restricting weight flexibility across firms. *European Journal of Operational Research, 170*, 735–757.
Liang, L., Wu, J., Cook, W. D., & Zhu, J. (2008). Alternative secondary goals in DEA cross-efficiency evaluation. *International Journal of Production Economics, 113*(2), 1025–1030.
Li, Y., Liang, L., Chen, Y., & Morita, H. (2008). Models for measuring and benchmarking Olympics achievements. *Omega, 36*, 933–940.
Liu, F.-H. F., & Peng, H. H. (2008). Ranking of units on the DEA frontier with common weights. *Computers & Operations Research, 35*, 1624–1637.
Liu, F-H. F., & Peng, H. H. (2009). A systematic procedure to obtain a preferable and robust ranking of units. *Computers & Operations Research, 36*, 1012–1025.
Meng, W., Zhang, D., Qi, L., & Liu, W. (2008). Two-level DEA approaches in research evaluation. *Omega, 36*, 950–957.
Podinovski, V. V. (2004a). Production trade-offs and weight restrictions in data envelopment analysis. *The Journal of the Operational Research Society, 55*, 1311–1322.
Podinovski, V. V. (2004b). Suitability and redundancy of non-homogeneous weight restrictions for measuring the relative efficiency in DEA. *European Journal of Operational Research, 154*, 380–395.
Ramón, N., Ruiz, J. L., & Sirvent, I. (2010a). A multiplier bound approach to assess relative efficiency in DEA without slacks. *European Journal of Operational Research, 203*, 261–269.
Ramón, N., Ruiz, J. L., & Sirvent, I. (2010b). On the choice of weights profiles in cross-efficiency evaluations. *European Journal of Operational Research, 207*(3), 1564–1572.
Ramón, N., Ruiz, J. L., & Sirvent, I. (2011). Reducing differences between profiles of weights: A peer-restricted cross-efficiency evaluation. *Omega, 39*(6), 634–641.
Ramón, N., Ruiz, J. L., & Sirvent, I. (2012). Common sets of weights as summaries of DEA profiles of weights: With an application to the ranking of professional tennis players. *Expert Systems with Applications, 39*, 4882–4889.
Roll, Y., Cook, W., & Golany, B. (1991). Controlling factor weights in DEA. *IIE Transactions, 23*, 2–9.
Roll, Y., & Golany, B. (1993). Alternative methods of treating factor weights in DEA. *Omega, 21*, 99–109.
Ruiz, J. L., & Sirvent, I. (2012). On the DEA total weight flexibility and the aggregation in cross-efficiency evaluations. *European Journal of Operational Research, 223*, 732–738.
Sarrico, C. S., & Dyson, R. G. (2004). Restricting virtual weights in data envelopment analysis. *European Journal of Operational Research, 159*, 17–34.
Sexton, T. R., Silkman, R. H., & Hogan, A. J. (1986). Data envelopment analysis: Critique and extensions. In R. H. Silkman (Ed.), *Measuring efficiency: An assessment of data envelopment analysis* (pp. 3–105). San Francisco: Jossey-Bass.
Silva, R. C., & Milioni, A. Z. (2012). The adjusted spherical frontier model with weight restrictions. *European Journal of Operational Research, 220*, 729–735.
Thompson, R. G., Langemeier, L., Lee, C., Lee, E., & Thrall, R. (1990). The role of multiplier bounds in efficiency analysis with application to Kansas farming. *Journal of Econometrics, 46*, 93–108.
Thompson, R. G., Lee, E., & Thrall, R. (1992). DEA/AR-efficiency of US independent oil/gas producers over time. *Computers and Operations Research, 19*, 377–391.
Thompson, R. G., Singleton, F. D., Jr, Thrall, R. M., & Smith, B. A. (1986). Comparative site evaluations for locating a high-energy physics lab in Texas. *Interfaces, 16*(6), 35–49.
Wang, Y.-M., & Chin, K. S. (2010a). A neutral DEA model for cross-efficiency evaluation and its extension. *Expert Systems with Applications, 37*(5), 3666–3675.
Wang, Y-M., & Chin, K. S. (2010b). Some alternative models for DEA cross-efficiency evaluation. *International Journal of Production Economics, 128*(1), 332–338.
Wang, Y-M., Chin, K.-S., & Poon, G. K. K. (2008). A data envelopment analysis method with assurance region for weight generation in the analytic hierarchy process. *Decision Support Systems, 45*, 913–921.
Wang, Y-M., Luo, Y., & Lan, Y.-X. (2011). Common weights for fully ranking decision making units by regression analysis. *Expert Systems with Applications, 38*, 9122–9128.

Wang, Y-M., Luo, Y., & Liang, L. (2009). Ranking decision making units by imposing a minimum weight restriction in the data envelopment analysis. *Journal of Computational and Applied Mathematics, 223*, 469–484.

Wong, Y. H., & Beasley, J. E. (1990). Restricting weight flexibility in data envelopment analysis. *Journal of the Operational Research Society, 41*(9), 829–835.

Wu, J., Liang, L., & Yang, F. (2009a). Achievement and benchmarking of countries at the Summer Olympics using cross efficiency evaluation method. *European Journal of Operational Research, 197*, 722–730.

Wu, J., Liang, L., & Yang, F. (2009b). Determination of cross-efficiency under the principle of ranking priority in cross-evaluation. *Expert Systems with Applications, 36*, 4826–4829.

Wu, J., Sun, J., & Liang, L. (2012). Cross efficiency evaluation method based on weight-balanced data envelopment analysis model. *Computers & Industrial Engineering, 63*, 513–519.

Zhiani Rezai, H., & Davoodi, A. (2011). Some remarks on the two-level DEA model. *Applied Mathematics Letters, 24*, 969–974.